

Union-free Hypergraphs and Probability Theory

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Let $F(n)$ denote the maximum number of distinct subsets of an n -element set such that there are no four distinct subsets: A, B, C, D with $A \cup B = C \cup D$. We prove that $2^{(n-\log 3)/3} - 2 \leq F(n) \leq 2^{(3n+2)/4}$. We use probability theory for the proof of both the lower and upper bounds. Some related problems are considered, too.

1. INTRODUCTION

In 1969 Erdős and Moser [4] raised the problem of estimating $f(n)$, the maximum number of distinct subsets of an n -element set such that all the $\binom{f(n)}{2}$ pairwise unions are different.

THEOREM 1. $2^{(n-3)/4} \leq f(n) \leq 1 + 2^{(n+1)/2}$. (1)

Notice that the upper bound is an immediate consequence of $\binom{f(n)}{2} \leq 2^n$. To prove the lower bound we use an algebraic construction which is a modification of a construction of Babai and Sós [1]. How a family of sets can fail to have the union-free property? There are essentially two possibilities:

- (a) there are four distinct sets A, B, C, D with $A \cup B = C \cup D$.
- (b) there are three distinct sets A, B, C with $A \cup B = A \cup C$.

We call families for which (a) never holds *weakly union-free*, and those for which (b) never holds *cancellative* (the second name indicates that $A \cup B = A \cup C$ implies $B = C$). We denote by $F(n)(G(n))$ the maximum number of subsets of an n -set in a weakly union-free (cancellative) family, respectively.

Our main result is the following:

THEOREM 2. $2^{(n-\log 3)/3} - 2 \leq F(n) \leq 2^{(3n+2)/4} \sim 2^{1/2} \cdot 1.68^n$. (2)

The lower bound is deduced by a non-constructive, probabilistic method. The proof of the upper bound uses information theory, it was inspired by the paper Kleitman, Shearer and Sturtevant [9]. For cancellative families we prove:

THEOREM 3. $(8/9)^{\varepsilon(n)/3} 3^{n/3} \leq G(n) < n 1.5^n \quad (n \geq 14)$, (3)

where $\varepsilon(n)$ is determined by $0 \leq \varepsilon(n) \leq 2$, $n + \varepsilon(n)$ is divisible by 3.

Erdős and Katona (cf. [8]) conjecture that the lower bound is exact. Their construction is simple: let X_1, \dots, X_q be pairwise disjoint sets with union of size n with $|X_i| = 2$ or 3 and with at most two sets of size 2 among the X_i . Let our family consist of all the transversals that is of those sets which intersect each X_i in one element. Clearly this family achieves the lower bound and it is cancellative.

2. RELATED AND OPEN PROBLEMS

Let k be an integer, $k \geq 2$. Let us denote by $f_k(n)$ the maximum number of k -subsets of an n -set forming a union-free family, $F_k(n)$, $G_k(n)$ are defined similarly. Then $f_2(n)$,

$F_2(n)$, $G_2(n)$ denote the maximum number of edges in a graph without a cycle of length 3 or 4, of length 4, of length 3, respectively. The problem of determining $F_2(n)$ was raised by Erdős [3] already 45 years ago, but it is still unsolved. However it is known that

$$F_2(n) = \left[1(1 + o(1)) \frac{n^{3/2}}{2} + o(1) \right]. \quad (4)$$

Recently the second author determined the exact value of $F_2(n)$ for $n = 4^s + 2^s + 1$. He proved: (cf. [7])

$$F_2(n) = 2^{s-1}(2^s + 1)^2. \quad (5)$$

For $f_2(n)$ it is only known that

$$\frac{1}{2 \cdot 2^{1/2}} n^{3/2} < f_2(n) < \frac{1}{2} n^{3/2}. \quad (6)$$

The determination of $G_2(n)$ is a special case of Turan's theorem ([11]):

$$G_2(n) = \lfloor n^2/4 \rfloor. \quad (7)$$

For $n = 3$ the authors proved in [8]:

$$f_3(n) = \lfloor n(n-1)/6 \rfloor, \quad (8)$$

and

$$F_3(n) = n(n-1)/3 \quad \text{for } n > n_0 \quad \text{and} \quad n \equiv 1 \pmod{6}. \quad (9)$$

