ON THE EXISTENCE OF COUNTABLE UNIVERSAL GRAPHS

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Abstract. Let Forb(G) denote the class of graphs with countable vertex sets which do not contain G as a subgraph. If G is finite, 2-connected, but not complete, then Forb(G) has no element which contains every other element of Forb(G) as a subgraph, i.e., this class contains no universal graph.

1. Introduction

Given a class, $\mathcal{G}$, of graphs we say that it has a *universal* element $U \in \mathcal{G}$ if any other graph $G \in \mathcal{G}$ is isomorphic to a (not necessarily induced) subgraph of U. The theory of universal graphs was initiated by Rado [16,17] who observed that there exists a countable graph containing all others as an induced subgraph. In this paper on *subgraph* we always mean not necessarily induced subgraph.

Given a cardinal κ and a family $\mathcal{F}$ of so-called *forbidden* subgraphs, let $\mathrm{Forb}_\kappa(\mathcal{F})$ be defined as the class of all graphs with at most κ vertices containing no subgraph isomorphic to any element of $\mathcal{F}$. The class of countable graphs $\mathrm{Forb}_\omega(\mathcal{F})$ is abbreviated as $\mathrm{Forb}(\mathcal{F})$.

It is known (folklore) that there is a universal element in $\mathrm{Forb}(K_n)$ for $n = 2, 3, \dots$, where K_n denotes the complete graph on n vertices. (For extensions to larger cardinals

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see [14].) In [10] it was shown that there is a universal element in $\mathrm{Forb}(P_n)$ and in $\mathrm{Forb}(\{C_n, C_{n+1}, \dots\})$, where P_n, C_n denote the path, cycle, respectively, on n vertices.

On the other hand, it is known that there is no universal element in $\mathrm{Forb}(C_n)$ (Hajnal and Pach [9] for $n = 4$, Cherlin and Komjáth [3] for all $n \geq 4$), or for $\mathrm{Forb}(K_{a,b})$ (Komjáth and Pach [11]), where $a, b \geq 2$, and $K_{a,b}$ denotes the complete bipartite graph with color classes of sizes a and b. The aim of this paper is to extend these negative results to all noncomplete 2-connected graphs.

2. Theorem

For an arbitrary (finite) graph $G = (V, \mathcal{E})$ with vertex set V, edge set $\mathcal{E}$, call two edges $e, f \in \mathcal{E}$ equivalent, in notation $e \sim f$, if there is a cycle containing both of them (or if $e = f$). It is well-known (see e.g., Lovász [15], page 43.) that this relation $\sim$ is indeed an equivalence, and for the equivalence classes $\mathcal{E}_1 \cup \mathcal{E}_2 \cup \dots = \mathcal{E}$ the following hold. Let $V_i = \cup\{e : e \in \mathcal{E}_i\}$, then each of the the graphs $(V_i, \mathcal{E}_i)$ is either a single edge or a maximal 2-connected subgraph of G (called *block*), and the sets $V_1, V_2, \dots$ form a (generalized hyper)forest, i.e., one can suppose that V_{i+1} intersects $V_1 \cup V_2 \cup \dots V_i$ in at most one element.

The *unification* of two vertices $x, y \in V$ of the graph G results a graph $G|_{xy}$ with vertex set $V \setminus \{x, y\} \cup \{z\}$, where z is a new vertex, the edges in $V \setminus \{x, y\}$ are unchanged, and z is connected to all vertices of $V \setminus \{x, y\}$ connected to either of x and y. Note that in case of $x, y \in V_1$, $|V_1| > 2$ (i.e., if they belong to the same non-trivial block), the other blocks of G are unchanged, (except x or y are replaced by z), while V_1 becomes $V_1 \setminus \{x, y\} \cup \{z\}$.

Theorem 2.1. *Let G be a finite graph and let B be a 2-connected, noncomplete block in G which is not isomorphic to a subgraph of another 2-connected block in G. Then* $\mathrm{Forb}(G)$ *has no universal element.*

Note that G has the above property if, e.g., itself is a 2-connected, noncomplete graph. Also, in more general, $\mathrm{Forb}(G)$ has no universal element if the cardinality of $V(B)$ is strictly larger than the size of any other block (and B contains 2 nonadjacent vertices).

For the proof we give a lemma on hypergraphs in the next Section. Section 4 contains the definition of 2^ω distinct G-free graphs, $G(\varepsilon)$, where $\varepsilon \in \{0,1\}^\omega$. In Section 5 it is shown that a countable G-free graph contains at most countable many of the $G(\varepsilon)$'s. The paper concludes with a list of further problems.

