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THE GREATEST ANGLE AMONG *n* POINTS IN THE *d*-DIMENSIONAL EUCLIDEAN SPACE

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There exists a pointset $\mathcal{P}$ of cardinality at least 1.15^d in E_d such that all angles determined by the triples of $\mathcal{P}$ are less than $\pi/2$. This disproves the old conjecture that $|\mathcal{P}| \leqslant 2d-1$.

1. Introduction

Many decades ago Erdös conjectured that if there are given 2^d+1 points in a d-dimensional Euclidean space at least one of the angles determined by the points is greater than $\pi/2$. A very simple and ingenious proof for this conjecture was given by Danzer and Grünbaum. The following problems remained open: Determine the largest α_d for which 2^d+1 points in E_d always determine an angle $\geqslant(\pi/2)+\alpha_d$. We can make no contribution to this problem at present. The second problem states: Denote by $f(d)$ the largest integer for which there are $f(d)$ points in E_d all the angles of which are $<\pi/2$. $f(2)=3$ is trivial and Croft proved $f(3)=5$. It was often conjectured that $f(d)<Cd$. We prove that $f(d)$ tends to infinity exponentially and at the moment cannot prove that $f(d)<(2-\varepsilon)^d$. As a matter of fact we cannot even prove that $f(d)<2^d-1$ ($d>2$). We prove that for every $\varepsilon>0$ there exists a δ_ε so that one can have $(1+\delta)^d$ points in E_d all the angles of which are $<(\pi/3)+\varepsilon$.

Erdös and Szekeres proved that 2^n points in E_2 always determine an angle $>\pi(1-1/n)$ and Szekeres proved that this result is best possible in the following sense: One can give, for every $\varepsilon>0$, 2^n points in E_2 no angle of which is greater than $\pi(1-(1/n))+\varepsilon$. It does not seem easy to get a result of the same precision for higher dimension but we will get inequalities giving estimates for the number of points in E_d which give an angle $>\pi-\varepsilon$.

2. Strictly antipodal polytopes

Let us denote by E_d the d-dimensional Euclidean space. Let $\mathcal{P}\subset E_d$ be a pointset, and let $a,b\in\mathcal{P}$. We shall say that a and b are an *antipodal pair* of $\mathcal{P}$

provided there exists a pair of parallel (distinct) supporting hyperplanes of $\mathcal{P}$ such that a belongs to one of them and b to the other. A pointset $\mathcal{P}$ is said to be *antipodal* provided each two of its points forms an antipodal pair of $\mathcal{P}$. (Clearly, in this case $\mathcal{P}$ is the vertex set of the polytope conv $\mathcal{P}$, the convex hull of $\mathcal{P}$.) Let us denote by $G(d)$ the maximal number of vertices in an antipodal d-polytope. Danzer and Grünbaum [4] showed the conjecture of Klee [11] that $G(d) = 2^d$.

Theorem 2.1 [4]. *If the pointset $\mathcal{P} \subset E_d$ is an antipodal, then $|\mathcal{P}| \leqslant 2^d$. Equality holds if and only if $\mathcal{P}$ consists of the vertices of a d-dimensional parallelotope.*

This result implies also that $F(d) = 2^d$ is the answer to the following problem of Erdös [5, 6]: What is the maximal possible number $F(d)$ of points in E_d such that all angles determined by the triples of them are less than or equal to 90°?

Now let the notion of an antipodal pair be modified by defining a pair $a, b \in \mathcal{P}$ as *k-antipodal* provided there exist parallel (distinct) supporting hyperplanes of $\mathcal{P}$, each of which intersects conv $\mathcal{P}$ in a set of dimension at most k, such that a belongs to one of the hyperplanes, and b to the other. In analogy to the above, we define k-antipodal polytopes and the numbers $G_k(d)$. Clearly, a d-polytope is antipodal if and only if it is $(d-1)$-antipodal.

A number of interesting problems concern 0-antipodality, which is called *strict antipodality*. While it is easy to show that $G_0(2) = 3$, the proof of $G_0(3) = 5$ is rather involved (Grünbaum [11]). For $d \geqslant 4$, it is known that $G_0(d) \geqslant 2d - 1$, and it has been conjectured that $G_0(d) = 2d - 1$ (Danzer–Grünbaum [4], Grünbaum [11, 12]). In this section we disprove this conjecture, giving a construction with more than 1.15^d points. (See Theorem 2.2.)