Bollobás [2] proved:

$$G_3(n) = \left\lfloor \frac{n}{3} \right\rfloor \left\lfloor \frac{n+1}{3} \right\rfloor \left\lfloor \frac{n+2}{3} \right\rfloor. \quad (10)$$

For $k \geq 4$ no exact values are known. The authors have established several bounds for $f_k(n)$ and $F_k(n)$, e.g. (cf. [6]):

$$f_4(n) = \lfloor 1 + o(1) \rfloor n^3/24. \quad (11)$$

For $G_k(n)$ Bollobás [2] conjectures that

$$G_k(n) = \prod_{0 \leq i < k} \left\lfloor \frac{n+i}{k} \right\rfloor. \quad (12)$$

It is easy to see that this is a lower bound for $G_k(n)$. We prove the conjecture for $n \leq 2k$.

PROPOSITION 2.1. *For $n \leq 2k$ we have*

$$G_k(n) = 2^{n-k}. \quad (13)$$

COROLLARY 2.2. *For $n \geq 2k$ we have*

$$G_k(n) \leq \binom{n}{k} 2^k / \binom{2k}{k}. \quad (14)$$

For the problems considered in detail in this paper the most important would be to determine $\lim_{n \rightarrow \infty} \log h(n)/n$ where h is any of f , F and G . For f and F it is not even proved yet that this limit exists, for G it follows from $G(n_1 + n_2) \geq G(n_1)G(n_2)$.

Let us note that equation (12) would imply $\lim_{n \rightarrow \infty} \log G(n)/n = 3^{1/3} = 1.44 \dots$. The upper bound of Theorem 3 gives 1.5.

3. THE PROOF OF THE UPPER BOUND OF THEOREM 2

Let $\mathcal{F} = \{F_1, \dots, F_m\}$ be any weakly union-free family of subsets of $\{1, \dots, n\}$. Let \mathbf{v}_i be the characteristic vector F_i : \mathbf{v}_i is a $(0, 1)$ -vector which has 1 in the j th position if and only if $j \in F_i$. The following proposition can be proved easily.

PROPOSITION 3.1. *The $\binom{m+1}{2}$ sums $\mathbf{v}_i + \mathbf{v}_{i'}$ ($1 \leq i \leq i' \leq n$) are all distinct $(0, 1, 2)$ -vectors of length n .*

Notice that this proposition already implies $\binom{m+1}{2} \leq 3^n$, in particular $m < 3^{(m+1)/2}$. However, we want to show that the considerably stronger inequality (2) is valid. Let us give weights to the vectors $\mathbf{v}_i + \mathbf{v}_{i'}$. Let the weight, $w(\mathbf{v}_i + \mathbf{v}_{i'})$ be 1 if $i = i'$ and 2 if $i \neq i'$. Then the total sum of weights is m^2 . Let us define a probability distribution \mathbf{x} on these sums by setting $p(\mathbf{x} = \mathbf{v}_i + \mathbf{v}_{i'}) = w(\mathbf{v}_i + \mathbf{v}_{i'})/m^2$. Then \mathbf{x} can be considered as a random vector $\mathbf{x} = (x_1, \dots, x_n)$ where x_j is the frequency distribution of 0s, 1s and 2s in the j th position. If d_j denotes the degree of j in \mathcal{F} , i.e., the number of sets containing j and $p_j = d_j/m$, then x_j is given by $p(x_j = 2) = p_j^2$, $p(x_j = 1) = 2p_j(1 - p_j)$, $p(x_j = 0) = (1 - p_j)^2$. Thus the information-theoretic entropy of x_j is:

$$H(x_j) = -p_j^2 \log p_j^2 - 2p_j(1 - p_j) \log 2p_j(1 - p_j) - (1 - p_j)^2 \log (1 - p_j)^2, \quad (15)$$

log means \log_2 . The next proposition can be proved by elementary analysis:

PROPOSITION 3.2. *The function in equation (15) takes its maximum value for $p_j = \frac{1}{2}$ where $H(x_j) = \frac{3}{2}$.*

The next proposition is from [10, p. 33].