3. Hypergraphs of large girth

A *hypergraph* H is a pair $H = (V, \mathcal{H})$, where V is a set (the set of *vertices*) and $\mathcal{H}$ is a family of subsets of V. A *cycle* of length ℓ (≥ 2) is a sequence of distinct vertices and edges $v_1, v_2, \dots v_\ell$, $E_1, E_2, \dots, E_\ell$ such that $\{v_i, v_{i+1}\} \subset E_i$ $(1 \leq i < \ell)$ and $\{v_\ell, v_1\} \subset E_\ell$. The length of the shortest cycle in H is called its *girth.* If the girth is at least 3 then $|E \cap E'| \leq 1$ for all pair of edges, i.e., the hypergraph is *linear* or *nearly disjoint.* A hypergraph consisting of only 2-element edges is called a *graph.* Note that if we replace the edge E by two smaller edges F_1, F_2 such that $|F_1 \cap F_2| \leq 1$, $F_1 \cup F_2 \subset E$ then the girth does not decrease.

Lemma 3.1. *For every k and g there exist a $t = t(k,g)$ and a countable hypergraph $H(k,g)$ with vertex set ω and with edge set $\{E_1, E_2, \dots\}$ with the following properties:*
— $|E_i| = k$,
— *the girth of H is at least g, (hence it is nearly disjoint, if $g \geq 3$),*
— $(i+t) \in E_i \subset \{i-t, i-t+1, \dots, i+t\}$.

Proof. One can easily make a probabilistic proof using Lovász Local Lemma (see [15] Exercise 2.18, or [1] Section 5) or one by the greedy algorithm. Here we present a (basically) constructive example.

Erdős and Sachs ([5], also see in Bollobás [2], Theorem 1.4' on p. 108.) proved that for every δ and g there exists a δ-regular graph of girth at least g on at most δ^g vertices. Duplicating the edges and vertices of such a graph G_1 one obtains a bipartite G_2 which is δ-regular and of girth at least g. (The standard duplication can be done as follows: take two disjoint copies of $V(G_1)$, call them V_1 and V_2; join two vertices $u_1 \in V_1$ and $v_2 \in V_2$ if $uv \in \mathcal{E}(G_1)$.) An explicit construction of such a G_2 can be found in [7].

Proposition 3.2. *For every k and g there exist a natural number t and a $2(k-1)$-regular graph $G(k,g)$ of girth at least $2g$ with vertex set $[t] = \{1, 2, \dots, t\}$ such that its edge set can be decomposed into t $(k-1)$-stars with t distinct centers.*

This means, that there are functions $e_j : [t] \to [t]$ $(1 \leq j \leq k-1)$ such that the edge set of the stars $S_i = \{(i, e_1(i)), (i, e_2(i)), \dots, (i, e_{k-1}(i))\}$ $(1 \leq i \leq t)$, where $e_{j_1}(i) \neq e_{j_2}$ if $j_1 \neq j_2$, form a partition of $\mathcal{E}(G(k,g))$.

Proof of the Proposition. Start with a $2(k-1)$-regular bipartite graph G of girth at least $2g$. (The existence of such a graph is mentioned before the Proposition). We may suppose that its color classes consist of $\{1, 2, \dots, s\}$ and $\{s+1, s+2, \dots, 2s\}$ (and that $s < 2(2k)^{2g}$). König's theorem says that the edge set of a regular bipartite graph can be decomposed into perfect machings, $\mathcal{M}_1 \cup \cdots \cup \mathcal{M}_{2(k-1)} = \mathcal{E}$, (and here each $\mathcal{M}_j$ consists of s pairwise disjoint edges). Define $t = 2s$ and for $1 \leq i \leq s$ and let $e_j(i) = x$ be the other endpoint of the edge in $\mathcal{M}_j$ covering the vertex i $\quad (1 \leq j \leq k-1)$, while for the stars with centers $s < i \leq 2s$ one can use the matchings $\mathcal{M}_{s+1}, \dots, \mathcal{M}_{2s}$. $\square$

Using the construction of the above Proposition one can define the hypergraph $H(k,g)$ as follows. Write i in the form $i = at + r$ where a is a nonnegative integer, $1 \leq r \leq t$, and define $E_i = \{i+t\} \cup \{at + e_j(r) : 1 \leq j \leq k-1\}$. We claim that this hypergraph has the desired properties. Since $i - t \leq i - r = at < at + t = i - r + t < i + t$ the third property follows. We claim that the girth of H is at least g.

