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On Maximal Intersecting Families of Finite Sets

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Let r be a positive integer. A finite family $\mathscr{H}$ of pairwise intersecting r-sets is a maximal clique of order r, if for any set $A \notin \mathscr{H}$, $|A| \leqslant r$ there exists a member $E \in \mathscr{H}$ such that $A \cap E = \varnothing$. For instance, a finite projective plane of order $r-1$ is a maximal clique. Let $N(r)$ denote the minimum number of sets in a maximal clique of order r. We prove $N(r) \leqslant \frac{3}{4}r^2$ whenever a projective plane of order $r/2$ exists. This disproves the known conjecture $N(r) \geqslant r^2 - r + 1$.

1. THE STATEMENT OF THE RESULTS

Let r be a positive integer. We say that the hypergraph $\mathscr{H}$ is a maximal clique of order r if

(1) $|E| = r$ for each $E \in \mathscr{H}$;

(2) $E \cap F \neq \varnothing$ for any $E, F \in \mathscr{H}$,

(3) for any set $A \notin \mathscr{H}$, $|A| \leqslant r$ we have $A \cap E = \varnothing$ for some $E \in \mathscr{H}$.

For example, the following hypergraphs are maximal cliques of order r:

(a) the r-subsets of a given $(2r-1)$-set;

(b) the systems of lines of a finite projective plane of order $r-1$.

Let us set
$N(r) = \min\{|\mathscr{H}| : \mathscr{H}$ is a maximal clique of order $r\}$. The determination of the value of $N(r)$ is one of the few questions dealing with the problem of determination of the minimal cardinality of set-families satisfying certain restrictions in which no set can be added to it without violating these restrictions. This type of problem was raised by Erdös and Kleitman in [2, p. 282 (b)]. There has been very little progress in these investigations up to the present moment.

Example (b) shows that for an infinite number of r's, $N(r) \leqslant r^2 - r + 1$ holds. Meyer [4, 5] (cf. Erdös [1], 11th problem) conjectured that $N(r) = r^2 - r + 1$ whenever a projective plane of order $r-1$ exists and proved that $N(3) = 7$. In what follows we give a better upper bound for $N(r)$ for

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some special values of r, using families derived from the projective plane; in particular we give counterexamples to Meyer's conjecture.

THEOREM 1. *If there exists a projective plane of order n, then* $N(2n) \leqslant 3n^2$, *i.e., for an infinite number of r's* $N(r) \leqslant \frac{3}{4}r^2$ *holds.*

This result raises the question of the magnitude of $N(r)$ for other values of r. The following construction, originally constructed for the case $n = 2$, and later generalized for all values of n greater than 2 by Babai and the author, gives other counterexamples for the conjecture.

PROPOSITION 1. *If there exists a projective plane of order n then we have* $N(n^2 + n) \leqslant n^4 + n^3 + n^2$.

Theorem 1 and Proposition 1 provide presumably infinite families of counterexamples to the conjecture of Meyer, namely when n and $2n - 1$ (n and $n^2 - n + 1$, respectively) are simultaneously prime powers.

The exact value of $N(r)$ or at least its order of magnitude is unknown. It is not even clear whether or not $N(r) = O(r^2)$ holds. I can prove only $N(r) < r^{c \cdot r^{7/12}}$. We should mention that the best known lower bound, which is due to Erdös and Lovàsz [3], says $N(r) \geqslant (8r/3) - 3$.

PROPOSITION 2. *If* $\mathscr{H}$ *is a maximal clique of order r then either* $|\mathscr{H}| > r^2$ *or* $|V(\mathscr{H})| > r^2/(2 \log r)$, *where* $V(\mathscr{H}) = \bigcup \{H: H \in \mathscr{H}\}$.

We have the following

Conjecture. If $\mathscr{H}$ is a maximal clique then $|\mathscr{H}| \geqslant |V(\mathscr{H})|$.

Our conjecture in view of Proposition 2 would imply $N(r) > r^2/(2 \log r)$. Let us set $\overline{N}(r) = \max\{|\mathscr{H}|: \mathscr{H}$ is a maximal clique of order $r\}$. Erdös and Lovàsz [3] give an example showing $\overline{N}(r) \geqslant [r \mid (e - 1)]$. On the other hand they prove $\overline{N}(r) \leqslant r^r$. More exactly they prove that $|\mathscr{H}| \leqslant r^r$ holds if $\mathscr{H}$ satisfies (1), (2), and

(3′) for any set A, $|A| = r - 1$ we have $A \cap E = \varnothing$ for some $E \in \mathscr{H}$.

