A MINIMAL CUTSET OF THE BOOLEAN LATTICE WITH ALMOST ALL MEMBERS

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Abstract. Two almost explicit constructions are given satisfying the title.

1. Preliminaries. Let $[n]$ denote the set of the first n positive integers, $2^{[n]}$ its power set. Sometimes $2^{[n]}$ will be called the *Boolean lattice* and denoted by $\mathbf{B}_n$. The collection of all k-subsets of a set S is denoted by $\binom{S}{k}$. A family $\mathcal{L} = \{L_0, L_1, \dots, L_t\} \subset 2^{[n]}$ is called a *chain* if its members contain each other, $L_0 \subset L_1 \subset \cdots \subset L_t$. Such a chain is *maximal* if $t = n$, in which case $|L_i| = i$ for all i. The family $\mathcal{C} \subset 2^{[n]}$ is a *cutset* of the Boolean lattice if $\mathcal{C} \cap \mathcal{L} \neq \emptyset$ for all maximal chains $\mathcal{L}$. A *minimal cutset* $\mathcal{C}$ is a cutset with the property that for every $C \in \mathcal{C}$ some maximal chain avoids $\mathcal{C} \setminus \{C\}$. For example the whole k-th level of the Boolean lattice $\binom{[n]}{k}$ is a minimal cutset. But there are minimal cutsets of much larger size, e.g. the following family

$$\{C \subset [n] : |C \cap \{1,2\}| = 1\} \tag{1.1}$$

has size 2^{n-1}. Denote the maximum size of a minimal cutset of $\mathbf{B}_n$ by $c(n)$. Ko-Wei Lih asked whether $c(n) = 2^{n-1}$ in general. It is easy to see that

$$c(n+1) \geq 2c(n). \tag{1.2}$$

(Indeed, if $\mathcal{C}$ is a minimal cutset of $\mathbf{B}_n$ then $\mathcal{C} \cup \{C \cup \{n+1\} : C \in \mathcal{C}\}$ is a minimal cutset of $\mathbf{B}_{n+1}$.) The inequality (1.2) implies that there is a limit of the sequence $c(n)/2^n$ whenever n tends to infinity. This limit is at least 1/2 by (1.1). In [L] Ko-Wei Lih gives a construction for $n = 6$ due to Jin-Fa Chern in which $|\mathcal{C}| = 33 > 2^{n-1}$. (Unfortunately, his example contains a misprint. To fix it, the set $\{1,2,4,5,6\}$ should be replaced by $\{1,2,3,5,6\}$.) It is natural to ask whether the answer is asymptotic to 2^n. In this note we give an almost explicit construction proving that

$$\lim_{n\to\infty} c(n)/2^n = 1. \tag{1.3}$$

"Almost explicit" means that we will define a large cutset (of size $(1 - o(1))2^n$) and prove that by deleting only $o(2^n)$ members of it one can obtain a minimal cutset.

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2. An Almost Deterministic Construction. Let $k \geq 3$ be an integer, and suppose that n is divisible by k. Let $S_1 \cup \ldots S_{n/k}$ be a partition of $[n]$ into k-element parts. Define the family $\mathcal{C}$ as follows.

$$\mathcal{C} = \{C \subset [n] : 0 < |S_i \cap C| < k \text{ for all } S_i\} \\ \cup \{C \subset [n] : \exists S_i \text{ and } S_j \text{ with } |S_i \cap C| = 0, |S_j \cap C| = k\}.$$

We claim that $\mathcal{C}$ is a cutset. Indeed, if $\emptyset = L_0 \subset L_1 \subset \ldots L_n$ is a maximal chain then define t as the largest integer such that L_t is still disjoint from some S_i. Then L_{t+1} intersects all S_i. If L_{t+1} does not contain any S_j, then then it belongs to the first part of $\mathcal{C}$. If L_{t+1} contains some S_j, then L_t belongs to the second part of $\mathcal{C}$.

