Nonexistence of universal graphs without some trees

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0. Introduction

Recent years have seen considerable progress in the theory of *universal graphs*, i.e., when one investigates the existence of universal elements of various classes of countable graphs. The first such result was given by R. Rado [14,15], who proved the existence of a universal countable graph, a countable graph which isomorphically embeds every countable graph. A well-known argument gives that there exists a universal countable K_n-free graph where K_n is the complete graph on n vertices. Hajnal and Pach [9] showed the nonexistence of a universal countable C_4-free graph, and this opened the way toward proving non-existence results on classes characterized by the exclusion of some finite subgraphs. Komjáth and Pach [11] generalized this to the case when $K_{a,b}$, the complete bipartite graph on a, b vertices is excluded. Only for $a = 1$, $b \leq 3$ is there a universal graph. Cherlin and Komjáth [3] gave another generalization of the Hajnal-Pach theorem, they showed that there is no universal countable C_n-free graph where C_n is the circuit of length $n \geq 4$. In [10], however, it was shown that there does exist a universal graph if all odd circuits up to a certain length are excluded (so we cannot exclude C_5 alone but we can C_3 and C_5). A recently proved counterpart to this is the result of Cherlin and Shi [6], if C_{n_1}, $C_{n_2}, \dots, C_{n_k}$ are excluded, then there is no universal graph except in the above case when $\{n_1, n_2, \dots, n_k\}$ is a set of the first some odd numbers. It is also proved in [10] that there is a universal graph if P_n, the path on n edges or if all circuits from a certain length onward are excluded.

Another very general result is in [7] where we showed that there is no universal countable G-free graph if G is a 2-connected noncomplete graph.

At the other end of the spectrum stand the trees. After the above mentioned result on paths Goldstern and Kojman [8] proved that there is no universal graph when a *bridge* is omitted, that is, a tree with $n \geq 5$ edges and two vertices of degree 3 in distance $n - 4$. Here we show nonexistence for a general class of trees; when there is a unique vertex of largest, and ≥ 4, degree which has a neighbor of degree one. Our result, nevertheless, does not contain the above result on bridges.

For our proof of nonexistence we need to show that for $r \geq 3$ there exist r-regular, r-connected graphs with arbitrarily large girth (Lemma 3). This statement and the statement of Lemma 4 can be deduced from known properties of random regular graphs, i.e., in the model when one member is taken from the set of all r-regular graphs on a certain vertex set (see [1], Cor.2.19. on p.53, on short circuits, and Theorem 7.32. on p.174, on connectivity). Our proof seems to be different, more in the elementary context, via a packing statement (which is, perhaps, new).

We mention one more interesting subclass of results and problems. For certain cases when the existence of universal countable G-free graphs is to be shown one can proceed as

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follows. First, a case analysis shows that from a certain structural point of view finitely many different classes of G-free graphs exist then it is observed that each has a universal element, finally the vertex disjoint union of them is taken. If G, the omitted graph, is disconnected, the last step cannot be executed, so for this case the appropriate notion—it seems—is the following. The class of countable G-free graphs has *finite complexity* if there are finitely many G-free graphs, $X_1, \dots, X_n$ such that every countable G-free graph can be embedded into some X_i. This type of problems was first investigated in [12] it was shown that if $G = K_3 + K_3$, the disjoint union of two triangles, then the corresponding class is of finite complexity. A conjecture of [12], that this can be extended to arbitrary $K_{n_1} + \cdots + K_{n_r}$ has recently been established by Cherlin and Shi [5].

1. Notation, definitions

A *graph* is any set of two-element subsets of some set (called the set of *vertices*). That is, our graphs are undirected, loopless, and (with one exception) without double edges. If x is a vertex then $d(x)$, the *degree* of x is the number of edges incident to x. If x, y are vertices, then $d(x, y)$, the *distance* of x and y is the length of the shortest path between x and y. For simplicity we say that a graph *has girth* g if there is no circuit of length $\leq g$ (has girth at least g would be the proper name). Given two graphs X_1 on V_1 and X_2 on V_2 and embedding of X_1 into X_2 is an injection $f : V_1 \to V_2$ such that if x and y are joined in X_1 then $f(x)$, $f(y)$ are joined in X_2. If G, X are graphs, X is *G-free* if there is no embedding of G into X. If $\mathcal{H}$ is a class of graphs then $X \in \mathcal{H}$ is universal if every $Y \in \mathcal{H}$ embeds into X. We notice that usually when the existence of a universal element is proved we usually are able to show that there is some element which isomorphically embeds every element. On the other hand proofs of nonexistence (like the one in the present paper) usually show that elements with the weaker embedding property fail to exist.