As in the case of antipodal pairs, $G_0(d)$ may be considered as the affine variant of the following Euclidean problem due to Erdös [6]: Determine the maximal possible number $f(d)$ of points in E_d such that all angles determined by triples of them are acute. Examples show that $f(d) \geqslant 2d - 1$, and clearly $G_0(d) \geqslant f(d)$. In contrast to the situation in the case $G(d)$, it is not known whether $G_0(d) = f(d)$ for $d \geqslant 4$. (Direct proofs of $f(3) = 5$ were given by Croft [3] and Schütte [15].) The following theorem implies that $f(d) > 1.15^d$.

Theorem 2.2. *There exists a pointset $\mathcal{P}$ in E_d of cardinality 1.15^d such that all angles determined by triples of $\mathcal{P}$ are acute.*

Proof. We select the points of $\mathcal{P}$ from the vertices of the d-dimensional cube. As usual, the d-dimensional 0–1 vectors correspond to the subsets of a d-element set X. More precisely, if $a \in \{0; 1\}^d$ then let $A = A(a) = \{i : a_i = 1\}$, $X := \{1, 2, \ldots, d\}$.

Lemma 2.3. *The points $a, b, c \in \{0;1\}^d$ determine a right angle at the point c if and only if*

$$A \cap B \subset C \subset A \cup B, \tag{2.4}$$

where the sets $A, B, C \subset X$ are associated with the vertices a, b, c.

(This lemma is a trivial consequence of Pythagoras' Theorem.) As the angles determined by the triples of the cube are less than or equal to $\pi/2$, the construction of the desired $\mathscr{P}$ will be completed if we find a set system $\mathscr{F}$ over X no three different members of which satisfy (2.4), and whose cardinality is greater than 1.15^d. Let $h(d)$ denote the greatest cardinality of such $\mathscr{F}$, i.e., $h(d) := \max\{|\mathscr{F}| : \mathscr{F} \subset 2^X$, for all $A \neq B \neq C \in \mathscr{F}$, $A \cap B \not\subset C$ or $C \not\subset A \cup B\}$.

Lemma 2.5. $h(d) > (2/\sqrt{3})^{d-1}$.

Proof. Let us choose independently the coordinates of the d-dimensional 0–1 vectors $a_1, a_2, \ldots, a_{2m}$ with probability $\text{Prob}(a_{ij} = 0) = 1/2$, $\text{Prob}(a_{ij} = 1) = 1/2$, $1 \leqslant i \leqslant 2m$, $1 \leqslant j \leqslant d$ and $m = \lfloor (2/\sqrt{3})^{d-1} \rfloor$. Then,

$$\text{Prob}(a, b, c \text{ hold for } (2.4)) = (3/4)^d. \tag{2.6}$$

Indeed, (2.4) means that for all $1 \leqslant i \leqslant d$ neither $a_i = b_i = 1$, $c_i = 0$ or $a_i = b_i = 0$, $c_i = 1$ hold. The independency of the coordinates yields (2.6). Hence the expectation number

$$E(\text{the number of the triples } (a, b, c) \text{ satisfying } (2.4)) =$$
$$= 2m(2m-1)(2m-2)(3/4)^d < m.$$

Hence the expected number of vectors to remain after the omission of the points which are vertices of a right angle is greater than $2m - m = m$, and for that set of vectors the conditions are already satisfied. □

Finally, since $f(d) \geqslant \max\{2d-1, h(d)\}$, (2.2) follows. □

A bit more complicated random process gives

$$h(d) > (\sqrt[4]{2} - o(1))^d \sim 1.189\ldots^d. \tag{2.7}$$

Instead of Lemma 2.5, we can use the following theorem due to Frankl and the authors [9] to prove $G_0(d) \geqslant f(d) > (1+c)^d$.

Theorem 2.8 [9]. *There exists a set-system $\mathscr{F}$ over a d-element underlying set in which no set is covered by the union of two others and $|\mathscr{F}| > 1.13^d$.*

We have the following.

Conjecture 2.9. *There exists an absolute constant $c > 0$ (not depending on d) such that*

$$f(d) \leqslant G_0(d) < (2-c)^d. \quad (?)$$

The following fact says that it is not possible to choose more than $2\sqrt{3}^d$ points from the vertices of a cube such that all angles among them are less than $\pi/2$. Even more, denote by $G_0(\{0;1\}^d)$ the greatest cardinality of a strictly antipodal $\mathcal{P} \subset \{0;1\}^d$.

Fact 2.10. $h(d) \leqslant G_0(\{0;1\}^d) < \sqrt{2}(\sqrt{3})^d$.

