

CROSS-INTERSECTING FAMILIES OF FINITE SETS

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ABSTRACT. It is proved that if \mathcal{A} is a family of a -element sets and \mathcal{B} is a family of b -element sets on the common underlying set $[n]$, and $A \cap B \neq \emptyset$ for all $A \in \mathcal{A}$, $B \in \mathcal{B}$ (i.e., cross-intersecting), and $n \geq a + b$, $|\mathcal{A}| \geq \binom{n-1}{a-1} - \binom{n-b-1}{a-1} + 1$, and $|\mathcal{B}| > \binom{n-1}{b-1} - \binom{n-a-1}{b-1} + 1$, then there exists an element $x \in [n]$ such that it belongs to all members of \mathcal{A} and \mathcal{B} . This is an extension of a result of Hilton and Milner who generalized the Erdős-Ko-Rado theorem for non-trivial intersecting families. Several problems remain open.

1. NON-TRIVIAL CROSS-INTERSECTING FAMILIES

For a positive integer n , let $[n] = \{1, 2, \dots, n\}$, for integers $1 \leq a \leq b$ let $[a, b] = \{a, a+1, \dots, b\}$. For a set S let $\binom{S}{k}$ denote the collection of k -element subsets of S and let 2^S denote the collection of all subsets of S . A family of sets \mathcal{F} is called *intersecting* if $A \cap B \neq \emptyset$ hold for all $A, B \in \mathcal{F}$.

Let $\mathcal{F} \subseteq \binom{[n]}{k}$ be an intersecting family. Erdős, Ko and Rado [3] proved that $|\mathcal{F}| \leq \binom{n-1}{k-1}$ holds for $n \geq 2k$. Moreover, in case of equality $\cap \mathcal{F} \neq \emptyset$ (for $n > 2k$). An intersecting family \mathcal{G} is called *non-trivial* if $\cap \mathcal{G} = \emptyset$. Define the following non-trivial families. $\mathcal{G}^1 = \{G \in \binom{[n]}{k} : 1 \in G, G \cap [2, k+1] \neq \emptyset\} \cup \{[2, k+1]\}$ and $\mathcal{G}^2 = \{G \in \binom{[n]}{k} : |[3] \cap G| \geq 2\}$. For $k = 2$, $\mathcal{G}^1 \equiv \mathcal{G}^2$; for $k = 3$, $|\mathcal{G}^1| = |\mathcal{G}^2|$; while for $k \geq 4$, $n > 2k$, $|\mathcal{G}^1| > |\mathcal{G}^2|$. Hilton and Milner [10] proved the following generalization of the Erdős-Ko-Rado theorem. If $n > 2k$ and $\mathcal{G} \subseteq \binom{[n]}{k}$ is a non-trivial intersecting family then

$$(1) \quad |\mathcal{G}| \leq |\mathcal{G}^1| = \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1.$$

Moreover, equality is possible only for $\mathcal{G} = \mathcal{G}^1$ or \mathcal{G}^2 . A short proof was given in [6].

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Two families \mathcal{A} and \mathcal{B} are called *cross-intersecting* if $A \cap B \neq \emptyset$ hold for all $A \in \mathcal{A}$, $B \in \mathcal{B}$. Here we extend (1) for two families.

Theorem. *If $\mathcal{A} \subset \binom{[n]}{a}$, $\mathcal{B} \subset \binom{[n]}{b}$, $A \cap B \neq \emptyset$ for all $A \in \mathcal{A}$, $B \in \mathcal{B}$ (i.e., cross-intersecting), and $n \geq a + b$, $|\mathcal{A}| > \binom{n-1}{a-1} - \binom{n-b-1}{a-1}$, and $|\mathcal{B}| > \binom{n-1}{b-1} - \binom{n-a-1}{b-1}$, then one of the following two cases holds*

- (i) *there exists an element $x \in [n]$ such that x belongs to all members of \mathcal{A} and \mathcal{B} ; or*
- (ii) *$|\mathcal{A}| = \binom{n-1}{a-1} - \binom{n-b-1}{a-1} + 1$ and $|\mathcal{B}| = \binom{n-1}{b-1} - \binom{n-a-1}{b-1} + 1$.*

