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THE NUMBER OF WELL-ORIENTED REGIONS

INTRODUCTION

Consider a system $\mathscr{L}$ of $n > 1$ oriented straight lines in the Euclidean plane, in a general position. (That is, any two lines have a common point, but any point is contained in, at most, two lines.)

This arrangement divides the plane into finite and infinite convex polygonal regions. The orientation of the lines induces an orientation on the sides of the polygons. We call a region *well-oriented* if its sides form an oriented circle or path.

L. Fejes Tóth [1] raised the question: What is the maximum number of well-oriented regions in an arrangement of n lines. He conjectured

$$\text{(1)} \qquad \frac{\text{Number of well-oriented regions}}{\text{Total number of regions}} \leqslant \frac{4}{7},$$

where equality holds only in case $n = 3$ (see Figure 1). The aim of this paper is to prove this conjecture.

THEOREM *Let $\mathscr{L}$ be an arrangement of n oriented lines in a general position in the plane* ($n > 1$). *Then for the number $f(\mathscr{L})$ of well-oriented regions*

$$f(\mathscr{L}) \leqslant \left[\left(\frac{n+1}{2}\right)^2\right]$$

holds, and this bound is best possible.

After some preliminaries we prove the Theorem in the last section.

Denote by $R(\mathscr{L})$ the set of regions in $\mathscr{L}$. It is well-known that $|R(\mathscr{L})| = \binom{n}{2} + \binom{n}{1} + \binom{n}{0}$, and from this (1) follows immediately. Moreover $\max_{\mathscr{L}} \times f(\mathscr{L})/|R(\mathscr{L})|$ tends to $\frac{1}{2}$, while $n \to \infty$, where the maximum is taken over all arrangements $\mathscr{L}$ having n lines.

Let us note that $f(\mathscr{L}) \geqslant 2$ is true for every $\mathscr{L}$, as was pointed out by I. Palásti [2].

CONSTRUCTION

The following arrangement shows the sharpness of the theorem. Choose $n+1$ distinct points on a semicircle arc: $A_0, A_1, \ldots, A_n$ (in cyclic order). Consider the following n oriented lines $\overrightarrow{A_0A_1}$, $\overrightarrow{A_2A_1}, \ldots, \overrightarrow{A_{2i}A_{2i+1}}$, $\overrightarrow{A_{2i+2}A_{2i+1}}, \ldots$ (see Figure 2). It is easy to compute that the number of well-oriented regions in this system equals $[\frac{1}{4}(n+1)^2]$.

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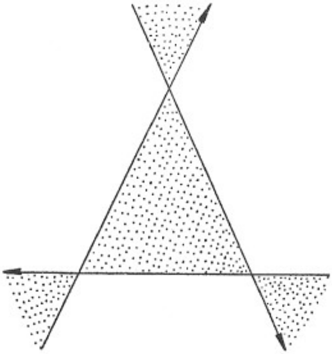

Fig. 1.

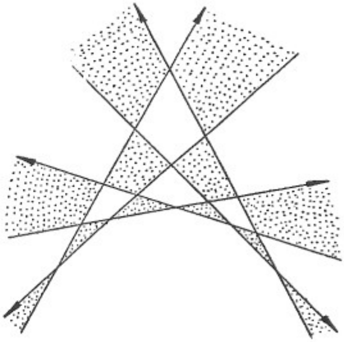

Fig. 2.

Uniformly Oriented Lines

Let $\mathscr{K}$ be a system of lines of general position and let 0 be a point which does not belong to any line. We orient each line in the same way around the point 0, say, clockwise. In this case the lines are said to be *uniformly oriented around* 0. This notion is basic to the proof, since such a system contains only very few well-oriented regions. More precisely:

(2) *If $\mathscr{K}$ is uniformly oriented around* 0 *and* 0 *is contained in an unbounded region R then there are exactly two well-oriented regions in $R(\mathscr{K})$ (namely R and the unbounded region R′ opposite to R).*

