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NOTE
ON A TURÁN TYPE PROBLEM OF ERDŐS

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Let $\mathbf{L}^k$ be the graph formed by the lowest three levels of the Boolean lattice $\mathbf{B}_k$, i.e., $V(\mathbf{L}^k) = \{0, 1, \ldots, k, 12, 13, \ldots, (k-1)k\}$ and 0 is connected to i for all $1 \le i \le k$, and ij is connected to i and j $(1 \le i < j \le k)$.

It is proved that if a graph $\mathbf{G}$ over n vertices has at least $k^{3/2}n^{3/2}$ edges, then it contains a copy of $\mathbf{L}^k$.

1. Preliminaries, Results

A *hypergraph*, $\mathbf{H}$, is a pair $(V, \mathcal{E})$, where $\mathcal{E}$ is a family of subsets of V. The elements of V are called *vertices*, the $E \in \mathcal{E}$ are called hyperedges. A hypergraph is called *t-uniform*, or a t-graph, if $|E| = t$ holds for every $E \in \mathcal{E}$. The 2-graphs are called graphs. For $X \subset V$ we set $\mathcal{E}[X] = \{E : X \subset E \in \mathcal{E}\}$. The degree, $\deg(\mathbf{H}, X)$, or briefly $\deg(X)$, is the cardinality of $\mathcal{E}[X]$, $\deg(\{x\})$ is abbreviated as $\deg(x)$. The set $N(x) = \cup\mathcal{E}[x] \setminus \{x\}$ is called the *neighbourhood* of x. The family of all t-subsets of a k-set is called the complete t-graph and is denoted by $\mathbf{K}_t^k$.

Given a graph $\mathbf{F}$, what is $T(n, \mathbf{F})$, the maximum number of edges of a graph with n vertices not containing $\mathbf{F}$ as a subgraph? This is one of the basic problems of extremal graph theory, the so called Turán problem. The Erdős-Stone-Simonovits theorem([9], [11], for a survey see Bollobás' book [1]) says that the order of magnitude of $T(n, \mathbf{F})$ depends on the chromatic number of $\mathbf{F}$, namely $\lim_{n\to\infty} T(n, \mathbf{F})/\binom{n}{2} = 1 - (\chi(\mathbf{F}) - 1)^{-1}$. This theorem gives a sharp estimate, except for bipartite graphs. The case of bipartite graphs seems to be more difficult, and only a very few $T(n, \mathbf{F})$ are known. Even the exact value of $T(n, \mathbf{C}_4)$ is known only for a quite rare sequence of n's [12]. For every bipartite graph $\mathbf{F}$ which is not a forest there is a positive contant c (not depending on n) such that

$$\Omega(n^{1+c}) \le T(n, \mathbf{F}) \le O(n^{2-c})$$

holds for all $n > n_0$. The first problem is to determine the right exponent of n.

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Erdős, Rényi and T. Sós [8] and Brown [2] proved that

$$T(n, \mathbf{C}_4) = \frac{1}{2}(1 + o(1))n^{3/2}, \tag{1.1}$$

$$c_3 n^{5/3} < T(n, \mathbf{K}_{3,3}) < c_4 n^{5/3}. \tag{1.2}$$

Conjecture 1.3. (Erdős [5], also see in [10], [14]) *Let* $\mathbf{F}$ *be a bipartite graph such that each induced subgraph has a vertex of degree at most 2. Then* $T(n, \mathbf{F}) = O(n^{3/2})$.

The aim of this note is to make a small contribution to this direction. Let $k \geq 2, s \geq 1$ be integers, and define the following bipartite graph $\mathbf{L}^{k,s}$ with classes X and Y. $X = \{x_0\} \cup \{x_{ij}^{\alpha} : 1 \leq i < j \leq k, \alpha = 1, \dots s\}$ and $Y = \{y_1, \dots y_k\}$. Join x_0 to each vertex of Y, and join $x_{i,j}^{\alpha}$ to y_i and y_j. $\mathbf{L}^k$ stands for $\mathbf{L}^{k,1}$. All $\mathbf{L}^{k,s}$ contain four-cycles, so $\Omega(n^{3/2}) \leq T(n, \mathbf{L}^{k,s})$. Erdős [4] proved that $T(n, \mathbf{L}^3) = O(n^{3/2})$, and conjectured (see in [4], [6], [7]) that this holds for all $\mathbf{L}^k$, (according to the Conjecture 1.3.)