We can describe the extremal families in case (ii). Namely, either

- (ii/1) $n = a + b$, $|\mathcal{A}| = \binom{a+b-1}{a-1}$, $|\mathcal{B}| = \binom{a+b-1}{b-1}$, and for every partition of $X \cup Y = [n]$ with $|X| = a$, $|Y| = b$, either $X \in \mathcal{A}$ or $Y \in \mathcal{B}$; or
- (ii/2) $a = b = k$, $\mathcal{A} = \mathcal{B} \cong \mathcal{G}^i$ for some $i \in \{1, 2\}$ (see (1)); or
- (ii/3) $a, b \geq 2$ and for some a -set A_0 and b -set B_0 with $A_0 \cap B_0 \neq \emptyset$, and for some element $x \notin A_0 \cup B_0$, we have $\mathcal{A} = \{A : x \in A \in \binom{[n]}{a}, A \cap B_0 \neq \emptyset\} \cup \{A_0\}$ and $\mathcal{B} = \{B : x \in B \in \binom{[n]}{b}, B \cap A_0 \neq \emptyset\} \cup \{B_0\}$.

An easy corollary of the Theorem was used to answer a problem of Trotter about the order dimension of two levels of the Boolean lattice, see [9].

2. PROOF

We prove the Theorem by induction on $a + b$. The cases $a = 1$ or $b = 1$ are trivial.

The case $n = a + b$ is easy. Indeed, consider all the $\binom{a+b}{a}$ (ordered) partitions of $[n]$ into $X \cup Y = [n]$, with $|X| = a$, $|Y| = b$. For each such partition either $X \notin \mathcal{A}$ or $Y \notin \mathcal{B}$, implying $|\mathcal{A}| + |\mathcal{B}| \leq \binom{a+b}{a}$. (The case $a = b$ requires a little more care). The lower bounds for $|\mathcal{A}|$ and $|\mathcal{B}|$ give $|\mathcal{A}| \geq \binom{n-1}{a-1} - \binom{n-b-1}{a-1} + 1 = \binom{a+b-1}{a-1}$, and $|\mathcal{B}| \geq \binom{a+b-1}{b-1}$. So both inequalities hold with equality and we get case (ii/1). From now on, we suppose that $n > a + b$, $b \geq a \geq 2$.

Consider the case when for some $x \in [n]$ we have $x \in \cap \mathcal{A}$. If there exists a $B \in \mathcal{B}$ with $x \notin B$, then $\mathcal{A} \subset \{F \in \binom{[n]}{a} : x \in F, F \cap B \neq \emptyset\}$, implying $|\mathcal{A}| \leq \binom{n-1}{a-1} - \binom{n-b-1}{a-1}$, a contradiction. We obtain that $x \in \cap \mathcal{B}$, leading to case (i). From now on, we suppose that $\cap \mathcal{A} = \emptyset$. By a similar argument this implies that $\cap \mathcal{B} = \emptyset$ holds, too.

Consider the case when \mathcal{A} itself is an intersecting family. Then (1) implies that $|\mathcal{A}| \leq \binom{n-1}{a-1} - \binom{n-a-1}{a-1} + 1$, which is not more than $\binom{n-1}{a-1} - \binom{n-b-1}{a-1}$ for $b > a \geq 2$, a contradiction.

We obtain that $a = b$ and $|\mathcal{A}| = \binom{n-1}{a-1} - \binom{n-a-1}{a-1} + 1$. If there exists a set $B \in \mathcal{B} \setminus \mathcal{A}$, then $\mathcal{A} \cup \{B\}$ is a non-trivial intersecting family of size larger than the bound given by the Hilton-Milner theorem (1), a contradiction. So $\mathcal{B} \subset \mathcal{A}$, implying $\mathcal{A} = \mathcal{B}$, we obtain case (ii/2).