2. Lemmas

If two graphs, X and Y are given on the same vertex set, an *alternating* (X, Y)*-path of length* h is a sequence of distinct vertices $(x_0, \dots, x_h)$ such that $\{x_0, x_2\}, \{x_2, x_4\}, \dots$ are in X while $\{x_1, x_3\}, \{x_3, x_5\}$ are in Y.

An *alternating* (X, Y)*-circuit of length two* is a pair (x, y) of vertices such that x, y are joined both in X and Y. For an even number $h > 2$ an *alternating* (X, Y)*-circuit of length* h is a sequence of distinct vertices $(x_0, \dots, x_{h-1})$ such that x_{2i} and x_{2i+1} are joined in X and x_{2i+1} and x_{2i+2} are joined in Y (with the understanding that $x_h = x_0$).

We notice that if both X and Y have degree $\leq c$ then the number of vertices reachable from any given vertex of V by an alternating (X, Y)-path of length $\leq h$ is at most $1 + c + \cdots + c^h \leq 2c^h$ (for $c \geq 2$).

The following Lemma is a generalization of a theorem of Catlin and Sauer-Spencer (see [2,16]).

Lemma 1. *If $c \geq 2$, h is even, X and Y are graphs of degree $\leq c$ on the same vertex set V with $|V| \geq 9c^h$ then there is a permutation $\pi : V \to V$ such that there is no alternating $(\pi(X), Y)$ circuit of length $\leq h$.*

Proof. Let $V = \{x_1, \ldots, x_n\}$. By induction on $1 \le i \le n$ we show that there is an injection $\varphi : \{x_1, \ldots, x_i\} \to V$ such that there is no alternating $(\varphi(X), X)$-circuit of length $\le h$. The case $i = n$ gives the desired result. For $n = 1$ our statement is trivial.

In the proof if $A \subseteq \{x_1, \ldots, x_i\}$ then $\varphi[A] = \{\varphi(a) : a \in A\}$ and $\varphi(X)$ is the graph formed with the edges of the form $\{\varphi(x_j), \varphi(x_k)\}$ where $1 \le j < k \le i$ and x_j, x_k are joined in X.

Assume that we have found a $\varphi : \{x_1, \ldots, x_{i-1}\} \to V$ as required but we cannot extend it to an appropriate $\varphi' : \{x_1, \ldots, x_i\} \to V$. Set $S = \{x_j : j < i, \{x_j, x_i\} \in X\}$, $U = V - \mathrm{Rng}(\varphi)$. The fact that for a certain $y \in U$ we cannot select $\varphi'(x_i) = y$ means that there is some alternating $(X, \varphi(X))$-path of length $\le h-1$ from y to an element of $\varphi[S]$. y is, therefore, available from S in at most $h-1$ steps, so we get that $|U| \le |S| \cdot 2c^{h-1} \le 2c^h$.

We conclude that $i - 1 \ge |V| - 2c^h \ge 7c^h$.

Fix a vertex $y \in U$. Our idea is to show that there exists an x_j (with $1 \le j < i$) such that the following function $\varphi' : \{x_1, \ldots, x_i\} \to V$ is appropriate. $\varphi'(x_j) = y$, $\varphi'(x_i) = \varphi(x_j)$, otherwise φ' is the same as φ.

Fix x_j and assume that φ' creates an alternating $(\varphi'(X), X)$-circuit, C of length $\le h$. It must obviously contain y, $z = \varphi(x_j)$, or both. We are going to distinguish cases and in each case give a bound on the number of vertices x_j that may fall into that case.

Case 1. *The length of C is two.*

In this case there is a common edge, e, of X and $\varphi'(X)$. If e is between z and y then x_i and x_j are joined in X, there are at most c possibilities for x_j. If e is between z and some element of $\varphi[S]$ then $\varphi(x_j)$ is joined to $\varphi[S]$, the number of possibilities is at most $|S|c \le c^2$. If e is between y and some other vertex t, then t must be a neighbor of y (at most c possibilities) and x_j is a neighbor of $\varphi^{-1}(t)$, at most c^2 possibilities. In Case 1., we have altogether $\le 2c^2 + c$ possibilities for x_j.