A member C of a cutset $\mathcal{C}$ is *essential* if $\mathcal{C} \setminus \{C\}$ is not cutset. Define

$$\mathcal{C}_0 = \{C \subset [n] : 0 < |S_i \cap C| < k \text{ for all } S_i \text{ and } \exists S_i, S_j \text{ with } |S_i \cap C| = 1, |S_j \cap C| = k-1\}.$$

We claim that every member of $\mathcal{C}_0$ is essential in $\mathcal{C}$. Indeed, if $C \in \mathcal{C}_0$ with $|S_i \cap C| = 1$ and $|S_j \cap C| = k-1$, then every maximal chain containing $C \setminus S_i, C$ and $C \cup S_j$ avoids $\mathcal{C} \setminus \{C\}$. Starting with an arbitrary cutset one can always obtain a minimal cutset by deleting the unnecessary members one by one. But we can never delete an essential set. So all minimal cutsets contained in $\mathcal{C}$ contain $\mathcal{C}_0$. We have

$$(2.1) \quad |\mathcal{C}_0| = (2^k-2)^{n/k} - 2(2^k-k-2)^{n/k} + (2^k-2k-2)^{n/k} \\ > 2^n((1-\frac{2}{2^k})^{n/k} - 2(1-\frac{k+2}{2^k})^{n/k}) > 2^n(1-\frac{2n}{k2^k} - 2\exp[-\frac{n}{2^k}]).$$

Here we used the inequalities $(1-x)^y \leq \exp[-xy]$, which holds for $-\infty \leq x \leq 1$ and $y \geq 0$, and $1-xy \leq (1-x)^y$, which holds for $0 \leq x \leq 1$ and $y \geq 1$. If $n \sim 2^k \log k$, (i.e., $k \sim \log n - \log\log\log n$) then the (2.1) gives the following.

COROLLARY 2.1. *For sufficiently large* n

$$c(n) > 2^n(1 - \frac{4\log\log n}{\log n}).$$

We shall improve this result in Theorem 4.1.

3. Filters and Ideals. A subfamily $\mathcal{F}$ of $2^{[n]}$ is called a *filter* if $F \in \mathcal{F}$ and $F \subset F' \subset [n]$ imply $F' \in \mathcal{F}$. Starting with any subfamily $\mathcal{S} \subset 2^{[n]}$ one can obtain a filter $\mathcal{F}(\mathcal{S})$ as follows. $\mathcal{F}(\mathcal{S}) = \{F \subset [n] : \exists S \in \mathcal{S} \text{ such that } S \subset F\}$. $\mathcal{F}(\mathcal{S})$ is the filter *induced* by $\mathcal{S}$. A family $\mathcal{I}$ is called an *ideal* if $I \in \mathcal{I}$ and $I' \subset I$ imply $I' \in \mathcal{I}$ as well. For an arbitrary family $\mathcal{S} \subset 2^{[n]}$ we associate an ideal $\mathcal{I}(\mathcal{S})$ in the following way. $\mathcal{I}(\mathcal{S}) = \{I \subset [n] : \exists S \in \mathcal{S} \text{ such that } I \cap S = \emptyset\}$. $\mathcal{I}(\mathcal{S})$ is the ideal *induced* by $\mathcal{S}$. (Warning! This definition differs from the usual one.) In this way $\mathcal{F}(\mathcal{S})$ and $\mathcal{I}(\mathcal{S})$ consist of complementary pairs, i.e. $A \in \mathcal{F}(\mathcal{S})$ if and only if $([n] \setminus A) \in \mathcal{I}(\mathcal{S})$.

The *neighborhood* $N(\mathcal{G})$ of a family $\mathcal{G}$ is defined as the family of those subsets in $[n]$ whose Hamming distance from $\mathcal{G}$ is exactly 1, i.e. $N(\mathcal{G}) = \{N \subset [n] : N \notin \mathcal{G} \text{ and } \exists G \in \mathcal{G} \text{ such that } |N \triangle G| = 1\}$. Note that $\mathcal{G} \cap N(\mathcal{G}) = \emptyset$. The *complement* $\overline{\mathcal{G}}$ of the family $\mathcal{G}$ is defined as $\overline{\mathcal{G}} = 2^{[n]} \setminus \mathcal{G}$. The following idea underlies the construction in Section 2.