The proof is not complicated. Here we prove another easy assertion which is a bit more general.

PROPOSITION 3. *Let us suppose for the hypergraph* $\mathscr{H}$ *that for every* $E_1, \ldots, E_{k+1} \in \mathscr{H}$ *we have* $|\bigcup_{i \neq j} (E_i \cap E_j)| \geqslant \max_{E \in \mathscr{H}} |E| =: r$. *Then* $|\mathscr{H}| \leqslant k^r$.

In the case of equality we can find pairwise disjoint k-element sets $S_1, \ldots, S_r$ such that

$$\mathscr{H} = \{A: |A| = r, |A \cap S_i| = 1 \text{ for } i = 1, \ldots, r\}.$$

This proposition, although not too difficult, gives a sharp bound.

2. The Proofs

Let $\mathscr{P}$ denote the system of lines of a projective plane of order n. Let us fix an arbitrary $E_0 \in \mathscr{P}$, $E_0 = \{x_0, \dots, x_n\}$. Let us set $\mathscr{L}_i = \{E - x_i : E \in \mathscr{P}, x_i \in E, E \neq E_0\}$. The family $\bigcup_{r=0}^{n} \mathscr{L}_i$ is the corresponding affine plane, $\mathscr{A}$, with $V(\mathscr{A}) = V(\mathscr{P}) - E_0$. The $\mathscr{L}_i$'s are different classes of parallel lines, $|\mathscr{L}_i| = n$. (See Fig. 1. On the figures the places of 0's are left empty; we mark only the incidences.)

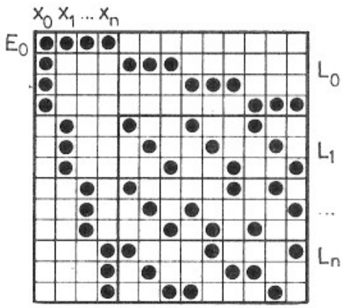

FIG. 1. cf. Lemma 1. ($n = 3$) Gamma-tableau of the projective plane of order 3.

LEMMA 1. *Let $S \subset V(\mathscr{A})$, $|S| \leqslant n$ such that for some i, $0 \leqslant i \leqslant n$, we have $S \cap E \neq \varnothing$ for every $E \in \bigcup_{j \neq i} \mathscr{L}_j$. Then $S \in \mathscr{L}_i$.*

The proof of the lemma. We have $|\mathscr{A} - \mathscr{L}_i| = n^2$. Every $\mathscr{L}_j$ consists of n pairwise disjoint sets, which cover $V(\mathscr{A})$. Thus the members of $\mathscr{A} - \mathscr{L}_i$ cover every point of $V(\mathscr{A})$ exactly n times. As S meets at most $|S| \cdot n$ lines of $\mathscr{A} - \mathscr{L}_i$, $|S| = n$ follows. Moreover we conclude that different points of S cover different lines. On the other hand for any two points of $V(\mathscr{A})$ there is exactly one $A \in \mathscr{A}$ containing them. So it holds for any two points of S as well. We conclude that the corresponding lines belong to $\mathscr{L}_i$, and consequently it is always the same line, yielding the assertion.

The proof of Theorem 1. Let us consider the so-called gamma-tableau of a projective plane $\mathscr{P}$ of order n. We may obtain it from an arbitrary incidence matrix C of $\mathscr{P}$ by interchanging rows and columns of C in such a way that the first $n + 1$ columns correspond to $x_0, \dots, x_n$; the first row is E_0. The next n rows are the remaining lines passing through x_0, then come the lines containing x_1, and so on. In this way, in the rows $2 + in$ on through $1 + (i + 1)n$ are the lines $\mathscr{L}_i \cup \{x_i\} = \{E \in \mathscr{P} : x_i \in E \neq E_0\}$ $(0 \leqslant i \leqslant n)$. Let C' denote the matrix which we obtain from C after deleting the first $n + 1$ rows and columns. Let C_0 denote the $n^2 \times n^2$ zero-matrix and C_1 the direct sum of n copies of J_n, the $n \times n$ matrix which has 1's in every position.

From these matrices we compose the following $3n^2 \times 3n^2$ 0–1 matrix:

$$A_{\mathscr{H}} = \begin{bmatrix} C_1 & C' & C_0 \\ C_0 & C_1 & C' \\ C' & C_0 & C_1 \end{bmatrix}$$

Let $\mathscr{H}$ be the hypergraph having $A_{\mathscr{H}}$ for its incidence matrix (cf. Fig. 2.)