¿From now on we assume that the length of C is > 2. Set $T = \{x_k : k < i, \{x_k, x_j\} \in X\}$.

Case 2. *C contains y but not z.*

In this case the $\varphi'(X)$ edge of C incidental to y must go to $\varphi[T]$ so (by removing this edge from C) there is an alternating $(X, \varphi(X))$-path of length $\le h-1$ from y to $\varphi[T]$, this gives $\le c^h$ possibilities.

Case 3. *C contains z but not y.*

Similarly, there is an alternating $(X, \varphi(X))$-path of length $\le h-1$ from z to $\varphi[S]$. As $|S| \le c$ this gives $\le c^h$ possibilities for z, and so for x_j.

Case 4. *C contains both y and z.*

In this case, C, a circuit of even length, is split by y and z into two paths, P_1 and P_2.

Subcase 4.1. *P_1 and P_2 are of odd length.*

Let P_1 be the alternating $(X, \varphi'(X))$-path from y to z, it is an $(X, \varphi(X))$-path, and this gives $\le 2c^{h-1}$ possibilities for z, and so for x_j.

Subcase 4.2. *P_1 and P_2 are of even length.*

Then one of them, say P_1, is a $(\varphi'(X), X)$-path from y to z. If its first vertex after y is u then $u \in \varphi[T]$ so z and u are joined in $\varphi(X)$, i.e., there is an alternating $(X, \varphi(X))$-circuit of length $|P_1|$, a contradiction to the inductive assumption.

To finish the proof we observe that at most $2c^h + 2c^{h-1} + 2c^2 + c < 7c^h$ vertices of the form x_j can fall under the various cases, a contradiction to our hypotheses. □

Lemma 2. *Given d, g, there is a number $v = v(d, g)$ such that if X, Y are two graphs of degree $\leq d$ and girth g on the same vertex set V with $|V| \geq v(d, g)$ then there is a permutation $\pi : V \to V$ such that the graph $\pi(X) \cup Y$ contains neither double edges nor circuits of length $< g$.*

Proof. Apply the previous Lemma to X', Y', where two vertices are joined in X' if their distance in X is $< g$, and likewise for Y'. Observe that the degree of X', Y' can be bounded by d and g, and any circuit in $\pi(X) \cup Y$ is either a circuit in $\pi(X)$ or is one in Y or gives rise to an alternating $(\pi(X'), Y')$-circuit. □

Lemma 3. *If $k \geq 2$, g are given, there are arbitrarily large k-regular, k-connected graphs with girth g.*

Proof. By induction on k. For $k = 2$ one can simply take a long enough circuit.

Assume now that we have an example X on some vertex set V for k, g and $|V| > v(k, g)$. We build an example for $k+1$, g on a set of size $2|V|$. We first take two copies of X, X_1 and X_2 on the disjoint vertex sets V_1 and V_2. By Lemma 1., there is a bijection $\varphi : V_1 \to V_2$ such that $\varphi(X_1) \cup X_2$ contains no short circuits.

To construct the desired graph Z take the edges of X_1, X_2, along with the matching F of the edges of the form $\{x, \varphi(x)\}$ for $x \in V_1$. Clearly, Z is $(k+1)$-regular and the girth of Z is g.

In order to show that Z is $(k+1)$-connected assume that $A \subseteq V_1$, $B \subseteq V_2$, $|A|+|B| \leq k$ and $A \cup B$ separates Z. If $B = \emptyset$ then the (connected) X_2 is in one of the remaining components but F adds every element of $V_1 - A$ to that component. A similar argument works if $A = \emptyset$. If, however, A, $B \neq \emptyset$, then by the properties of X, as $|A|$, $|B| < k$, both $V_1 - A$ and $V_2 - B$ induce connected graphs, and there is at least one edge between them, as $|A \cup B| \leq k < |V_1|$ so $A \cup B$ cannot cover F. □

Lemma 4. *Given $r \geq 3$, a, g, there exists a natural number $f(r, g, a)$ with the following properties. For every N there is an r-regular 3-connected graph X with at least N vertices, with girth g, such that if the vertex set V of X is decomposed as $V = A \cup B$ with $|A|$, $|B| \geq f(r, g, a)$, then the number of edges between A and B is at least a.*

Proof. We first argue that there exists a 2-connected graph Y with girth g on some vertex set of w elements containing exactly $2a$ vertices of degree $r-1$ and $w-2a$ vertices of degree r.