From now on, we suppose that there are two members $A_1, A_2 \in \mathcal{A}$ disjoint to each other, $A_1 \cap A_2 = \emptyset$. Without loss of generality we may suppose that $A_1 = [a]$, $A_2 = [a+1, 2a]$. We are going to obtain the sharp upper bounds of (ii) for the sizes of $|\mathcal{A}|$ and $|\mathcal{B}|$.

Following Erdős, Ko and Rado [3] we define a compression operation P_{ij} for all $1 \leq i < j \leq n$. However, here we will apply it to two families simultaneously, as it was first done for a similar problem in [8] (also see [5]). Then the rest of the proof is an extension of the ideas of the short proof for the Hilton-Milner theorem given in [6]. For a family $\mathcal{G} \subset 2^{[n]}$ let $P_{ij} : \mathcal{G} \rightarrow 2^{[n]}$ as follows

$$P_{ij}(G) = \begin{cases} (G \setminus \{j\}) \cup \{i\}, & \text{if } i \notin G, j \in G, \text{ and } (G \setminus \{j\}) \cup \{i\} \notin \mathcal{G}, \\ G & \text{otherwise.} \end{cases}$$

Let us set $P_{ij}(\mathcal{G}) = \{P_{ij}(G) : G \in \mathcal{G}\}$. Obviously, $|P_{ij}(\mathcal{G})| = |\mathcal{G}|$.

We claim that if \mathcal{F} and \mathcal{G} are cross-intersecting, then $P_{ij}(\mathcal{F})$ and $P_{ij}(\mathcal{G})$ are cross-intersecting, too. Suppose, on the contrary, that $P_{ij}(F) \cap P_{ij}(G) = \emptyset$ for some $F \in \mathcal{F}$, $G \in \mathcal{G}$. As $F \cap G \neq \emptyset$ the only possibility is that one of these sets, say F , is unchanged, $P_{ij}(F) = F$, but the other one is new, $P_{ij}(G) = (G \setminus \{j\}) \cup \{i\}$. As F and G are unchanged outside $\{i, j\}$, we get that $F \cap G = \{j\}$, $i \notin P_{ij}(F) = F$. Then the only reason that F is unchanged is that $F' = (F \setminus \{j\}) \cup \{i\} \in \mathcal{F}$. This leads to $F' \cap G = \emptyset$, a contradiction.

Apply repeatedly P_{ij} for \mathcal{A} and \mathcal{B} simultaneously for all pairs (i, j) with $1 \leq i \leq a+b < j \leq n$ until we get two families \mathcal{A}^* and \mathcal{B}^* having the property $P_{ij}(\mathcal{A}^*) = \mathcal{A}^*$ and $P_{ij}(\mathcal{B}^*) = \mathcal{B}^*$ for every such pair (i, j) . This can be reformulated the following way.

- (2) If $A \in \mathcal{A}^*$, $i \notin A, j \in A, i \leq a+b < j$ then $(A \setminus \{j\}) \cup \{i\} \in \mathcal{A}^*$ as well.
 If $B \in \mathcal{B}^*$, $i \notin B, j \in B, i \leq a+b < j$ then $(B \setminus \{j\}) \cup \{i\} \in \mathcal{B}^*$ as well.

We claim that \mathcal{A}^* and \mathcal{B}^* are not simply cross-intersecting, but they are cross-intersecting even on the first $a+b$ elements, i.e. for all $A \in \mathcal{A}^*$ and $B \in \mathcal{B}^*$ we have

(3)
$$A \cap B \cap [a+b] \neq \emptyset.$$

PROOF OF (3). Suppose, on the contrary, that $A \in \mathcal{A}^*$, $B \in \mathcal{B}^*$ with $A \cap B \cap [a+b] = \emptyset$, and suppose (A, B) is such that $|A \cap B|$ is minimal. Thus there exist a $j \in A \cap B$

(hence $j > a + b$) and an $i \in [a + b]$ such that $i \notin A \cup B$. Then (2) implies that $A \setminus \{j\} \cup \{i\} = A' \in \mathcal{A}^*$. However, $|A' \cap B| < |A \cap B|$, a contradiction. \square