We assert that $\mathscr{H}$ is a maximal clique of order $2n$. As $|\mathscr{H}| = 3n^2$, this it would imply Theorem 1.

It is evident that $\mathscr{H}$ satisfies (1). From the construction it is not hard to see that for $\mathscr{H}$, (2) holds as well. Let us now consider a set S, $|S| \leqslant 2n$ such that $S \cap E \neq \varnothing$ holds for every $E \in \mathscr{H}$. All we have to prove is $S \in \mathscr{H}$.

Let us partition $V(\mathscr{H})$ into B_1, B_2, B_3 according to the $n^2 \times n^2$ submatrices, and $\mathscr{H}$ into A_1, A_2, A_3. If $|S \cap B_1| < n$ then for some i, $0 \leqslant i < n$, we have $S \cap \{b_{in+j}: j = 1,..., n\} = \varnothing$, where b_q is the qth element of B_1. As S has nonempty intersection with each of the corresponding edges, i.e., with E_{in+j} for $j = 1,..., n$, but these edges are pairwise disjoint outside of B_1 and do not intersect B_3, we infer $|S \cap B_2| \geqslant n$. In essentially the same way $|S \cap B_2| < n$ implies $|S \cap B_3| \geqslant n$, and $|S \cap B_3| < n$ implies $|S \cap B_1| \geqslant n$. By symmetry reasons we may assume $|S \cap B_1| \geqslant n$, $|S \cap B_2| \geqslant n$. Consequently $|S| \leqslant 2n$ yields $|S \cap B_1| = n$, $|S \cap B_2| = n$, $|S \cap B_3| = 0$.

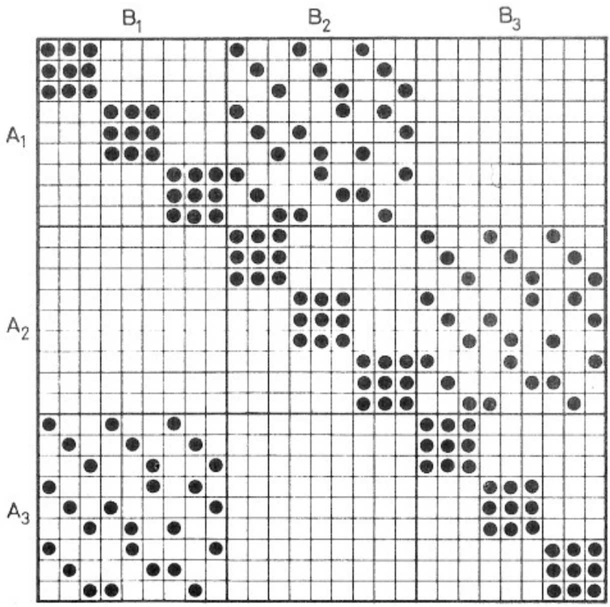

FIG. 2. cf. Theorem 1. ($n = 3$) Incidence matrix of a maximal clique of order 6.

Now we deduce that $S \cap (B_1 \cup B_3) = S \cap B_1$ covers the edges in C'. But C' consist of the line classes $\mathscr{L}_1, \ldots, \mathscr{L}_n$ of the affine space whence Lemma 1 yields $(S \cap B_1) \in \mathscr{L}_0$. This means that for some i, $1 \leqslant i \leqslant n$, $S \cap B_1$ coincides with $E_q \cap B_1$ for $q = in - n + j$, where $1 \leqslant j \leqslant n$. Consequently $S \cap B_2$ covers the line classes $\mathscr{L}_j$, $1 \leqslant j \leqslant n$, $j \neq i$, in B_2. Moreover considering the edges in A_2 we derive that $S \cap B_2$ covers $\mathscr{L}_0$ as well. Applying Lemma 1 we obtain that $(S \cap B_2) \in \mathscr{L}_i$; consequently $S = (S \cap B_1) \cup (S \cap B_2) \in A_1 \subset \mathscr{H}$. Q.E.D.

The proof of Proposition 1. Let C be the incidence matrix of a projective plane of order n. As any regular bipartite graph has a 1-factorization, one can color the nonzero elements of C using $n + 1$ colors in such a way that every color occurs in every row and column exactly once (cf. Fig. 3). Let us consider the line classes $\mathscr{L}_i$, $0 \leqslant i \leqslant n$, of the affine space $\mathscr{A}$ and let us construct an $n^2 \times n^2$ matrix K_i in such a way that each line of $\mathscr{L}_i$ occurs exactly n times as a row of K_i and the main diagonal of K_i consists merely of 1's. Now let us replace every 1 of color number i by a copy of K_i and every zero, an $n^2 \times n^2$ zero-matrix.

