

# Dimension Versus Size

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**Abstract.** We investigate the behavior of  $f(d)$ , the least size of a lattice of order dimension  $d$ . In particular we show that the lattice of a projective plane of order  $n$  has dimension at least  $n/\ln(n)$ , so that  $f(d) = O(d^2 \log^2 d)$ . We conjecture  $f(d) = \theta(d^2)$ , and prove something close to this for height-3 lattices, but in general we do not even know whether  $f(d)/d \rightarrow \infty$ .

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## 1. Introduction and Results

We will be concerned in this paper with how small a lattice can be relative to its dimension. For our purposes a *linear extension* of a poset  $P$  is an order-preserving bijection

$$\sigma: P \rightarrow \{1, \dots, |P|\}.$$

The (order) *dimension* of  $P$ , denoted  $\dim P$ , is the least  $s$  for which there exist linear extensions  $\sigma_1, \dots, \sigma_s$  of  $P$  such that for all  $p, q \in P$  with  $p \not\leq q$  there exists  $i \in \{1, \dots, s\}$  with  $\sigma_i(p) > \sigma_i(q)$ . For more information on dimension see [4] or [5].

A venerable theorem of Hiraguchi [3] states that if  $\dim P \geq 3$ , the size of  $P$  is a least twice its dimension (this bound being attained for  $\dim P = d$  by the poset of 1- and  $(d-1)$ -element subsets of a  $d$ -element set, ordered by containment).

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For lattices the situation is completely different, and far more complicated; the problem of finding lattice analogues of Hiraguchi's theorem was recently raised by B. Sands (see [2]). Let us denote by  $f(d)$  the least size of a  $d$ -dimensional lattice. Sands asked whether  $f(d) \geq 2^d$ , i.e., whether the Boolean algebra of order  $d$  is the smallest  $d$ -dimensional lattice. This was answered in the negative by Ganter *et al.* [2]. Let  $\pi_n$  denote the lattice of partitions of an  $n$ -set and  $L_n(q)$  the lattice of subspaces of an  $n$ -dimensional vector space over  $\text{GF}(q)$ . It is shown in [2] that

$$\frac{3}{8} \binom{n}{2} \leq \dim \pi_n \leq \binom{n}{2}, \quad (1.1)$$

$$\frac{2}{n+1} (2^n - 1) \leq \dim L_n(2) \leq 2^n - 1. \quad (1.2)$$

Either of these shows that  $f(d)$  grows much more slowly than  $2^d$ , and in particular (1.2) gives

$$f(d) < c^{\log^2 d}.$$

Ganter *et al.* in turn asked for better bounds on  $f(d)$ , and specifically whether  $f(d)$  is bounded by a polynomial in  $d$ . Here we answer this in the affirmative:

**THEOREM 1.3.** *If  $\mathcal{P}_n$  is the lattice of a projective plane of order  $n$ , then  $\dim \mathcal{P}_n > n/2 \ln(n)$ .*

**COROLLARY 1.4.**  $f(d) = O(d^2 \log^2 d)$ .

An upper bound of  $2n + 2$  on  $\dim \mathcal{P}_n$  was shown to us by K. Reuter and appears to us to be closer to the truth. Of course this would give  $f(d) = O(d^2)$  and we (somewhat recklessly) propose.

**CONJECTURE 1.5.**  $f(d) = \theta(d^2)$ .

In fact we cannot even show  $f(d)/d \rightarrow \infty$ , though this seems certain to be the case. We mention one small step in the direction of the conjecture (recalling that the *height* of a poset is one less than the size of a largest chain).