As $A_1 = [a]$ and $A_2 \subset [a + b]$ are unchanged during the above compressions, we have that $\cap \mathcal{A}^* = \emptyset$. This implies that $\cap \mathcal{B}^* = \emptyset$, too. Indeed, suppose, on the contrary, that $x \in B$ for all $B \in \mathcal{B}^*$. There exists an $A \in \mathcal{A}^*$ avoiding x , so $\mathcal{B}^* \subset \{F \in \binom{[n]}{b}, x \in F, F \cap A \neq \emptyset\}$, implying $|\mathcal{B}^*| \leq \binom{n-1}{b-1} - \binom{n-a-1}{b-1}$. This contradicts the lower bound condition on $|\mathcal{B}|$. So from now on we may suppose that $\cap \mathcal{A}^* = \cap \mathcal{B}^* = \emptyset$.

Define the families of traces $\mathcal{A}_u = \{A \cap [a + b] : A \in \mathcal{A}^*, |A \cap [a + b]| = u\}$, $\mathcal{B}_v = \{B \cap [a + b] : B \in \mathcal{B}^*, |B \cap [a + b]| = v\}$. If $|\mathcal{A}_u| \leq \binom{\ell-1}{u-1} - \binom{\ell-b-1}{u-1}$ holds for all u , where $\ell = a + b$, then we get

$$(4) \quad |\mathcal{A}| = |\mathcal{A}^*| \leq \sum_{u=1}^a |\mathcal{A}_u| \binom{n-\ell}{a-u}$$

$$(5) \quad \leq \sum_{u=1}^a \left(\binom{\ell-1}{u-1} - \binom{\ell-b-1}{u-1} \right) \binom{n-\ell}{a-u} = \binom{n-1}{a-1} - \binom{n-b-1}{a-1},$$

a contradiction. So for some $1 \leq u \leq a$ we have $|\mathcal{A}_u| > \binom{\ell-1}{u-1} - \binom{\ell-b-1}{u-1}$. In the same way, we can prove that there exists a $1 \leq v \leq b$ such that $|\mathcal{B}_v| > \binom{\ell-1}{v-1} - \binom{\ell-a-1}{v-1}$. We may also suppose that both u and v are chosen to be minimal, i.e.,

$$(6) \quad |\mathcal{A}_i| \leq \binom{\ell-1}{i-1} - \binom{\ell-b-1}{i-1} \quad \text{and} \quad |\mathcal{B}_j| \leq \binom{\ell-1}{j-1} - \binom{\ell-a-1}{j-1}$$

hold for all $1 \leq i < u$, $1 \leq j < v$. By (3) $\mathcal{A}^* \cup \mathcal{A}_u$ and $\mathcal{B}^* \cup \mathcal{B}_v$ are cross-intersecting families.

If $\mathcal{A}_1 \neq \emptyset$, $\{x\} \in \mathcal{A}_1$, then $x \in B$ for all $B \in \mathcal{B}^*$, contradicting $\cap \mathcal{B}^* = \emptyset$. So from now on we may suppose that $\mathcal{A}_1 = \emptyset$, and similarly $\mathcal{B}_1 = \emptyset$. If $x \in \cap \mathcal{A}_u$, then in the same way as above, we get $x \in \cap \mathcal{B}^*$, a contradiction. So from now on, we may suppose that $\cap \mathcal{A}^* = \cap \mathcal{A}_u = \emptyset$, and similarly $\cap \mathcal{B}^* = \cap \mathcal{B}_v = \emptyset$. Moreover, $u, v \geq 2$.