Theorem 1.4. $T(n, \mathbf{L}^{k,s}) < n\dfrac{k-1}{4} + n^{3/2}\sqrt{\dfrac{sk(k-1)^2 + 2(k-2)(k-1)}{8}}.$

To give a lower bound consider a $\mathbf{C}_4$-free graph $\mathbf{H}$ with maximum number of edges over $v = \lfloor n/(k-1) \rfloor$ vertices. Replace every vertex x with a $k-1$-element set $V(x)$. Join all vertices of $V(x)$ to all vertices of $V(y)$ if and only if $\{x, y\}$ is an edge of $\mathbf{H}$. The obtained graph is $\mathbf{L}^k$-free, so (1.1) yields

$$T(n, \mathbf{L}^k) \geq (1 + o(1))\frac{\sqrt{k-1}}{2}n^{3/2}.$$

Theorem 1.4 is implied by the following lemma.

Lemma 1.5. *Suppose that* $\mathcal{A}$, $|\mathcal{A}| = a$, *is a collection of subsets of the* n*-element set* S *with average size* b, *(that is,* $\sum |A_i|/a = b$*). Let* $k \geq t \geq 2$ *and* $d > g \geq 1$ *be integers, and suppose that*

$$\binom{d-1}{g}\binom{a}{t}\binom{k-1}{t-1} < \binom{n}{g}\binom{a\binom{b}{g}/\binom{n}{g}}{t-1}\frac{a\binom{b}{g}/\binom{n}{g} - (k-1)}{t}.$$

Then there exists k *members of* $\mathcal{A}$, $A_1, A_2, \dots, A_k \in \mathcal{A}$, *such that* $|\cap A_i| \geq g$, *and the size of the intersection of every* t *of them is at least* d.

The proof of this Lemma is postponed until the second Section. The definition of $\binom{x}{t}$ for real x, as usual, is $x(x-1)\dots(x-t+1)/t!$ when $x > t-1$ and 0 otherwise.

Proof of Theorem 1.4 from Lemma 1.5. Suppose that $\mathbf{G}$ is a graph on n vertices and e edges, where e has the value which is given by the right hand side of inequality in Theorem 1.4. Define $\mathcal{A}$ as the family of the neighbourhoods $N(x)$ ($x \in V(\mathbf{G})$). Then one can apply Lemma 1.5 to $\mathcal{A}$ with the values $a = n$, $b = 2e/n$, $k = k$, $t = 2$, $g = 1$ and $d = k - 1 + s\binom{k}{2}$. We obtain the sets $N(y_1), \dots N(y_k)$ with the following properties. There exists a vertex $x_0 \in \cap N(y_i)$, and for $1 \leq i <$

$j \le k$ one has $|N(y_i) \cap N(y_j)| \ge s\binom{k}{2} + k - 1$. Then one can find disjoint sets $V_{i,j} \subset N(y_i) \cap N(y_j) \setminus \{x_0, y_1 \dots y_k\}$ of size s, i. e., the subgraph of $\mathbf{G}$ induced on $\{x_0, y_1 \dots y_k\} \cup V_{i,j}$ contains a copy of $\mathbf{L}^{k,s}$. ∎

Another corollary of the Lemma, for example, that if $\sqrt{n}$ sets are given of average size $5\sqrt{n}$, then one can find four of them whose pairwise intersections have at least 4 elements. (Moreover they have a common element, as well.)

The Lemma also implies that if $\mathbf{G}[n, \sqrt{n}]$ is a bipartite graph with classes of sizes n and $\sqrt{n}$ and with $c(k,s)n$ edges, then it contains a copy of $\mathbf{L}^{k,s}$. (For this reformulation the author is indebted to P. Erdős.)