**PROPOSITION 1.6.** *If  $\mathcal{L}$  is a lattice of height 3 then  $\dim \mathcal{L} = O(|\mathcal{L}|^{1/2} \log |\mathcal{L}|)$ .*

## 2. Proofs

Let us denote by  $P$  and  $L$  the point and line sets of the projective plane associated with  $\mathcal{P}_n$ . For the lower bound in Theorem 1.3, note that as there are  $(n^2 + n + 1)n^2$  nonincident pairs  $(p, l) \in P \times L$ , it suffices to prove

**LEMMA 2.1.** *For any linear extension  $\sigma$  of  $\mathcal{L}$  there are at most  $n^3 \ln(n^2 + n + 1)$  pairs  $(p, l) \in P \times L$  for which  $\sigma(l) < \sigma(p)$ .*

*Proof.* We need the following useful result of Corradi (see [6, prob. 13.13]).

(2.2) If  $\mathcal{F}$  is a family of subsets of a set  $X$ , with  $F, G \in \mathcal{F} \Rightarrow |F| \geq k, |F \cap G| \leq \lambda$ , then

$$|X| \geq \frac{k^2 |\mathcal{F}|}{k + (|\mathcal{F}| - 1)\lambda}.$$

This implies (taking  $\mathcal{F} = L_0, X = P \setminus P_0$ ).

(2.3) If  $P_0 \subset P$  and  $L_0 \subset L$  satisfy  $p \notin l \forall p \in P_0, l \in L_0$ , then  $(|P_0| + n)(|L_0| + n) \leq n(n+1)^2$ .

Now number the lines of  $L$  so that

$$\sigma(l_1) < \dots < \sigma(l_{n^2+n+1}).$$

If  $\sigma(p) > \sigma(l_i)$  then  $p \notin \bigcup_{j=1}^i l_j$ , so by (2.3)

$$|\{p : \sigma(p) > \sigma(l_i)\}| \leq \left[ \frac{n(n+1)^2}{i+n} \right] - n.$$

The Lemma and Theorem follow after a little calculation for  $n \geq 5$ . For  $n \leq 4$ ,  $[n/\ln(n)] = 2$  and trivially  $\dim \mathcal{P}_n > 2$ .  $\square$

REMARK. As far as we know the correct upper bound in Lemma 2.1 could be  $O(n^3)$ , which would give  $\dim \mathcal{P}_n = \theta(n)$ , in agreement with Conjecture 1.5.

*Proof of Proposition 1.6.* We denote by 0 and 1 the minimum and maximum elements of  $\mathcal{L}$ , and by  $L_0(L_1)$  the set of elements covering 0 (covered by 1). Obviously we may assume  $L_0 \cap L_1 = \emptyset$ .

As in [1], to show  $\dim \mathcal{L} \leq s$  we need only find permutations  $\sigma_1, \dots, \sigma_s$  of  $L_0$  satisfying

(2.4) for all  $p \in L_0, l \in L_1$ , with  $p \prec l$  there exists  $i \in \{1, \dots, s\}$  such that  $\sigma_i(p) > \sigma_i(q)$  for all  $q \prec l$ .

Let  $n = \max\{|L_0|, |L_1|\}$ . If we choose  $\sigma_1, \dots, \sigma_r, r = 4n^{1/2} \ln(n)$ , at random, then with positive probability (2.4) holds for  $(p, l)$  whenever

$$|\{q \in L_0 : q \prec l\}| < 2\sqrt{n} \quad (2.5)$$

(see e.g. [1]). But this excludes only a small subset of  $L_1$ :

$$|\{l \in L_1 : l \text{ violates (2.5)}\}| < 2\sqrt{n}. \quad (2.6)$$

(To see this, note that  $L_1$  may be regarded as a collection of subsets of  $L_0$ , no two having more than one element in common, and apply (2.2).) We may thus choose  $\sigma_1, \dots, \sigma_r$  so that (2.4) holds whenever (2.5) is true, and add to these for each  $l$  violating (2.5) a permutation  $\sigma_l$  satisfying

$$\sigma_l(q) < \sigma_l(p) \quad \forall q < l, p \not< l$$

to obtain the desired set of  $O(|\mathcal{L}|^{1/2} \log |\mathcal{L}|)$  permutations.

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