Apply the induction hypothesis for the cross-intersecting families \mathcal{A}_u and \mathcal{B}_v with values ℓ, u, v in place of n, a, b . Only case (ii) can occur, so we get $|\mathcal{A}_u| = \binom{\ell-1}{u-1} - \binom{\ell-v-1}{u-1} + 1$. But $|\mathcal{A}_u| \geq \binom{\ell-1}{u-1} - \binom{\ell-b-1}{u-1} + 1$, which implies $v = b$ (because $u - 1 \geq 1$). Similarly, we get $u = a$. Now use (6) and the upper bound $|\mathcal{A}_a| \leq \binom{\ell-1}{a-1} - \binom{\ell-b-1}{a-1} + 1$ in (4). Instead of (5) we get $|\mathcal{A}| \leq \binom{n-1}{a-1} - \binom{n-b-1}{a-1} + 1$, as desired. Analogously, we get $|\mathcal{B}| \leq \binom{n-1}{b-1} - \binom{n-a-1}{b-1} + 1$, finishing the case (ii). \square

3. THE CASE OF EQUALITY

For brevity, we characterize the extremal families only if $b \geq 4$ and $a \geq 3$, the cases $a = b = 3$ and $a = 2$ are left to the reader. From Section 2 the only unsettled case is the relation of \mathcal{A}, \mathcal{B} and $\mathcal{A}^*, \mathcal{B}^*$ at the very end of the proof of case (ii). Equalities must hold in (4) and in (6) for all i and j for which $\binom{n-\ell}{a-i}$ and $\binom{n-\ell}{b-j}$ are positive, in particular for $i = a - 1$ and $j = b - 1$. We get $|\mathcal{A}_{a-1}| \geq \binom{\ell-1}{a-2} - \binom{\ell-(b-1)-1}{a-2} + 1$ and $|\mathcal{B}_{b-1}| > \binom{\ell-1}{b-2} - \binom{\ell-(a-1)-1}{b-2} + 1$. Apply the induction hypothesis for \mathcal{A}_{a-1} and \mathcal{B}_{b-1} . We get that there exists an element x belonging to all members of \mathcal{A}_{a-1} and \mathcal{B}_{b-1} . Suppose that $x \notin A_2$, then the cross-intersecting property (and the size of \mathcal{B}_{b-1}) imply that \mathcal{B}_{b-1} consists of all $(b - 1)$ element subsets of $[a + b]$ containing x and meeting A_2 . It is easy to see that A_2 is the only member of \mathcal{A}^* avoiding x . Then, necessarily, \mathcal{A}^* and \mathcal{B}^* have the structure described in (ii/3).

We claim that \mathcal{A} and \mathcal{B} must have had the same property before the compressions. Consider the way we got \mathcal{A}^* from \mathcal{A} by compressions, $\mathcal{A} = \mathcal{A}^0 \rightarrow \mathcal{A}^1 \rightarrow \dots \rightarrow \mathcal{A}^s = \mathcal{A}^*$. During the compressions for each family $\cap \mathcal{A}^\alpha = \emptyset$ holds, because A_1 and A_2 remained unchanged. This implies, in the familiar way, that at each step $\cap \mathcal{B}^\alpha = \emptyset$.

It is easy to prove that if there is a vertex v contained in all but one of the sets of \mathcal{A}^α , then the pair $(\mathcal{A}^\alpha, \mathcal{B}^\alpha)$ has the structure described in (ii/3). We claim that $\max_{v \leq n} \deg(v, \mathcal{A}^\alpha) = |\mathcal{A}| - 1$ is true for all $0 \leq \alpha \leq s$.

Suppose that $\max_{v \leq n} \deg(v, \mathcal{A}^{\alpha+1}) = |\mathcal{A}| - 1$. Then the pair $\mathcal{A}^{\alpha+1}, \mathcal{B}^{\alpha+1}$ has the structure of (ii/3) with special element c and special sets A' and B' . If for the compression P_{ij} for which $P_{ij}(\mathcal{A}^\alpha) = \mathcal{A}^{\alpha+1}$, we have $c \notin \{i, j\}$, then the degree of c is unchanged, $\deg(c, \mathcal{A}^\alpha) = \deg(c, \mathcal{A}^{\alpha+1}) = |\mathcal{A}| - 1$. Suppose that $i = c$ (the case $j = c$ is impossible). We claim that either $\deg(c, \mathcal{A}^\alpha) = |\mathcal{A}| - 1$ or $\deg(j, \mathcal{A}^\alpha) = |\mathcal{A}| - 1$. For brevity we discuss only the case $c, j \notin A' \cup B'$.

