

ARRANGEMENTS OF LINES WITH A LARGE NUMBER OF TRIANGLES

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ABSTRACT. An arrangement of lines is constructed by choosing n diagonals of the regular $2n$ -gon. This arrangement is proved to form at least $n(n-3)/3$ triangular cells.

1. Introduction. We shall use the terminology of Grünbaum [5]. By an arrangement \mathcal{A} of lines we mean a finite family of lines L_1, \dots, L_n in the real projective plane \mathbb{P} . The number of lines in \mathcal{A} is denoted by $n(\mathcal{A})$. If no point belongs to more than two of these lines L_i , the arrangement is called *simple*. With an arrangement \mathcal{A} there is associated a 2-dimensional *cell-complex* into which the lines of \mathcal{A} decompose \mathbb{P} . It is well known that in a simple arrangement \mathcal{A} the number of cells (or polygons) of that complex is $(n^2 - n + 2)/2$ ($n = n(\mathcal{A})$). We shall denote by $p_j(\mathcal{A})$ the number of j -gons among the cells of (the complex associated with) \mathcal{A} .

2. Constructions. Let us denote by $P(O)$ a fixed point on the circle \mathcal{C} with centre C . For any real α , let $P(\alpha)$ be the point obtained by rotating $P(O)$ around C , with angle α . Further denote by $L(\alpha)$ the straight line $P(\alpha)P(\pi - 2\alpha)$. In case $\alpha \equiv \pi - 2\alpha \pmod{2\pi}$, $L(\alpha)$ is the line tangent to \mathcal{C} at $P(\alpha)$. (See Figure 1.)

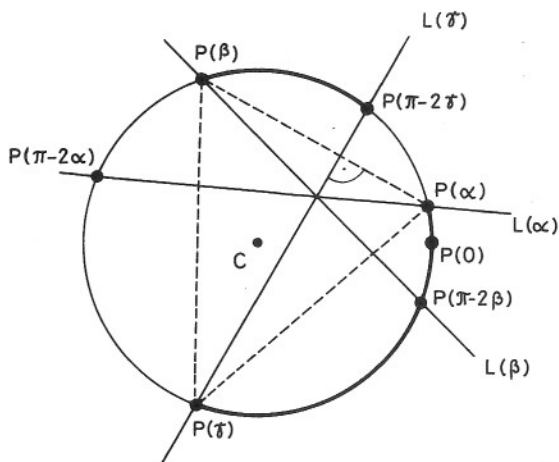


FIGURE 1

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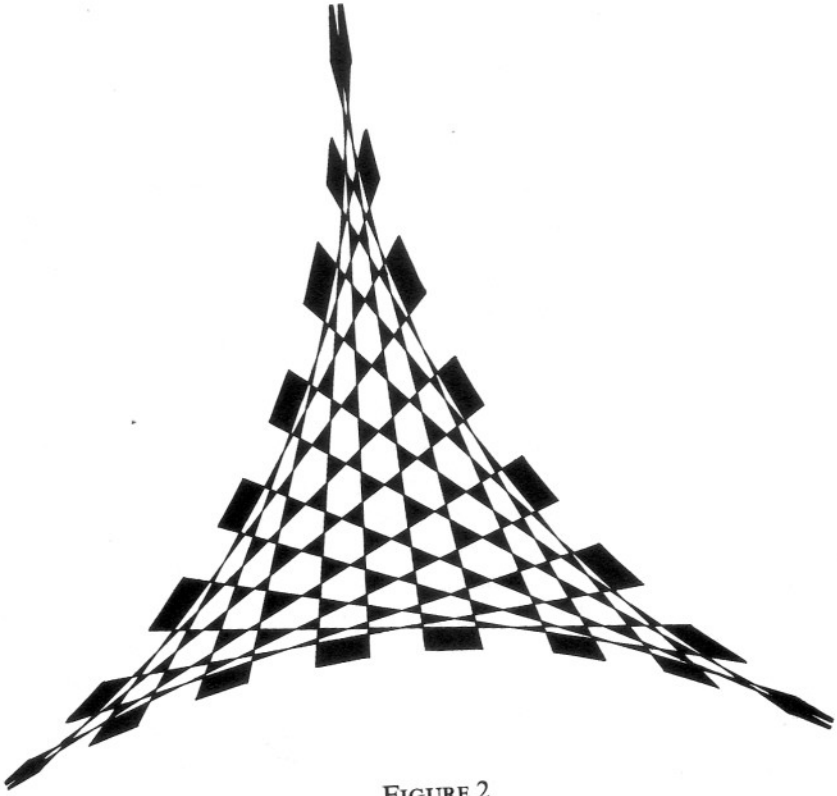