Proof. If $\{a,b\},\{c,d\} \subset \mathcal{P}$ are distinct pairs, then $a+b \neq c+d$ (since $a+b=c+d$ implies that the points a, c, b, d form a parallelogram which is contrary to the strict antipodality of $\mathcal{P}$). Thus

$$\binom{|\mathcal{P}|+1}{2} = |\{a+b : a, b \in \mathcal{P}\}| \leqslant |\{0;1;2\}^d| = 3^d. \qquad \square$$

Finally, we mention some more problems. Slightly generalizing the question about $G(d)$ one is led to the problem of determining $e(d,n)$, the maximal number of antipodal pairs among the vertices of a d-polytope with n vertices. It is not hard to show that $e(2,n) = \lfloor 3n/2 \rfloor$ (Grünbaum [11]), and that

$$e(3,n) \geqslant \lfloor \tfrac{1}{2}n \rfloor \lfloor \tfrac{1}{2}(n+1) \rfloor + \lfloor \tfrac{3}{4}n \rfloor + \lfloor \tfrac{1}{4}(3n+1) \rfloor.$$

Conjecture of Grünbaum [12]. *Does the relation*

$$\lim \frac{e(d,n)}{n^2} = \frac{1}{2} - \frac{1}{2^{d-1}} \quad (?) \qquad (2.11)$$

hold for all $d \geqslant 2$.

In analogy to the above, we define $e_k(d,n)$, the maximal number of k-antipodal pairs. Regarding $e_0(d,n)$, it is easy to prove that $e_0(2,n) = n$ (Grünbaum [11]), but even the 3-dimensional case seems to be very complicated. Clearly, $e_0(d,n) \geqslant d(d,n)$, the maximal number of diameters of an n-element set in E_d. It is known that $d(2,n) = n$ (Erdös [7]), $d(3,n) = 2n-2$ (Grünbaum [10], Heppes [13] and Straszewicz [16]), but for $d \geqslant 4$

$$\lim d(d,n)/n^2 = \tfrac{1}{2} - 1/(2\lfloor d/2 \rfloor)$$

(Erdös [7]).

Theorem 2.2 yields a new lower bound of the following problem: What is the order of the number c_d defined as follows: For a d-dimensional convex body K, denote by $c(K)$ the minimal number of translates of K the union of which covers K. Then c_d is defined as the maximum of $c(K)$ for all d-dimensional convex bodies K. It is easily seen that $c_d \geqslant G_0(d)$, but it is not known whether $c_d = G_0(d)$ for $d \geqslant 3$.

Corollary 2.12. $c_d \geqslant 1.15^d$.

The following conjecture would extend Theorem 2.1.

Conjecture 2.13. *There exists a positive constant* ε *(*ε *independent of* d*) for which the following is true: If the pointset* $\mathcal{P} \subset E_d$ *and* $|\mathcal{P}| \geqslant 2^d + 1$*, then* $\mathcal{P}$ *contains an angle greater than* $\pi/2 + \varepsilon$.

3. Pointsets with all angles small

Denote by $\alpha(\mathcal{P})$ the greatest angle determined by the triples of the pointset $\mathcal{P}$, and $\alpha_d(n) =: \inf\{\alpha(\mathcal{P}) : |\mathcal{P}| = n,\ \mathcal{P} \subset E_d\}$. We can write Theorems 2.1 and 2.2 and Conjecture 2.9 as follows:

$$\alpha_d(2^d+1) > \pi/2, \quad \alpha_d(2^d) = \pi/2 \qquad \text{(cf. 2.1)}, \tag{3.1}$$

$$\alpha_d(1.15^d) < \pi/2 \quad \text{(cf. 2.2)}, \tag{3.2}$$

$$\alpha_3(5) < \pi/2, \quad \alpha_3(6) = \pi/2 \qquad \text{(Croft [3] and Schütte [15])}, \tag{3.3}$$

$$\exists c > 0: \quad \alpha_d((2-c)^d) = \pi/2 \quad (?) \qquad \text{(cf. 2.9)}. \tag{3.4}$$

We were not able to establish an even weaker version of (3.4).

Conjecture 3.5. *For each* $\varepsilon > 0$ *there exists a* $c(\varepsilon) > 0$ *with*

$$\alpha_d((2-c)^d) > \pi/2 - \varepsilon.$$

But for ε large enough we can prove the following.