This can be proved with a simple modification of the proof of Lemma 2. Let T be a large enough connected $(r-1)$-regular graph of girth g. Take two disjoint copies of it, as in the construction of Lema 2., but rather than joining them by a matching we draw only $|T| - a \geq 2$ edges between them.

Set $f(g, r, a) = aw$. Given N as in the statement of the Lemma let Z be a $2a$-regular $2a$-connected graph with girth g on a set I with $|I| \geq N/w$.

For $i \in I$ let Y_i be a graph, isomorphic to Y, on some vertex set W_i such that the sets $\{W_i : i \in I\}$ are disjoint. We extend the union of those graphs $\bigcup\{Y_i : i \in I\}$ to an r-regular graph X on $W = \bigcup\{W_i : i \in I\}$ such that if i, $j \in I$ are joined in Z then there are $x \in W_i$, $y \in W_j$, joined in X which are of degree $r-1$ in Y_i, Y_j. This is obviously possible as the degree of i in Z is the same as the number of vertices with degree $r-1$ in Y_i.

Assume that $W = A \cup B$ is a decomposition in which $|A|$, $|B| \geq f(g,r,a) = aw$. We have to show that there are at least a edges between A and B. If there are at least a indices $i \in I$ such that $A \cap W_i \neq \emptyset$, $B \cap W_i \neq \emptyset$, then (as Y is connected) each such W_i contains an edge between A and B and we are done.

In the other case all but $\leq a-1$ of those sets are entirely in A or in B. If the number of edges between $\bigcup\{W_i : W_i \subseteq A\}$ and $\bigcup\{W_i : W_i \subseteq B\}$ is $< a$ then the removal of $< 2a$ points disconnects Z, a contradiction.

We finally show that X is a 3-connected graph. Assume that the removal of two vertices, x_1 and x_2, disconnects X.

If x_1 and x_2 are in distinct vertex sets, i.e., $x_1 \in W_i$, $x_2 \in W_j$ for some $i \neq j$, then, $X_i - \{x_1\}$, $X_j - \{x_2\}$ are connected, and as Z is connected, $X - \{x_1, x_2\}$ is connected, as well.

Assume now that x_1, $x_2 \in W_i$ for some i. Remember the way how X_i was created. We took two copies of some graph T on some sets W_i' and W_i'' and draw an appropriate partial matching F between them. Let $A_i \subseteq W_i'$, $B_i \subseteq W_i''$ be the sets covered by F. The vertices in $U = (W_i' - A_i) \cup (W_i'' - B_i)$ are covered by those edges of X which go between W_i and some other W_j, so, as $Z - \{i\}$ is connected, $U - \{x_1, x_2\}$ and the part outside W_i are in one connected component.

If $x_1 \in W_i'$, $x_2 \in W_i''$, then, again, as T is 2-connected, $W_i' - \{x_1\}$ and $W_i'' - \{x_2\}$ are both connected which establishes that $X - \{x_1, x_2\}$ is connected.

If, finally, x_1, $x_2 \in W_i'$ (say), then W_i'' is entirely in a component, but then the edges between B_i and A_i add all the vertices in A_i into that component which therefore contains everything. □

Lemma 5. *Same as the previous Lemma, except that we require X be connected and having four vertices a_1, a_2, b_1, b_2 of degree $r-1$, all other vertices of degree r, and $d(a_1,a_2)$, $d(b_1,b_2) \geq g-2$.*

Proof. Take the example of Lemma 4, select two pairs of neighboring vertices, $\{a_1, a_2\}$ and $\{b_1, b_2\}$ and remove the edges between a_1 and a_2, respectively b_1 and b_2. □

3. The main result

Theorem. *Assume that $r \geq 3$, T is a (finite) tree with two neighboring vertices of degree $r+1$, 1, respectively, and of degree $\leq r$ for the other vertices. Then there is no universal, countable, T-free graph.*

Proof. The idea is that we produce continuum many T-free graphs so that no two can be embedded into the same T-free graph. This is obviously nonsense as the vertex disjoint union of them certainly embeds both. But this will be just about the only way of doing this.