PROPOSITION 3.3 *If $\mathbf{x} = (x_1, \dots, x_n)$ is a random vector then*

$$H(\mathbf{x}) \leq \sum_{1 \leq j \leq n} H(x_j). \quad (16)$$

Let us now count $H(\mathbf{x})$.

$$H(\mathbf{x}) = -m \left(\frac{1}{m^2} \log(1/m^2) \right) - \left(\frac{m}{2} \right) \frac{2}{m^2} \log \left(\frac{2}{m^3} \right) = \log \left(\frac{m^2}{2} \right) + \frac{1}{m} \log 2 > \log \left(\frac{m^2}{2} \right). \quad (17)$$

Now combining expressions (5), (6) and Proposition 2.2 we obtain $m^2/2 < 2^{3n/2}$, yielding the upper bound of expression (2).

4. THE LOWER BOUND OF THEOREM 2

Let us consider a random $(0, 1)$ -matrix of size $2m$ by n where each element is 1 with independent probability p (we shall fix m and p later). Each row of the matrix is the characteristic vector of a subset of $\{1, \dots, n\}$. Let \mathcal{F} denote the collection of the corresponding (not necessarily distinct) sets. The probability that some 4 sets in \mathcal{F} satisfy (a) is $\{1 - 2(1 - p)^2[1 - (1 - p)^2]\}^n$. This quantity becomes 2^{-n} for $p = (1 - 2^{1/2})/2$. If we choose m at most $2^{(n - \log 3)/3}$ then the expected number of four-tuples in \mathcal{F} , satisfying (a) is at most m . Omitting one set from each of these four-tuples we omit at most m sets, i.e. at least m sets remain and since (a) is impossible for these sets, at most one of them appears twice. Consequently, $F(n) \geq 2^{(n - \log 3)/3} - 2$.

5. THE PROOF OF THEOREM 1

We only have to prove the lower bound. First let us note: arguing in the same way as for the lower bound of Theorem 2 but choosing $p = 1/3$ we can get as many as $(1 + o(1))(27/19)^{n/2}$ sets forming a union-free system, e.g. for $n > 1000$ we obtain

$$f(n) > \frac{1}{2}(27/19)^{n/2}. \quad (18)$$

The inequality is actually stronger than that in Theorem 1, however it is non-constructive and valid only for large values of n .

To give the other bound it will be enough to show that for every positive integer n we have

$$f(4n) \geq 2^n. \quad (19)$$

To prove this inequality, let us consider 4 pairwise disjoint n -element sets: X, X', Y, Y' and let us fix 4 embeddings of $\text{GF}(2^n)$ into $2^X, 2^{X'}, 2^Y, 2^{Y'}$, respectively: g, g', h, h' . Let 1 denote the element $(1, 1, \dots, 1)$ in $\text{GF}(2^n)$. Now let us define:

$$\mathcal{A} = \{g(a) \cup g'(1-a) \cup h(a^3) \cup h(1-a^3) : a \in \text{GF}(2^n)\}.$$

We have to show that \mathcal{A} is union-free. Suppose a, b, c, d are elements of $\text{GF}(2^n)$ for which the corresponding sets satisfy (a) or (b). Then $g(a) \cup g(b) = g(c) \cup g(d)$ and also $g'(1-a) \cup g'(1-b) = g'(1-c) \cup g'(1-d)$. The second equality yields $g'(a) \cap g'(b) = g'(c) \cap g'(d)$. We infer $a + b = c + d$. Similarly, from the equalities for h and h' , it follows that $a^3 + b^3 = c^3 + d^3$. However over a field of characteristic 2 we have: $a^3 + b^3 = (a+b)[(a+b)^2 + ab]$. Since $a + b = c + d$, we infer $ab = cd$ from $a^3 + b^3 = c^3 + d^3$. Thus $\{a, b\}$ and $\{c, d\}$ are both the set of roots of the equation $x^2 - (a+b)x + ab = 0$ i.e. $\{a, b\} = \{c, d\}$.