In such a way we obtain an $(n^4 + n^3 + n^2) \times (n^4 + n^3 + n^2)$ matrix which is the incidence matrix of a hypergraph that is a maximal clique of order $n^2 + n$.

The proof of this Proposition runs analogously to that of Theorem 1. Statements (1) and (2) can be seen easily. To prove (3) one divides $V(\mathscr{H})$ into the classes $B_1, B_2, \ldots, B_{n^2+n+1}$ according to the $n^2 \times n^2$ submatrices, and $\mathscr{H}$ into the classes $A_1, \ldots, A_{n^2+n+1}$.

Let S be a subset of $V(\mathscr{H})$ with at most $n^2 + n$ elements such that S meets any $E \in \mathscr{H}$. It can be proved that if $|B_i \cap S| < n$ for some i then $B_i \cap S = \varnothing$. Applying Lemma 1 first for the system $\{B_i\colon 1 \leqslant i \leqslant n^2 + n + 1\}$, next for the systems $\{B_i \cap E\colon E \in \mathscr{H}\}$ and finally for an A_i we get $S \in \mathscr{H}$. The details are left to the reader.

The proof of Proposition 2. $r = 1$ has no practical interest. For $r = 2$ the statement is true (because the only maximal clique of order 2 is a triangle). In what follows $r \geqslant 3$.

Let $B \subset V(\mathscr{H})$, $|B| = r$. Then either $B \in \mathscr{H}$ or $B \cap E = \varnothing$ for some $E \in \mathscr{H}$. Hence we obtain:

$$\binom{|V(\mathscr{H})|}{r} - |\mathscr{H}| \leqslant \sum_{\substack{E \in \mathscr{H} \\ B \subset V(\mathscr{H}),\, |B| = r,\, B \cap E = \varnothing}} 1 = |\mathscr{H}| \cdot \binom{|V(\mathscr{H})| - r}{r},$$

and putting $|V(\mathscr{H})| = v$

$$|\mathscr{H}| \geqslant \frac{\binom{v}{r}}{1 + \binom{v-r}{r}} =: f(v).$$

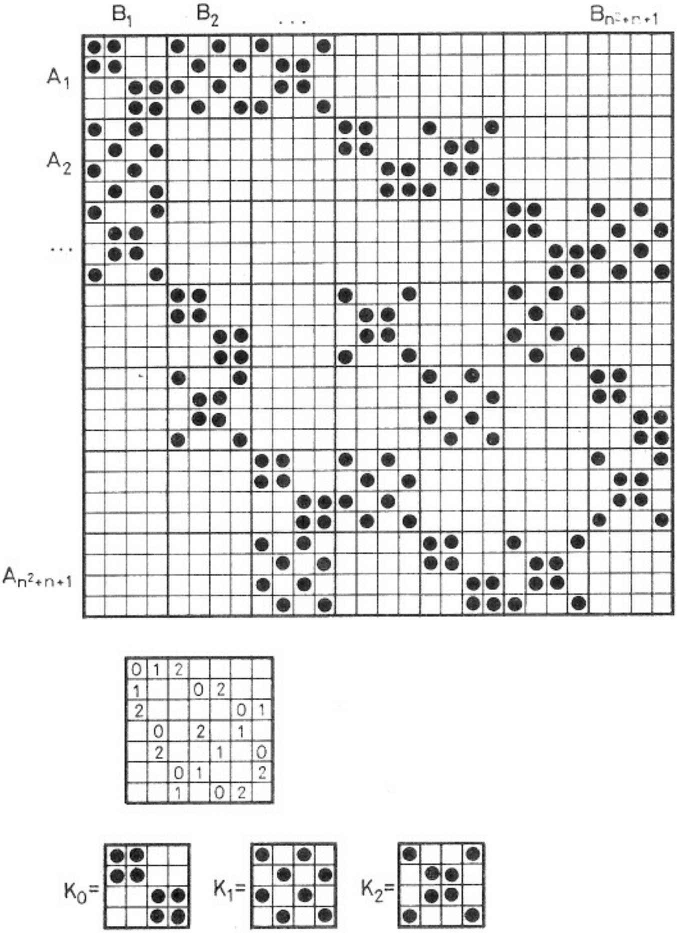

FIGURE 3

Clearly $v \geqslant 2r - 1$. If $v = 2r - 1$ or $2r$ then $|\mathscr{H}| = \left({r \atop 2r-1}\right) > r^2$. In the interval $[2r+1; \infty)$ the function $f(v)$ is monotone decreasing, as one can easily see by derivation. We have

$$f(v) = \frac{\binom{v}{r}}{\binom{v-r}{r}} \cdot \frac{1}{1 + 1\Big/\binom{v-r}{r}} = \prod_{i=0}^{r} \left(1 + \frac{r}{v-i-r}\right) \cdot \frac{1}{1 + 1\Big/\binom{v-r}{r}}.$$