Theorem 3.6. *If* $0 < c < 1$ *then*

$$\alpha_d((1+c)^d) > \pi/3 + c/4 - o(1). \tag{3.6a}$$

Further, there exists a construction showing that

$$\alpha_d((1+c)^d) < \pi/3 + \sqrt{c}. \tag{3.6b}$$

Clearly $|\mathcal{P}| \geq 3$ implies $\alpha(\mathcal{P}) \geq \pi/3$. What Theorem 3.6 says is that there are exponentially many $(1+c)^d$ points in E_d with all angles less than 61°, but that 1.4^d points always determine an angle larger than 72°.

Proof. We start with the construction.

Lemma 3.7. *There exists a k-uniform set system $\mathcal{F}$ over d elements such that $|F_1 \cap F_2| < \varepsilon k$ for each distinct $F_1, F_2 \in \mathcal{F}$ and $|\mathcal{F}| > (1+0.4\varepsilon^2)^d$.*

Proof. We choose $F_1, F_2, \ldots, F_i, \ldots$ recursively. Suppose $F_1, \ldots, F_i$ are already chosen. Let $\mathcal{F}_i =: \{F \subset X : |F| = k, |F \cap F_j| \geq x\}$ for all $1 \leq j \leq i$. We can select an $F_{i+1} \notin \mathcal{F}_i$ if $|\mathcal{F}_i| < \binom{d}{k}$. Thus this process can be continued for at least as many as

$$\binom{d}{k} \Big/ \binom{k}{x}\binom{d-x}{k-x}$$

steps. Putting $k = d\varepsilon/4$ and $x = k\varepsilon$ we prove (3.7) using Stirling's formula. □

Returning to the proof of (3.6b), put $\varepsilon = \sqrt{2.5c}$, and let $\mathcal{F}$ be the set system defined in (3.7). For the distances of the two vertices f_1, f_2 corresponding to $F_1, F_2 \in \mathcal{F}$ we have

$$\sqrt{2k(1-\varepsilon)} < |f_1 - f_2| \leq \sqrt{2k},$$

i.e., the distances defined by $\mathcal{F}$ are almost equal. Now a simple calculation gives (3.6b).

To prove (3.6a) let $\mathcal{P} = \{P_1, P_2, \ldots, P_m\} \subset E_d$ be a pointset with every angle less than $\pi/3 + x$. Hence the ratio of the smallest and largest sides of a triangle $P_iP_jP_k$ is greater than $\sin(\pi/3 - 2x)/\sin(\pi/3 + x) \geq 1 - 2x$. So if the largest distance in $\mathcal{P}$ is 1, then the smallest is at least $(1-2x)^2 \geq 1 - 4x$ $(x \leq 1/4)$.

Let S be the smallest ball containing $\mathcal{P}$ with center 0. By the Yung theorem [12], the radius of S is less than $1/\sqrt{2 + (2/d)} < 1/\sqrt{2}$. Project the points of $\mathcal{P}$ from 0 to the surface of S. The image of P_i is Q_i. It is easily seen that $|P_iP_j| < 1/\sqrt{2}$ implies $|Q_iQ_j| \geq |P_iP_j|$. So if x is small enough $(x < 0.07)$ then any two point Q_iQ_j have a distance at least $1 - 4x$. So we can apply Böröczki's quite sharp estimation [1] about the density of a packing of the d-dimensional sphere by congruent balls, to get $|\mathcal{P}| = |\mathcal{Q}| < (1-4x)^{-d} 2d$. However we can give a simple straightforward proof of this last step.

Lemma 3.8. *If $\mathcal{Q}$ is a pointset on the surface of the sphere S of radius $1/\sqrt{2}$, and for all $Q_1, Q_2 \in \mathcal{Q}$ we have $|Q_1Q_2| > 1 - y$ then $|\mathcal{Q}| < d \cdot 2d(1-y)^{-d}$.*

Proof. Any ball with radius $(1-y)/\sqrt{2}$ contains at most $d+1$ points of $\mathcal{Q}$. Taking averages,

$$|\mathcal{Q}| \leqslant (d+1)\frac{\text{surface area of } S}{\begin{array}{c}\text{maximal surface area of the intersection of } S\\ \text{with a sphere of radius } (1-y)/\sqrt{2}\end{array}}$$

$$< (d+1)(1-y)^{-d}d. \qquad \square$$

4. The greatest angle among n points

Erdös and Szekeres proved [8] that 2^n points in the plane always determine an angle $> \pi(1-(1/n))$ and Szekeres [17] proved that this result is best possible in the following sense: One can give, for every $\varepsilon > 0$, 2^n points in E_2 no angle of which is greater than $\pi(1-(1/n))+\varepsilon$. I.e.,

$$\alpha_2(2^n) = \pi\left(1-\frac{1}{n}\right) \tag{4.1}$$

([8] and [17]). So in general,

$$\alpha_2(n) = \pi\left(1-\frac{1}{\log_2 n}\right) + \mathrm{O}\left(\frac{1}{(\log_2 n)^2}\right). \tag{4.2}$$

In this section we are concerned with the d-dimensional version of this fact.