Suppose, on the contrary, that H contains a cycle with vertices $x_1, x_2, \dots, x_\ell$ and edges $E_{i_1}, E_{i_2}, \dots, E_{i_\ell}$ with $\ell < g$. Write $i_\nu = a_\nu t + r_\nu$ and $x_\nu = b_\nu t + s_\nu$, where $1 \leq r_\nu, s_\nu \leq t$. Since x_1 and x_2 are vertices in E_{i_1}, they are from the form $a_1 t + e_j(r_1)$ or $i_1 + t$. Consider the vertices s_1 and s_2 in the graph $G(k,g)$. If $x_1 = a_1 t + e_{j_1}(r_1)$ and $x_2 = a_1 t + e_{j_2}(r_1)$ then $s_1 = e_{j_1}(r_1)$ and $s_2 = e_{j_2}(r_1)$. Thus, s_1 and s_2 are distinct since x_1 and x_2 are distinct and $P_1 := (s_1, r_1, s_2)$ is an $s_1 - s_2$-path of length 2 in $G(k,g)$. If $x_1 = a_1 t + e_j(r_1)$ and $x_2 = i_1 + t$ then $s_1 = e_j(r_1)$ and from $x_2 = a_1 t + r_1 + t = (a_1+1)t + r_1$ follows $s_2 = r_1$. Again $s_1 \neq s_2$ and $P_1 := (s_1, s_2)$ is an $s_1 - s_2$-path of length 1 in $G(k,g)$. Finally, if $x_1 = i_1 + t$ and $x_2 = a_1 t + e_j(r_1)$ we get $s_1 \neq s_2$ and an $s_1 - s_2$-path P_1 of length 1 in $G(k,g)$ in the same way. Using the paths P_ν generated by x_ν, $x_{\nu+1}$ and E_{i_ν} one gets a closed walk in G of length at most 2ℓ with vertex set $\{r_1, r_2, \dots, r_\ell\} \cup \{s_1, \dots, s_\ell\}$. The girth of G exceeds 2ℓ, so this walk cannot contain a proper cycle. Hence it is a closed walk along the edges of a subtree of G. We will get a contradiction as follows.

Suppose that, e.g., s_2 is an endpoint of that tree with the pendant edge $\{w, s_2\}$. Then the paths P_1 and P_2 both contain $\{w, s_2\}$. The edge sets of the star-decomposition of $G(k,g)$ are pairwise disjoint so this implies that the hyperedges E_{i_1} and E_{i_2} were obtained from the same star, i.e., $r_1 = r_2$. However, $E_{i_1} \neq E_{i_2}$, so $r_1 = r_2$ implies that $t(a_1 - a_2) = i_1 - i_2$ is at least t. We get that these two edges are disjoint, contradicting the fact $x_2 \in E_{i_1} \cap E_{i_2}$. $\square$

4. Finitely determined G-free graphs

For the proof of Theorem 2.1 here we define continuum many G-free graphs, $G(\varepsilon)$, where $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots)$ is a 0–1 sequence. Let $k = 2|V(G)| - 1$, $g = |V(G)| + 1$, and let xu be an edge of G such that $u \in V(B)$, and denote by z the new vertex of $G|_{xy}$. Consider the hypergraph $H(k,g)$ supplied by Lemma 3.1. The vertex set of $G(\varepsilon)$ is defined as $V(G(\varepsilon)) = \omega = V(H(k,g))$. Now we are ready to define the edge set of $G(\varepsilon)$. Let $f_i : V(G) \to E_i$ and $g_i : V(G|_{xy}) \to E_i$ be injective functions with $f_i[V(G)] \cap g_i[V(G|_{xy})] = \emptyset$, $g_i(z) = i + t$, and $f_i(x), f_i(u)$ are the smallest elements in E_i. Now let

$$\begin{aligned}\mathcal{E}(G(\varepsilon)) = &\bigcup_i \{g_i(a)g_i(b) : ab \in \mathcal{E}(G|_{xy})\} \\ &\cup \bigcup_{i:\varepsilon_i=1} \{f_i(a)f_i(b) : ab \in \mathcal{E}(G - xu)\} \\ &\cup \bigcup_{i:\varepsilon_i=0} \{f_i(x)f_i(u)\}.\end{aligned}$$

Remember that the E_i are pairwise nearly disjoint. Thus, with $X_i := f_i[V(G)]$ and $Y_i := g_i[V(G|_{xy})]$, the induced subgraph $G(\varepsilon)[Y_i]$ is isomorphic to $G|_{xy}$ for every i, $G(\varepsilon)[X_i]$ is isomorphic to G_{xu} if $\varepsilon_i = 1$, and $G(\varepsilon)[X_i]$ contains only one edge if $\varepsilon_i = 0$.