OBSERVATION 3.1. *Suppose that $\mathcal{I}$ is an ideal and $\mathcal{F}$ is a filter such that there are no two sets $I \in \mathcal{I} \setminus \mathcal{F}$ and $F \in \mathcal{F} \setminus \mathcal{I}$ such that*

$$I \subset F \text{ and } |F \setminus I| = 1. \tag{3.1}$$

Then $\mathcal{C} = (\overline{\mathcal{I}} \cap \overline{\mathcal{F}}) \cup (\mathcal{I} \cap \mathcal{F})$ is a cutset. Moreover, all members of $N(\mathcal{I}) \cap N(\mathcal{F})$ are essential. □

If we use an arbitrary family $\mathcal{S}$ to induce an ideal and a filter , then we obtain

LEMMA 3.2. *If for every S and $S' \in \mathcal{S}$ one has $|S \cap S'| \neq 1$, then the ideal $\mathcal{I}(\mathcal{S})$ and the filter $\mathcal{F}(\mathcal{S})$ fulfill Observation 3.1.*

Proof. Indeed, if $F \in \mathcal{F}(\mathcal{S}) \setminus \mathcal{I}(\mathcal{S})$ then there exists an $S_1 \in \mathcal{S}$ such that $S_1 \subset F$ and F intersects all members of $\mathcal{S}$. Moreover if $I \in \mathcal{I}(\mathcal{S}) \setminus \mathcal{F}(\mathcal{S})$ then there exists an $S_2 \in \mathcal{S}$ such that $S_2 \cap I = \emptyset$ and I does not contain any member of $\mathcal{S}$. So in this case $|F \setminus I| = 1$ would imply $S_1 \cap S_2 = F \setminus I$, a contradiction. □

4. A Random Construction. In view of Lemma 3.2, all that we need in order to construct a large minimal cutset is to find a suitable family $\mathcal{S}$ that has a filter $\mathcal{F}(\mathcal{S})$ with a big neighborhood. In this section we describe a *random* family $\mathcal{S}$ satisfying

$$|S \cap S'| \neq 1, \tag{4.1}$$

such that for some positive constant c

$$|N(\mathcal{F}(\mathcal{S}))| > 2^n\left(1 - c\frac{(\log n)^{3/2}}{\sqrt{n}}\right). \tag{4.2}$$

Of course, the same lower bound holds for $|N(\mathcal{I}(\mathcal{S}))|$ as well, thus

$$|N(\mathcal{F}(\mathcal{S})) \cap |N(\mathcal{I}(\mathcal{S}))| > 2^n\left(1 - 2c\frac{(\log n)^{3/2}}{\sqrt{n}}\right).$$

So Lemma 3.2 yields that $\mathcal{C} = (\overline{\mathcal{I}(\mathcal{S})} \cap \overline{\mathcal{F}(\mathcal{S})}) \cup (\mathcal{I}(\mathcal{S}) \cap \mathcal{F}(\mathcal{S}))$ is a cutset with a large number of essential sets.

THEOREM 4.1. *There exists a $c > 0$ such that $c(n) > 2^n\left(1 - c\frac{(\log n)^{3/2}}{\sqrt{n}}\right)$.*

Proof. To find such a family $\mathcal{S}$ our method is a modified version of what was used in **[FKK]** and in **[K]** to construct a small filter with large neighborhoods. Suppose that n is divisible by 8, and let $B_1 \cup \ldots B_{n/2}$ be a partition of the underlying set into pairs. Let k be an integer $k \sim \sqrt{n/\log n}$. For every $K \in \binom{[n/2]}{k}$ let ξ_K be a random variable with

$$\mathrm{Prob}(\xi_K = 1) = \frac{(1000\log n)^{3/2}}{\sqrt{n}} \binom{n/8}{k}^{-1} = p$$
$$\mathrm{Prob}(\xi_K = 0) = 1 - p.$$

These random variables are to be chosen totally independently. Let $\mathcal{S}$ be the random family defined by

$$\mathcal{S} = \{\cup_{i \in K} B_i : \xi_K = 1\}.$$

Of course, $\mathcal{S}$ satisfies (4.1). We next show that the expected size of $N(\mathcal{F}(\mathcal{S}))$ is as large as it was given in (4.2). This implies the existence of a family $\mathcal{S}$ wich fulfils both (4.1) and (4.2), proving Theorem 4.1.