Let $\mathcal{X} = \{X \subset [n] : |X| = a - 1, X \cap B' \neq \emptyset, c, j \notin X\}$ and similarly $\mathcal{Y} = \{Y \subset [n] : |Y| = b - 1, Y \cap A' \neq \emptyset, c, j \notin Y\}$. Define four families, $\mathcal{X}^c = \{X \in \mathcal{X} : X \cup \{c\} \in \mathcal{A}^\alpha\}$, $\mathcal{X}^j = \{X \in \mathcal{X} : X \cup \{j\} \in \mathcal{A}^\alpha\}$ and $\mathcal{Y}^c = \{Y \in \mathcal{Y} : Y \cup \{c\} \in \mathcal{B}^\alpha\}$, $\mathcal{Y}^j = \{Y \in \mathcal{Y} : Y \cup \{j\} \in \mathcal{B}^\alpha\}$. The families \mathcal{X}^c and \mathcal{X}^j form a partition of \mathcal{X} . A similar statement holds for the \mathcal{Y} 's.

We claim that if $X_1, X_2 \in \mathcal{X}$ and they differ only in one element (and $X_1 \cup X_2 \neq A'$), then they both belong to the same part of \mathcal{X} , implying that \mathcal{X} is equal to either \mathcal{X}^c or \mathcal{X}^j , the other part is empty. Indeed, find a $Y \in \mathcal{Y}$ with $Y \cap (X_1 \cup X_2) = \emptyset$. Then Y belongs to some of the \mathcal{Y}^z 's and then both of the X_i 's must belong to the same \mathcal{X}^z . \square

Note that in the above argument about the extremal families we corrected a small error in [6] (the families \mathcal{A}_2 and \mathcal{B}_2 could be empty for small n , $a + b < n < 2a + b - 2$).

The bounds in the Theorem are best possible in the following sense. Let $\mathcal{A}_0 = \{A \in \binom{[n]}{a} : 1 \in A, [2, b+1] \cap A \neq \emptyset\}$, and let $\mathcal{B}_0 = \{B \in \binom{[n]}{b} : 1 \in B\} \cup \{[2, b+1]\}$. Then \mathcal{A}_0 and \mathcal{B}_0 are cross-intersecting, $\cap \mathcal{B}_0 = \emptyset$, $|\mathcal{A}_0| = \binom{n-1}{a-1} - \binom{n-b-1}{a-1}$ and $|\mathcal{B}_0| = \binom{n-1}{b-1} + 1$, which is much larger than $\binom{n-1}{b-1} - \binom{n-a-1}{b-1}$. Another analogous example is $\mathcal{A}_1 = \{A \in \binom{[n]}{a} : 1 \in A\} \cup \{[2, a+1]\}$, and $\mathcal{B}_1 = \{B \in \binom{[n]}{b} : 1 \in B, B \cap [2, a+1] \neq \emptyset\}$.

The above proof of the Theorem can be easily modified in such a way that it includes the proof of the Hilton-Milner theorem, too. (Induction on $n + a + b$, and a more careful choice of the operations P_{ij}). Another short proof for the Hilton-Milner theorem, based on the Kruskal-Katona theorem [12], [11], was given by Alon [1]. Other powerful applications of compressions can be found in the survey of Frankl [4], and in the book of Bollobás [2].

4. PROBLEMS

In this section we always suppose that $\mathcal{A} \subset \binom{[n]}{a}$, $\mathcal{B} \subset \binom{[n]}{b}$ are cross-intersecting families with $b \geq a$, $n \geq a + b$. The maximum of $|\mathcal{A}||\mathcal{B}|$ was studied earlier, Pyber [14] proved that for $n \geq 2b + 2a$

$$(7) \quad |\mathcal{A}||\mathcal{B}| \leq \binom{n-1}{a-1} \binom{n-1}{b-1}.$$

This result was extended by Matsumoto and Tokushige [13] for all $n \geq \max\{2a, 2b\}$.