6. THE PROOF OF THE BOUNDS (13) AND (14)

Let \mathcal{A} be a cancellative family and let A be a member of \mathcal{A} with maximal cardinality, say k . Then $A \cup B \neq A \cup C$ implies $B \cap (\{1, \dots, n\} - A) \neq C \cap (\{1, \dots, n\} - A)$ for $B, C \in (\mathcal{A} - \{A\})$. Thus

$$|\mathcal{A}| \leq 1 + 2^{n-k}. \quad (20)$$

Now assume that \mathcal{A} is k -uniform that is all its members have the same size: k . Then $B \cap (\{1, \dots, n\} - A) = \emptyset$ is impossible for $B \in (\mathcal{A} - \{A\})$, yielding equation (13), as an upper bound. To show that we have equality, let us partition $\{1, \dots, n\}$ into k sets X_1, \dots, X_k such that $2k - n$ of them have size 1 and the remaining ones 2. Let \mathcal{A} be the complete k -partite graph that is

$$\mathcal{A} = \{A : |A \cap X_i| = 1 \text{ for every } 1 \leq i \leq k\}.$$

We prove inequality (14) by a simple averaging argument. Suppose that \mathcal{A} is a k -uniform, cancellative hypergraph on $X = \{1, \dots, n\}$, $n \geq 2k$. Let Y be a random $2k$ -element subsets of X . Set $\mathcal{A}_Y = \mathcal{A} \cap \binom{Y}{k}$. Then \mathcal{A}_Y is cancellative. Thus equation (13) implies

$$|\mathcal{A}_Y| \leq 2^k. \quad (21)$$

Denoting by $E(|\mathcal{A}_Y|)$ the expected number of edges in \mathcal{A}_Y , we have

$$E(|\mathcal{A}_Y|) = |\mathcal{A}| \binom{2k}{k} / \binom{n}{k}. \quad (22)$$

Since the expectation can not be greater than the maximum, expressions (21) and (22) imply inequality (14).

7. THE PROOF OF THEOREM 3

We need the following simple inequality:

$$\binom{2k}{k} > 2^{2k}/(2k)^{1/2}, \quad \text{if } k \geq 7. \quad (23)$$

To prove expression (23), notice that it holds for $k = 7$. Then apply induction. Passing from k to $k + 1$ the LHS of expression (23) grows by a factor of $4(2n + 1)/(2n + 2)$, while the RHS by a factor of $4(2n/2n + 2)^{1/2}$. Now, comparing these two, expression (23) follows from $2n + 1 > (2n(2n + 2))^{1/2}$.

Suppose now that \mathcal{A} is a cancellative family on $\{1, \dots, n\}$. Let A be a member of \mathcal{A} having maximal size. If $|A| \geq n/2$ then inequality (20) yields expression (3). Thus we may suppose $|A| < n/2$. Let a_k denote the number of k -element subsets in \mathcal{A} . By definition we have:

$$a_k \leq G_k(n) \quad \text{and} \quad |\mathcal{A}| = \sum_{0 \leq k \leq n/2} a_k.$$

Thus inequality (14) implies

$$|\mathcal{A}| \leq \sum_{0 \leq k \leq n/2} \binom{n}{k} 2^k / \binom{2k}{k}.$$

Using expression (23), for $n \geq 14$ we infer

$$|\mathcal{A}| < n^{1/2} \sum_{0 \leq k \leq n} 2^{-k} \binom{n}{k} = n^{1/2} \left(\frac{3}{2}\right)^n.$$

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