Let N be an arbitrary but fixed member of $2^{[n]}$. Denote the number of blocks B_i which are contained in N by n_2, and let $N_2 = \{i : B_i \subset N\}$. Similarly, let $N_1 = \{i : |B_i \cap N| = 1\}$, and $|N_1| = n_1$. We give an exact formula for the probability that N belongs to $N(\mathcal{F}(\mathcal{S}))$. N belongs to $N(\mathcal{F}(\mathcal{S}))$ if and only if $\xi_K = 0$ for all $K \in \binom{N_2}{k}$ and $\xi_K = 1$ for some k-set K with $|K \setminus N_2| = 1$ and $(K \setminus N_2) \subset N_1$. Since the variables ξ_K are independent, we obtain that

$$\begin{aligned} \mathrm{Prob}(N \in N(\mathcal{F}(\mathcal{S}))) &= (1-p)^{\binom{n_2}{k}}(1 - (1-p)^{n_1\binom{n_2}{k-1}}) \\ &\geq (1 - p\binom{n_2}{k})(1 - \exp[-pn_1\binom{n_2}{k-1}]) \end{aligned} \tag{4.3}$$

Now suppose that N is a *typical* member of $\mathbf{B}_n$. More exactly, define the collection $\mathcal{N}$ of typical sets N by

$$\mathcal{N} = \{N \in 2^{[n]} : |n_2(N) - \frac{n}{8}| < \sqrt{n\log n} \text{ and } |n_1(N) - \frac{n}{4}| < 0.1n\}.$$

Then the well-known de Moivre-Laplace formula (see, e.g. in [**R**, p. 151]) gives that

$$|\mathcal{N}| > 2^n(1 - \frac{1}{n}). \tag{4.4}$$

There exists some positive constant c such that for every typical set N,

$$p\binom{n_2}{k} = \frac{(1000\log n)^{3/2}}{\sqrt{n}} \frac{\binom{n_2}{k}}{\binom{n/8}{k}} < c\frac{(\log n)^{3/2}}{\sqrt{n}} \tag{4.5}$$

and

$$pn_1\binom{n_2}{k-1} = \frac{(1000\log n)^{3/2}}{\sqrt{n}} \frac{kn_1}{n_2-k+1} \frac{\binom{n_2}{k}}{\binom{n/8}{k}} > 2\log n. \tag{4.6}$$

(Here we used the inequalities for $(1-x)^y$ from Section 2.) Then (4.5) and (4.6) imply the following lower bound in (4.3). If $N \in \mathcal{N}$ then

$$\text{Prob}(N \in N(\mathcal{F}(\mathcal{S}))) > 1 - c\frac{(\log n)^{3/2}}{\sqrt{n}}. \tag{4.7}$$

Then (4.4) and (4.7) give that the expected size $E(N(\mathcal{F}(\mathcal{S})))$ fulfils (4.2). Hence there exists a family $\mathcal{S}$ satisfying (4.2). □

5. Problems, Remarks. It is a natural question how close $c(n)$ can be to 2^n. Obviously, $2^n - c(n) \geq 2^n/n$. Kostochka **[K]** proved that for every filter $\mathcal{F}$ one has $2^n - |N(\mathcal{F})| > 0.011 \cdot 2^n(\log n)^{3/2}/\sqrt{n}$. So the method presented in this note cannot give a better bound than Theorem 4.1.

Another possible direction for the further research is to extend the investigation to other (popular) posets. (Cf. **[GRS]**, **[N]**, **[SW]**).

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