Uniformly Oriented Cutting Lines

Let T be a convex region whose boundary consists of (finitely many) oriented lines and let $\mathscr{K}$ be a system of lines in a general position, uniformly oriented around 0. If each line intersects T in a segment and the lines have no intersection points on the boundary of T, then we shall say that $\mathscr{K}$ is a *uni-*

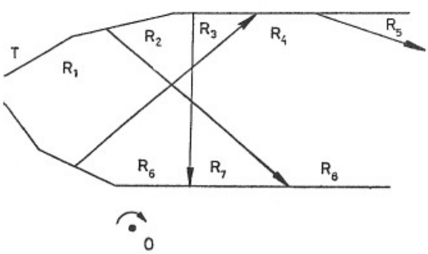

Fig. 3.

formly oriented system of cutting lines. (See Figure 3.) $\mathscr{K}$ divides T into convex subregions, denote the system of them by $R_T(\mathscr{K})$.

LEMMA *Let T be a convex domain, and $\mathscr{K}$ be a system of cutting lines well-oriented around 0. Suppose that $0 \notin T$ or 0 belongs to an unbounded region of $R_T(\mathscr{K})$. Then the number of well-oriented regions of $R_T(\mathscr{K})$ is at most the number of cutting lines.*

Proof. If $R \in \mathscr{R}_T(\mathscr{K})$ and the boundary of R is disjoint from that of T, then R cannot be well-oriented in T. We list the regions, lying beside the boundary of T in cyclic order: $R_1, R_2, \ldots, R_{2|\mathscr{K}|}$. (Certain regions may occur in this sequence several times.) Any two consecutive members on this list are neighbouring, that is, they have a segment in common. Using that the lines are in a general position, no two neighbouring regions are well-oriented at the same time. Hence, the number of well-oriented regions is at most $|\mathscr{K}|$. Q.E.D.

The proof of the theorem. Let 0 be an arbitrary point in an unbounded region R of $\mathscr{L}$. According to the two possible directions of rotation around 0, the lines of $\mathscr{L}$ can be divided into two uniformly oriented line systems $\mathscr{L}^+$ and $\mathscr{L}^-$ (see Figure 4.: the lines belonging to $\mathscr{L}^-$ are thick, the well-oriented regions are shadowed).

Now we give an upper bound for the number of well-oriented regions of $\mathscr{R}(\mathscr{L})$, contained in any fixed $T \in \mathscr{R}(\mathscr{L}^-)$. We may apply our Lemma with $\mathscr{K} = \mathscr{L}^+$, unless T has no cutting lines belonging to $\mathscr{L}^+$. In the latter case, if T contains at least one well-oriented region then (by (2)) T must be equal to R or to the unbounded region R' opposite to R. Change the roles of $\mathscr{L}^+$ and $\mathscr{L}^-$ to exclude this possibility. Thus

$$(3) \qquad f(\mathscr{L}) = \sum_{T \in \mathscr{R}(\mathscr{L}-)} f_T(\mathscr{L}^+) \leqslant \sum_{T \in \mathscr{R}(\mathscr{L}-)} (\text{the number of lines of } \mathscr{L}^+, \text{ cutting } T).$$

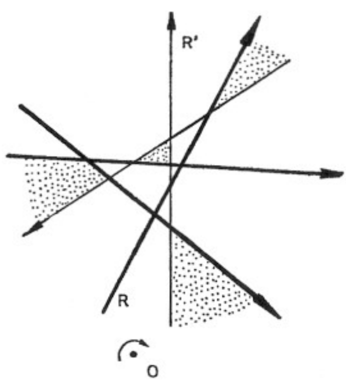

Fig. 4.

Since each line of $\mathscr{L}^+$ cuts exactly $|\mathscr{L}^-|+1$ regions belonging to $\mathscr{R}(\mathscr{L}^-)$, the right-hand side of (3) is equal to $(|\mathscr{L}^-|+1)|\mathscr{L}^+|$, from which

$$f(\mathscr{L}) \leqslant (|\mathscr{L}^-|+1)(n-|\mathscr{L}^-|) \leqslant \tfrac{1}{4}(n+1)^2$$

follows immediately. Q.E.D.

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References

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