Theorem 4.3. *We have*

$$\pi\left(1-\frac{4}{\sqrt[d-1]{\log_2 n}}\right) < \alpha_d(n) < \pi\left(1-\frac{1}{\sqrt[d-1]{\log_2 n}}\right).$$

The proof shows that one cannot hope to get an essentially better result without a new estimation for the sphere-packing problem in d-dimension, which is far better than the existing ones. The proof used adapted the proofs in [8] and [17]. We need two facts.

(4.4). In the d-dimensional space there exist more than $(1/\rho)^{d-1}$ lines going through the point 0, such that any two of them determine an angle greater than ρ.

(4.5). In the d-dimensional space there exist fewer than $(4/\rho)^{d-1}$ lines going through 0, such that any other line going through 0 determines an angle less than $\rho/2$ with some of them.

The facts (4.4) and (4.5) are in fact equivalent to the well-known sphere-packing and sphere-covering problems. There are much better estimations, but (4.4) and (4.5) are very easy to check, and the better known estimations would not eliminate the difference between the constants in (4.3).

The proof of the upper bound. Let $e_1, e_2, \ldots, e_m$, with $m > (1/\rho)^{d-1}$, be a system of lines determined by (4.4). We are going to construct 2^m points recursively such that every angle in the triples is less than $\pi - \rho$.

Let $\mathscr{P}_1 = \{A, B\}$ where $A, B \in e_1$. Translating $\mathscr{P}_1$ in the direction e_2 we get A', B' and if A' and B' are far enough then AB' and $A'B$ are *almost* parallel to e_2. Then translate the parallelogram $AA'BB'$ in the direction e_3 far enough ... and so on. After $m-1$ translations we get a construction showing $\alpha_d(2^m) < \pi - \rho$.

The proof of the lower bound. Let $f_1, \ldots, f_m$, with $m < (4/\rho)^{d-1}$, be a set system of lines determined by (4.5), and let $\mathscr{P} \subset E_d$ be a pointset with more than 2^m points. Consider the complete graph with vertex set $\mathscr{P}$ and colour its edges in the following way. For $U, V \in \mathscr{P}$ the edge UV gets colour i, $1 \leqslant i \leqslant m$, provided the angle between UV and f_i is less than $\rho/2$.

Lemma 4.6 (see [8]). *If the edges of the complete graph with more than 2^m vertices are coloured with m colours then there exists an odd circuit whose edges are of the same colour.*

Lemma 4.6 implies that there are $U, V, W \in \mathscr{P}$ and an f_i such that the angles (UV, f), (VW, f) and (WU, f) are less than $\rho/2$. But then the greater angle in the triangle UVW is at least $\pi - \rho$. □

One more open problem:

Let $\mathscr{P}$ be a pointset in E_d and $0 < \alpha < \pi$. Define $f(\mathscr{P}, < \alpha)$ (resp. $f(\mathscr{P}, > \alpha)$) as the number of angles in $\mathscr{P}$ smaller (resp. greater) than α. Put

$$f_d(n, < \alpha) := \min\{f(\mathscr{P}, < \alpha) \colon |\mathscr{P}| = n,\ \mathscr{P} \subset E_d\},$$

and

$$f_d(< \alpha) := \lim_{n \to \infty} f_d(n, < \alpha) / \tfrac{1}{2} n (n-1)(n-2).$$

$f_d(< \alpha)$ and $f_d(> \alpha)$ show at least what percent of the angles are smaller (or greater) than α for a pointset $\mathscr{P}$ in E_d. Convay, Croft, Erdös and Guy in [2] investigate f_2 and f_3. They show, for instance, that

$$\tfrac{1}{9} \leqslant f_2(> \tfrac{1}{2}\pi) \leqslant \tfrac{4}{27} \tag{4.7}$$

([2]). In higher dimensions no estimation is known for f_d. Even for $d \leqslant 3$, $f_d(<\alpha)$ is known for only a finite number of α.

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