Conjecture 1. *The inequality (7) holds for all $n > a + b$.*

In the case $n = a + b$ one can construct cross-intersecting families \mathcal{A}_2 and \mathcal{B}_2 of sizes $|\mathcal{A}_2| = \lfloor \frac{1}{2} \binom{a+b}{a} \rfloor$ and $|\mathcal{B}_2| = \lceil \frac{1}{2} \binom{a+b}{a} \rceil$. Then $|\mathcal{A}_2||\mathcal{B}_2|$ exceeds the right hand side of (7) for $b > a$. Considering the example $\mathcal{A}_1, \mathcal{B}_1$ given at the end of the previous Section we propose the following stronger form of Conjecture 1.

Conjecture 2. *If $|\mathcal{A}||\mathcal{B}| > |\mathcal{A}_1||\mathcal{B}_1|$ and $n > a + b$, then $\cap \mathcal{A} = \cap \mathcal{B} \neq \emptyset$.*

If we have individual lower bounds, like in our Theorem, then we might get more. Define the cross-intersecting families $\mathcal{A}_3 = \{A \in \binom{[n]}{a} : 1 \in A, [2, a+1] \cap A \neq \emptyset\} \cup \{[2, a+1]\}$ and $\mathcal{B}_3 = \{B \in \binom{[n]}{b} : 1 \in B, [2, a+1] \cap B \neq \emptyset\} \cup \{B \in \binom{[n]}{b} : 1 \notin B, [2, a+1] \subset B\}$.

Conjecture 3. *If $|\mathcal{A}| \geq |\mathcal{A}_3| = \binom{n-1}{a-1} - \binom{n-a-1}{a-1} + 1$ and $|\mathcal{B}| \geq |\mathcal{B}_3| = \binom{n-1}{b-1} - \binom{n-a-1}{b-1} + \binom{n-1-a}{b-a}$ and $n > a + b$, then either $\cap \mathcal{A} \neq \emptyset$, or $|\mathcal{A}| = |\mathcal{A}_3|$, $|\mathcal{B}| = |\mathcal{B}_3|$. Moreover, for $a + b > 6$, $|\mathcal{A}| \cong \mathcal{A}_3$, $\mathcal{B} \cong \mathcal{B}_3$ are the only extrema.*

I can settle the last two conjectures for $n > n_0(a, b)$. The proof is a simple application of the delta-system method. However, it would be interesting to lower $n_0(a, b)$ to $a + b$ (if it is true). The case $a = b$ seems to be especially interesting.

The maximum of $|\mathcal{A}| + |\mathcal{B}|$ was determined by Hilton and Milner [10], (see also Simpson [15]). Their result was extended by Frankl and Tokushige [7] as follows. For $n \geq a + b$, $b \geq a$, one has $|\mathcal{A}| + |\mathcal{B}| \leq \binom{n}{b} - \binom{n-a}{b} + 1$. They also proved a number of interesting inequalities, for example, if we also suppose that $|\mathcal{A}| \geq \binom{n-1}{a-1}$, then in the case $b > a$, one can get the stronger bound $|\mathcal{A}| + |\mathcal{B}| \leq \binom{n-1}{a-1} + \binom{n-1}{b-1}$. The method of proof in [7] is completely different from ours, and those results and our Theorem do not seem to imply one another. Let $\mathcal{A}_4 = \binom{[a+1]}{a}$ and $\mathcal{B}_4 = \{B \in \binom{[n]}{b} : |[1, a+1] \cap B| \geq 2\}$.

Conjecture 4. *Suppose that \mathcal{A} and \mathcal{B} are cross-intersecting and $\cap \mathcal{A} = \cap \mathcal{B} = \emptyset$. Then for $n \geq a + b$, $b \geq a$ one has $|\mathcal{A}| + |\mathcal{B}| \leq |\mathcal{A}_4| + |\mathcal{B}_4|$.*

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