Using that $1 + a/(b-a) > e^{a/b}$ whenever $b > a > 0$, we get

$$f(v) > \exp\left(\sum_{i=0}^{r-1} \frac{r}{v-i}\right) \cdot \exp\left[-\frac{1}{\binom{v-r}{r}}\right]$$

$$= \exp\left[\frac{r^2}{v} + \sum_{i=0}^{r-1} \frac{r \cdot i}{v(v-i)} - \frac{1}{\binom{v-r}{r}}\right]$$

$$> \exp\left[\frac{r^2}{v} + \frac{r^2(r-1)}{2v^2} - \frac{1}{\binom{v-r}{r}}\right].$$

If $v = r^2/(2 \log r)$ then $(2v^2/r^2(r-1)) < r < \binom{r}{r+1} \leqslant \binom{r}{v-r}$. So we get that

$$|\mathscr{H}| \geqslant f(v) \geqslant f(r^2/(2 \log r)) > r^2$$

if $v \leqslant r^2/(\log r)$, that was to be proved.

The proof of Proposition 3. Let $c(r, k)$ denote the maximum cardinality a hypergraph $\mathscr{H}$ satisfying the assumptions of the proposition can have. We apply induction on k; once k is fixed we apply induction on r to prove $c(r, k) = k^r$. The cases $k = 1$ or $r = 1$ are trivial. Let E_0 be an arbitrary edge of $\mathscr{H}$ which satisfies the assumptions. We have

$$|\mathscr{H}| = \sum_{X \subset E_0} |\{E : E \in \mathscr{H}, E \cap E_0 = X\}|. \tag{i}$$

For the hypergraphs $\{E - E_0 : E \in \mathscr{H}, E \cap E_0 = X\} =: \mathscr{H}_X$ we apply the inductional hypothesis $r' \leqslant r - |X|$, $k' = k - 1$. We deduce

$$|\mathscr{H}_X| \leqslant (k-1)^{r-|X|} \tag{ii}$$

Combining (i) and (ii) we obtain

$$|\mathscr{H}| \leqslant 1 + \sum_{X \subsetneq E_0} (k-1)^{r-|X|} \leqslant \sum_{i=0}^{r} \binom{r}{i} (k-1)^{r-i} = k^r,$$

as desired.

It is quite clear that equality occurs only in the case described in the statement of Proposition 3.

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References

1. P. Erdös, Problems and results in graph theory and combinatorial analysis, *in* "Proceedings, Fifth British Combinatorial Conference, Aberdeen, 1975" (C. St. J. A. Nash-Williams and J. Sheehan, Eds.), pp. 169–192, Utilitas Math. Publ., Winnipeg, 1976.
2. P. Erdös and D. J. Kleitman, Extremal problems among subsets of a set, *Discrete Math.* **8** (1974), 281–294; (see also "Proceedings Second Chapel Hill Conference, University of North Carolina, August 1970," pp. 146–170).

3. P. Erdös and L. Lovàsz, Problems and results on 3-chromatic hypergraphs and some related questions, *in* "Infinite and Finite sets" (A. Hajnal, R. Rado, and V. T. Sòs, Eds.), Proc. Coll. Math. Soc. J. Bolyai, Vol. 10, pp. 609–627, North-Holland, Amsterdam, 1974.
4. J.-C. Meyer, Quelques problèmes concernant les cliques des hypergraphes h-complets et q-parti h-complets, *in* "Proceedings, Hypergraph Seminar, Columbus, Ohio, 1972" (C. Berge and D. H. Ray-Chaudhuri, eds.), pp. 127–139, Lecture Notes in Mathematics No. 411, Springer-Verlag, New York/Berlin, 1974.
5. J.-C. Meyer, 23rd unsolved problem, *in* "Proceedings, Hypergraph Seminar, Columbus, Ohio, 1972" (C. Berge and D. H. Ray-Chaudhuri, Eds.), pp. 285–286, Lecture Notes in Mathematics No. 411, Springer-Verlag, New York/Berlin, 1974.

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