HAMILTONIAN KNESER GRAPHS

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Abstract. The Kneser graph $K(n,k)$ has as vertices the k-subsets of $\{1,2,...,n\}$. Two vertices are adjacent if the corresponding k-subsets are disjoint. It was recently proved by the first author [2] that Kneser graphs have Hamilton cycles for $n \geq 3k$. In this note, we give a short proof for the case when k divides n.

§ 1. Preliminaries.

Suppose that $n \geq k \geq 1$ are integers and let $[n] := \{1,2,...,n\}$. We denote the set of all k-subsets of a set S by $\binom{S}{k}$. The *Kneser graph* $K(n,k)$ has as vertices the k-subsets of $[n]$, that is, $V(K(n,k)) = \binom{[n]}{k}$. Two vertices are adjacent if the corresponding k-subsets are disjoint. Using a rather involved induction (on k), it was recently proved by Ya-Chen Chen that

Theorem 1 [2]. *The Kneser graph $K(n,k)$ has a Hamilton cycle for $n \geq 3k$.*

The aim of this note is to present a short proof when k divides n.

It is widely conjectured that all Kneser graphs but the Petersen graph, $K(5,2)$, have Hamilton cycles. Lovász [3] conjectures that every (finite) connected, vertex-transitive graph has a Hamilton path. For further results and an extensive list of references see [2].

§ 2. Proof of Theorem when $n = pk$.

We use some simple, new ideas for this case. First, we use Baranyai's partition theorem to partition the vertices of the Kneser graph into subsets which induce complete subgraphs; then we use Gray codes to join these subsets together to obtain a Hamilton cycle.

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Suppose that k divides n, and let $n/k = p$. Observe that $\binom{n}{k} = p\binom{n-1}{k-1}$. Let us denote $\binom{n-1}{k-1}$ by m. A *Baranyai partition* of the complete hypergraph $\binom{[n]}{k}$ is a family of m partitions of $[n]$, such that for any given i (with $1 \le i \le m$), one has that $A_i^1 \cup \cdots \cup A_i^p = [n]$, that $|A_i^1| = \cdots = |A_i^p| = k$, and that each k-subset of $[n]$ occurs among the A_i^j's exactly once. The existence of such a partition was proved in [1].

A *Gray code*, $\mathcal{C}(a,b)$, is a list $(D_1, D_2, ..., D_m)$ of the members of $\binom{[a]}{b}$, such that $|D_i \cap D_{i+1}| = |D_m \cap D_1| = b-1$ for $1 \le i < m$, where now $m := \binom{a}{b}$. It is easy to see (by induction) that Gray codes exist for all $a \ge b \ge 1$ (see [4]).

Theorem. *Suppose that n/k is an integer at least 3, then $K(n,k)$ has a Hamilton cycle.*

Proof. Set $n = pk$ and $m = \binom{n-1}{k-1}$. Consider a Baranyai partition

$$\binom{[n]}{k} = \bigcup_{i=1}^{m} \{A_i^1, A_i^2, ..., A_i^p\}.$$

We may suppose that the element n is in A_i^p, for every i with $1 \le i \le m$. We obtain that

$$\{A_1^p \setminus \{n\}, ..., A_m^p \setminus \{n\}\} = \binom{[n-1]}{k-1}.$$

Without loss of generality (permute the m partitions if necessary), we may suppose that $A_1^p \setminus \{n\},\ A_2^p \setminus \{n\}, \ldots, A_m^p \setminus \{n\}$ form a Gray code $\mathcal{C}(n-1, k-1)$. Let x_i be the element in A_i^p but not in A_{i+1}^p, so that $\{x_i\} = A_i^p \setminus A_{i+1}^p$, for $1 \le i < m$, and let $\{x_m\} = A_m^p \setminus A_1^p$.

Without loss of generality (permute the disjoint $A_{i+1}^1, A_{i+1}^2, \ldots, A_{i+1}^{p-1}$ if necessary, here we shall use $p-1 \ge 2$), we may suppose that $x_i \notin A_{i+1}^1$ (and that $x_m \notin A_1^1$). Note that $A_i^p \subset A_{i+1}^p \cup \{x_i\}$. Since A_{i+1}^1 is disjoint from A_{i+1}^p and does not contain x_i, we have that

$$A_i^p \cap A_{i+1}^1 = \emptyset.$$

Now,

$$A_1^1,\ A_1^2, \ldots, A_1^p,\ A_2^1,\ A_2^2, \ldots, A_2^p,\ \ldots,\ A_{m-1}^p,\ A_m^1,\ A_m^2, ..., A_m^p$$

form a Hamilton cycle of $K(n,k)$. □

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