FIGURE 2

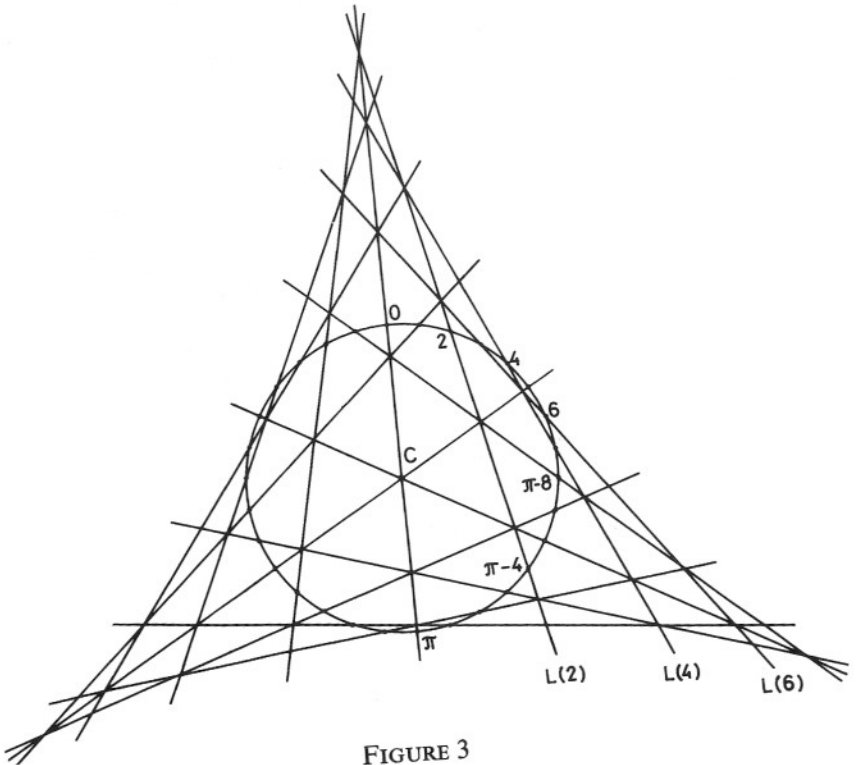


FIGURE 3

EXAMPLE 1. Given any integer $n \geq 3$, put $\mathcal{A}_n = \{L((2i+1)\pi/n): i = 0, 1, \dots, n-1\}$. (See Figure 2.)

EXAMPLE 2. Given any integer $n \geq 3$, put $\mathcal{B}_n = \{L(2i\pi/n): i = 0, 1, \dots, n-1\}$. (See Figure 3.)

Remark that our set of lines $\{L(\alpha): 0 \leq \alpha < 2\pi\}$ may be regarded as a set of tangents to the arcs of a hypocycloid of third order, drawn in a circle of centre C and radius 3. The line $L(\alpha)$ is tangent to the arc of the cycloid at the α th point. (See Figure 4.) However, we shall not rely upon this fact in what follows.

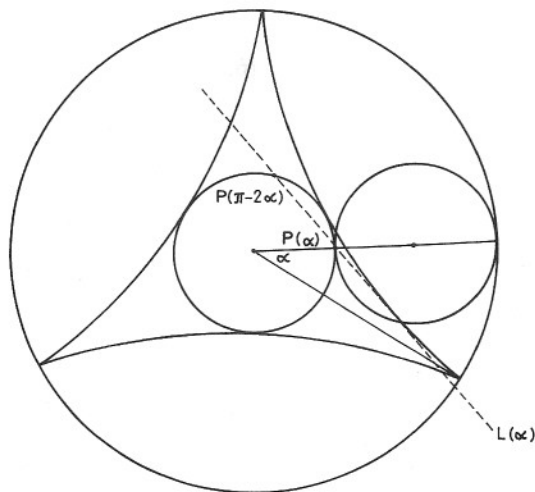


FIGURE 4

LEMMA. The lines $L(\alpha)$, $L(\beta)$ and $L(\gamma)$ are concurrent if and only if $\alpha + \beta + \gamma \equiv 0 \pmod{2\pi}$.

PROOF. If $\alpha + \beta + \gamma \equiv 0 \pmod{2\pi}$, then the sum of lengths of (directed) arcs $(P(\alpha), P(\gamma))$ and $(P(\beta), P(\pi - 2\gamma))$ is equal to π . This implies that $L(\gamma)$ is perpendicular to the line $P(\alpha)P(\beta)$. Hence, the lines $L(\alpha)$, $L(\beta)$ and $L(\gamma)$ are the altitudes of the triangle $P(\alpha)P(\beta)P(\gamma)$ (see Figure 1), consequently they meet at one point.