We claim that $G(\varepsilon)$ is G-free. It is sufficient to see that B is not a subgraph of it. But this is obvious, because, by definition, any cycle, C, of length at most $|V(G)|$ must be contained entirely in some E_i. As every two edges of B are contained in a short cycle, any copy of B must be contained completely in some E_i. But B is neither a subgraph of $G - xu$ nor of $G|_{xy}$, implying our claim.

The graph $G(\varepsilon)$ is finitely determined, i.e., any embedding into a G-free graph the location of the vertices $\{1, 2, \ldots, t\}$ determines the rest of its vertices. This will be made explicit in the next section.

5. The nonexistence of universal graphs

In this section we prove that there is no universal G-free graph. Suppose, on the contrary, that U is a countable G-free graph containing all $G(\varepsilon)$'s. Let $\varphi[\varepsilon] : \omega \to V(U)$ be an embedding of $G(\varepsilon)$ into U. There are only countable many t-subsets of $V(U)$, so there exist $\varepsilon \neq \varepsilon'$ such that the initial segments of the embeddings of $G(\varepsilon)$ and $G(\varepsilon')$ are identical, i.e.,

$$\varphi[\varepsilon](j) = \varphi[\varepsilon'](j) \quad \text{for all } 1 \leq j \leq t. \tag{5.1}$$

The functions $\varphi[\varepsilon]$ and $\varphi[\varepsilon']$ are abbreviated as φ and φ'. We claim that $\varphi(i) = \varphi'(i)$ must hold for all $i > 0$, i.e., the vertex sets of these two graphs are embedded in the same way. Indeed, (5.1) holds for each $1 \leq i \leq t$; for larger i we use induction. Suppose that $i > t$ and the equation (5.1) holds for every $1 \leq j < i$. The images of $Y_{i-t} \setminus \{i\}$ under φ and in φ' are identical by the induction hypothesis and by the fact that the elements of this set are contained in $\{1, 2, \dots, i-1\}$. The set $\varphi(Y_{i-t})$ contains a copy of $G \setminus \{x\}$, the set $\varphi(Y_{i-t})$ contains a copy of $G \setminus \{y\}$, (and these copies are compatible). So $\varphi(i) \neq \varphi'(i)$ would result a copy of G, a contradiction.

Finally, let $\varepsilon_\ell = 1$, $\varepsilon'_\ell = 0$. Then, on the set $\varphi(X_\ell)$ we get a subgraph of U isomorphic to $G - xu$. Considering $\varphi'(X_\ell)$ we get an edge of U (joining the images of the first two vertices of X_i) However, $\varphi = \varphi'$, so these edges altogether give a copy of G. Hence U is not G-free, this is a contradiction. □

6. Further problems, conjectures

Let $c(\mathcal{G})$, the *complexity* of a class of graphs $\mathcal{G}$, be defined as the least cardinality of a subset $\mathcal{G}_0 \subset \mathcal{G}$ with the property that any element of $\mathcal{G}$ is isomorphic to a subgraph of some $G_0 \in \mathcal{G}_0$. Obviously, $\mathcal{G}$ has an universal element if and only if $c(\mathcal{G}) = 1$. Let $\mathcal{F}_k$ (and $\mathcal{G}_k$) denote the class of all countable graphs containing no k vertex-disjoint (edge-disjoint, respectively) cycles. It was proved (Komjáth and Pach [13]) that $c(\mathcal{F}_k) = c(\mathcal{G}_k) = \omega$ for every $1 < k < \omega$. In the above sections we have proved that every G-free graph (in the case of Theorem 2.1) can contain only countable many of $G(\varepsilon)$'s, hence $c(\text{Forb}(G))$ is continuum. (This was shown for the complete bipartite graphs $K_{a,b}$ with $a, b \geq 2$ in [12]).

The case of disconnected G is more complicated, Cherlin and Shi [4] showed that $1 < c(\text{Forb}(K_{n_1} + K_{n_2} + \cdots + K_{n_s})) < \infty$ for all $s \geq 2$, $n_1, \dots, n_s \geq 2$. (This was proved first in [12] for $n_1 = \cdots = n_s = 2$ and for $n_1 = n_2 = 3$, $s = 2$).

Another seemingly hard problem is the case of trees. It is easy to prove (see in [12]) that for the star S_r with $r \geq 4$ edges the class $\text{Forb}(S_r)$ is not universal, while $\text{Forb}(S_3)$ contains a universal element. Call a graph B a *bridge* if it is obtained from a path by adding 2 and 2 pendant edges to both endvertices. Goldstern and Kojman [8] have proved very recently that $\text{Forb}(B)$ is not universal.

We are able to decide whether $\text{Forb}(G)$ is universal or not for several more classes of graphs; e.g., among those connected graphs of 5 vertices exactly 5 are universal. This, and further constructions, are the subject of a forthcoming paper [6].

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