We start with the (obvious) observation that if S is a finite tree with all degrees at most r then every r-regular graph with sufficiently large girth embeds S. Let g be large enough that every r-regular graph of girth g embeds T minus the edge specified in the description of T.

Set $a = 4r^{2g}+1$ and let $B_0, B_1, \dots$ be disjoint sets with a graph X_i on B_i as described in Lemma 5., and with $|B_0| > 4f(r,g,a)$, $|B_{i+1}| > |B_i| + 4f(r,g,a)$. If $F(0) < F(1) < \cdots$ is an increasing sequence of natural numbers, let $X(F)$ be the following graph. Its vertex set is $A_0^F \cup A_1^F \cup \cdots$ where $A_i^F = B_{F(i)}$. We include the graphs on $X_{F(i)}$ into $X(F)$ and if a_1^i, a_2^i, b_1^i, b_2^i are the vertices specified in Lemma 5, then we also add the following edges; $\{a_1^0, a_2^0\}$, $\{b_1^0, a_1^1\}$, $\{b_2^0, a_2^1\}, \dots, \{b_1^i, a_1^{i+1}\}, \{b_2^i, a_2^{i+1}\}, \dots$. Clearly, $X(F)$ is r-regular, and by the selection of the vertices $a_1^i, a_2^i, b_1^i, b_2^i$, it has girth g. As for every sequence F there is such a graph, we indeed have continuum many graphs.

Assume that some $X(F)$ is embedded by some mapping π into U, a T-free graph. Then $X(F)$ must occupy a full connected component, as otherwise there is a vertex v in $X(F)$ so that $\pi(v)$ is joined (in U) to a vertex w which is not in the image of $X(F)$. But then we can identify $\pi(v)$ with the $(r+1)$-degree vertex of T, w with its 1-degree neighbor, and we can build the rest of T using $\pi(X(F))$, so U is not T-free. It suffices, therefore, to show, that if $F \neq F'$ then $X(F)$ and $X(F')$ may not be mapped onto the same vertex set, i.e., if $X(F')$ is mapped onto $X(F)$, then a copy of T is produced. By an argument as above, it suffices to show that by mapping $X(F')$ onto the vertex set of $X(F)$ there will be two vertices to be joined which have distance $> g$ in $X(F)$.

Let $\pi : A_0^F \cup A_1^F \cup \cdots \to A_0^{F'} \cup A_1^{F'} \cup \cdots$ be a bijection. Pick i with $F(i) \neq F'(i)$. Assume first that there is an index j that if

$$D = \pi\big(A_i^{F'}\big) \cap \big[A_0^F \cup A_1^F \cup \cdots \cup A_j^F\big],$$

$$E = \pi\big(A_i^{F'}\big) \cap \big[A_{j+1}^F \cup A_{j+2}^F \cup \cdots\big]$$

then $|D|$, $|F| > f(r,g,a)$.

There are at most $2r^g$ vertices in $A_0^F \cup A_1^F \cup \cdots \cup A_j^F$ in distance at most $g-1$ from b_1^j, b_2^j, and likewise, there are at most $2r^g$ vertices in $A_{j+1}^F \cup \cdots$ of distance $\leq g-1$ from a_1^{j+1}, a_2^{j+1}, so all but $4r^{2g}$ of the pairs (x,y) with $x \in A_0^F \cup A_1^F \cup \cdots \cup A_j^F$, $y \in A_{j+1}^F \cup \cdots$ are pairs of vertices in distance $> g$. By the statement of Lemma 5., there is such a pair in $\pi(X')$, and this edge can be used to build a copy of T.

If there is no index j as claimed, then all but $\leq 2f(r,g,a)$ elements of $\pi(A_i^{F'})$ are in the same A_j. A similar argument shows that all but $\leq 2f(r,g,a)$ elements of A_j^F must be in some $\pi(A_k^{F'})$, but then $k = i$ and $\big||A_j^F| - |A_i^{F'}|\big| \leq 4f(r,g,a)$, a contradiction. □

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