The reverse can be proved similarly.

3. Triangles in a simple arrangement. Grünbaum [5] (cf. Theorem 2.21) pointed out that the maximal number of triangles in a simple arrangement $p_3^s(n) = \max\{p_3(\mathcal{A}): n(\mathcal{A}) = n, \mathcal{A} \text{ is simple}\}$ can be estimated by $p_3^s(n) \leq \lfloor n(n-1)/3 \rfloor$ for even n , and $p_3^s(n) \leq \lfloor n(n-2)/3 \rfloor$ if n is odd. Moreover, he conjectured that this latter inequality holds for all n , $n \not\equiv 4 \pmod{6}$. The exact value of $p_3^s(n)$ is known only for some small values of n . (Cf., e.g., Simmons [15, 16] for the case $n = 15$, Grünbaum [5] for $n = 20$ and Harborth [8] for $n = 17$.)

As far as we know, the best lower bound by now, $p_3^s(n) > (5/16 + o(1))n^2$, was due to Füredi and J. Pach (unpublished). We are now in a position to establish a sharper lower bound.

PROPERTY 1. $p_3(\mathcal{A}_n) \geq n(n-3)/3$; hence $p_3^s(n) = n^2/3 + \mathcal{O}(n)$.

PROOF. Let (L_i, L_j) denote the intersection point of lines

$$L_i = L((2i+1)\pi/n) \quad \text{and} \quad L_j = L((2j+1)\pi/n).$$

Using the Lemma, we obtain that only $L((2n-2i-2j-2)\pi/n)$ may cross L_i and L_j at (L_i, L_j) . But this line does not belong to \mathcal{A}_n , by definition. Therefore, the lines $L_i, L_j, L_{n-i-j-1}$ and $L_i, L_j, L_{n-i-j-2}$, resp., necessarily form triangular cells.

4. Quadrangles in a simple arrangement. Grünbaum conjectured that $p_4(\mathcal{A}) \geq 1$ for any simple arrangement with $n(\mathcal{A}) > 16$. (See 2.12 in [5], cf. [15, 16].) As far as we know one cannot find in the literature any example of a simple arrangement containing $\leq o(n^2)$ quadrangles. In view of this, Grünbaum's conjecture is surprisingly modest. It is easy to prove that

PROPERTY 2. *The simple arrangement \mathcal{A}_n contains only 3 or 5 quadrangles for odd n , and $p_4(\mathcal{A}_n) = \mathcal{O}(n)$ is valid for all values of n .*

The results of §§3 and 4 are collected in Table 1. (We remark that $p_j(\mathcal{A}_n) = 0$ for $j \geq 7$.)

TABLE 1

$n \geq 5$	$p_3(\mathcal{A}_n)$	$p_4(\mathcal{A}_n)$	$p_5(\mathcal{A}_n)$	$p_6(\mathcal{A}_n)$
$n \equiv 0 \pmod{6}$	$\frac{1}{3}(n^2 - 3n)$	$n/2 + 6$	$n - 6$	$\frac{1}{6}(n^2 - 6n + 6)$
$n \equiv \pm 1 \pmod{6}$	$\frac{1}{3}(n^2 - 3n + 5)$	5	$2n - 9$	$\frac{1}{6}(n^2 - 9n + 20)$
$n \equiv \pm 2 \pmod{6}$	$\frac{1}{3}(n^2 - 3n + 8)$	$n/2$	$n - 2$	$\frac{1}{6}(n^2 - 6n + 2)$
$n \equiv 3 \pmod{6}$	$\frac{1}{3}(n^2 - 3n + 9)$	3	$2n - 9$	$\frac{1}{6}(n^2 - 9n + 24)$

5. Triangles in arbitrary arrangements. Grünbaum conjectures that for any arrangement \mathcal{A} of n lines, $p_3(\mathcal{A}) \leq n(n-1)/3$ holds. (See [5].) Let $p_3(n) = \max\{p_3(\mathcal{A}) : n(\mathcal{A}) = n\}$. The best upper bound was given by Purdy [11, 12], who proved that

$$p_3(n) \leq \frac{7}{18}n(n-1) + \frac{1}{3} \quad \text{for } n \geq 6.$$

The best lower bound, $p_3(n) \geq 4 + n(n-3)/3$, is due to Strommer [18]. His result uses a construction of Burr, Grünbaum and Sloane [1].