2. Proof of the Lemma, and more Corollaries

Let $m \ge k \ge t \ge 2$ be integers. Define $T(m,k,t)$ as the minimum number of t-sets of an m-element set S such that every k-subset of S contains a t-set. The determination of $T(m,k,t)$ is the classical Turán problem, and with the notations of the previous Section one has $T(m,k,t) = \binom{m}{t} - T(m, \mathbf{K}_t^k)$. We have

$$T(m,k,t) \ge \binom{m}{t-1} \frac{m-k+1}{t} \binom{k-1}{t-1}^{-1}. \tag{2.1}$$

This lower bound is due to de Caen [3].

Suppose on the contrary, that among every k members of $\mathcal{A}$ containing g common elements one can find t of them with intersection size at most $d-1$. If the intersection of t members of $\mathcal{A}$ has at least g but less than d elements, then they are called a subsystem of *type 0*. Let $X \subset S$, $|X| = g$ and consider the family $\mathcal{A}[X]$. The indirect assumption implies that the number of subfamilies of $\mathcal{A}[X]$ of type 0 is at least $T(\deg(X), k, t)$. On the other hand, every subfamily of $\mathcal{A}$ of type 0 can appear at most $\binom{d-1}{g}$ times in some $\mathcal{A}[X]$. Then (2.1) and the Jensen's inequality give that

$$\begin{aligned}\binom{d-1}{g}\binom{a}{t} &\ge \sum_{X \subset S} T(\deg(X), k, t) \ge \sum_{X \subset S} \binom{\deg(X)}{t-1} \frac{\deg(X) - k + 1}{t} \binom{k-1}{t-1}^{-1} \\ &\ge \frac{\binom{n}{g}}{\binom{k-1}{t-1}} \binom{\sum \deg(X)/\binom{n}{g}}{t-1} \frac{\sum \deg(X)/\binom{n}{g} - (k-1)}{t} \\ &\ge \frac{\binom{n}{g}}{\binom{k-1}{t-1}} \binom{a\binom{b}{g}/\binom{n}{g}}{t-1} \frac{a\binom{b}{g}/\binom{n}{g} - (k-1)}{t}.\end{aligned}$$ ∎

Define the bipartite graph $\mathbf{L}_t^{k,s}$ over $X \cup Y$ as follows. $X = \{x_0\} \cup \{x_I^\alpha :$ where I is a t subset of $\{1, 2, \dots k\}$ and $1 \le \alpha \le s\}$, and $Y = \{y_1, \dots y_k\}$. Join x_0 to each y_i, and join x_I^α to y_i if $i \in I$. So $\mathbf{L}_2^{k,s} = \mathbf{L}^{k,s}$. Then Lemma 1.5 also implies that there exists a constant $c_t^{k,s}$ such that

$$T(n, \mathbf{L}_t^{k,s}) \le c_t^{k,s} n^{2 - \frac{1}{t}}. \tag{2.2}$$

The exponent of n in this bound is best possible for $t = 3$ as well by (1.2). Inequality (2.2) is a generalization of an estimate of $T(n, \mathbf{K}_{t,t})$ due to Erdős, Kővári, T. Sós and Turán [13], and was also conjectured by Erdős [7].

If we use Lemma 1.5 with $g = t$, ($a = n$, $d = s\binom{k}{t} + k$), then we obtain that

$$T(n, \mathbf{G}_t^{k,s}) \leq O(n^{2-\frac{1}{t}}),$$

where $\mathbf{G}_t^{k,s}$ is obtained from $\mathbf{L}_t^{k,s}$ by replacing x_0 by t new vertices and joining each of them to Y. For example $\mathbf{G}_2^{k,1}$ is a graph with vertex-set $\{0, 0', 1, \ldots, k, 12, 13, \ldots, (k-1)k\}$ and 0 and 0′ are connected to i for all $1 \leq i \leq k$, and ij is connected to i and j $(1 \leq i < j \leq k)$.

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