PROPERTY 3. $p_3(\mathcal{B}_n) \geq 4 + n(n-3)/3$.

More exactly $p_3(\mathcal{B}_n) = n(n-3)/3 + 6 - 2\epsilon/3$, where $\epsilon = 0, 2, 2$ according to whether $n \equiv 0, 1, 2 \pmod{3}$. Further, $p_4(\mathcal{B}_n) = n - 6 + \epsilon$, and $p_j(\mathcal{B}_n) = 0$ for $j \geq 5$. The proof is easy.

6. The orchard problem. Given a vertex V in an arrangement \mathcal{A} , denote by $t(V, \mathcal{A})$ the multiplicity of V , i.e. the number of lines of \mathcal{A} incident to V . Further, let $t_j(\mathcal{A})$ denote the number of vertices of multiplicity j ($2 \leq j \leq n$). We use the

notation $t_j(n) = \max\{t_j(\mathcal{A}): n(\mathcal{A}) = n\}$. The "orchard problem" has been investigated for about 150 years. It can be formulated as follows: find the value of $t_3(n)$. Significant progress has been made by Burr, Grünbaum and Sloane [1]. They proved that $t_3(n) \geq 1 + \lfloor n(n-3)/6 \rfloor$ by construction using elliptic integrals. Moreover, they conjectured that this result is sharp if n is large enough. (Confer [1] for a complete (historical) bibliography of the subject.) Our construction (see Example 2) is much simpler, but this is only a special case of their idea. (The equation of the poles of $L(\alpha)$'s is $(x^2 + y^2)(3x - 1) = 4x^3$. This can be transformed into the form $y^2 = 4x^3 - (1/12)x - (1/216)$. Cf. [1].)

PROPERTY 4. $t_2(\mathcal{B}_n) = n - 3 + \varepsilon$, $t_3(\mathcal{B}_n) = 1 + \lfloor n(n-3)/6 \rfloor$ and $t_j(\mathcal{B}_n) = 0$ for $j \geq 4$.

It should be noted that recently Szemerédi and Trotter [19] proved that there exist c and c' positive real numbers such that $cn^2/k^3 < \max_{\mathcal{A}} \sum_{i \geq k} t_i(\mathcal{A}) < c'n^2/k^3$ for all $n > k^2$.

7. Two-coloring of arrangements. It is easy to prove by induction that the cells of a (not necessarily simple) arrangement \mathcal{A} in the Euclidean plane can be colored by two colors (e.g., black and white) so that any two regions with a common side get different colors. Let $b = b(\mathcal{A})$ and $w = w(\mathcal{A})$ denote the numbers of black and white polygons. Without loss of generality we can assume that $b \geq w$. L. Fejes Tóth [3] raised the following question: What is the maximum of the ratio b/w ? Palásti [10] proved that $b/w < 2$ for $n(\mathcal{A}) \leq 9$. Upper bounds were given by Grünbaum [6], Simmons and Wetzel [17] and for higher dimensions by Purdy and Wetzel [13]. However, the exact value of $\max b/w$ is known only for some small values of n with $n \leq 16$.

Grünbaum proved that $b \leq 2w - 2$ for all arrangements \mathcal{A} with $n(\mathcal{A}) \geq 3$. For $n = 3, 5, 9$ and 15 equality holds. Our Example 1 shows

PROPERTY 5. The number of black regions $b(\mathcal{A}_n) = (n^2 + \varepsilon)/3$, where $\varepsilon = 0, 2, 2$ if $n \equiv 0, 1, 2 \pmod{3}$. So we get $b(\mathcal{A}_n) = 2w(\mathcal{A}_n) - (n + 2 - \varepsilon)$.

8. Gallai points. Let \mathcal{A} be an arrangement of n lines of the projective plane such that it does not contain a common point (i.e., $t_n(\mathcal{A}) = 0$). T. Gallai [4] proved that in this case there exist two lines from \mathcal{A} whose intersection point has multiplicity 2. This statement was improved by Kelly and Moser [9] ($t_2(\mathcal{A}) \geq 3n/7$) and recently by S. Hansen [7] ($t_2(\mathcal{A}) \geq \lfloor n/2 \rfloor$).

The following question was posed by P. Erdős [2]. Let us suppose that the arrangement \mathcal{A} does not contain a point with multiplicity more than 3. Then does there exist a Gallai triangle, i.e. three lines from \mathcal{A} such that their three intersection points have multiplicity 2, or not? Our construction \mathcal{B}_n shows that the answer is negative for $n \geq 4$, $n \not\equiv 0 \pmod{9}$. Another problem on Gallai points can be found in [2].

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