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EMPTY SIMPLICES IN EUCLIDEAN SPACE

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ABSTRACT. Let $P = \{p_1, p_2, \ldots, p_n\}$ be an independent point-set in $\mathbb{R}^d$ (i.e., there are no $d + 1$ on a hyperplane). A simplex determined by $d + 1$ different points of P is called empty if it contains no point of P in its interior. Denote the number of empty simplices in P by $f_d(P)$. Katchalski and Meir pointed out that $f_d(P) \geq \binom{n-1}{d}$. Here a random construction P_n is given with $f_d(P_n) < K(d)\binom{n}{d}$, where $K(d)$ is a constant depending only on d. Several related questions are investigated.

1. **Introduction**. We call a set P of n points ($n \geq d + 1$) in the d-dimensional Euclidean space $\mathbb{R}^d$ *independent* if P contains no $d + 1$ on a hyperplane. We call a simplex determined by $d + 1$ different points of P *empty* if the simplex contains no point of P in its interior and denote the number of empty simplices of P by $f_d(P)$, or briefly $f(P)$.

Katchalski and Meir [11] asked the following question: Given an independent set P of n points in $\mathbb{R}^d$, what can one say about the values of $f(P)$? If P consists of the vertices of a convex polytope, then clearly $f(P) = \binom{n}{d+1}$. So the interesting question is to find a lower bound for $f(P)$. Define

$$f_d(n) = \min\{f(P) : |P| = n, \quad P \subset \mathbb{R}^d \text{ independent}\}.$$

They proved that there exists a constant $K > 0$ such that for all $n \geq 3$,

$$\binom{n-1}{2} \leq f_2(n) \leq Kn^2, \tag{1}$$

and in general, for every independent $P \subset \mathbb{R}^d$, $|P| = n$

$$\binom{n-1}{d} \leq f_d(P). \tag{2}$$

(The case $d = 1$ has no importance, obviously $f_1(P) = n - 1$.) The aim of this paper is to give bounds for $f_d(n)$ and to consider several related questions.

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Our paper is organized as follows. In section 2 we state the upper bound for $f_d(n)$. Section 3 contains the results about the number of empty k-gons in the plane. In section 4 we deal with a related question: how many points are needed to pin the interiors of the empty simplices? Finally sections 5–12 contain the proofs.

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2. **Random constructions.**

THEOREM 2.1. *Let $A \subset R^d$ be a convex, bounded set with nonempty interior. Choose the points $p_1, \ldots, p_n$ randomly and independently from A with uniform distribution. Then we have for the expected value of $f(P)$*

$$E(\#\text{ empty simplices in } P) \leqq K\binom{n}{d}.$$

Here K is very large:

$$K = 2^{\binom{d}{2}} d!\, d^{d^2} \pi^{(d-1)/2}\left[\Gamma\left(\frac{d}{2}+1\right)\right]^{-1}\left(\prod_{i=1}^{d-1}\Gamma\left(\frac{i}{2}+1\right)\right)^2 < (2d)^{2d^2}$$

but independent of the shape of A! It is very likely that this value can be decreased, e.g., when A is a ball we can prove $K < d^{d^2}$.

COROLLARY 2.2. $f_d(n) < d^{d^2}\binom{n}{d}$.

The example of Katchalski and Meir gives in (1) that $K < 200$. Corollary 2.2 yields $K \leq 16$. The following random construction gives a much better upper bound. Let $I_1, I_2, \ldots, I_n$ be parallel unit intervals on the plane, $I_i = \{(x, y): x = i, 0 \leq y \leq 1\}$. Choose the point p_i randomly from I_i with uniform distribution. Let $P_n = \{p_1, \ldots, p_n\}$. Then

THEOREM 2.3. $E(f_2(P_n)) = 2n^2 + 0(n \log n)$.

On the other hand we have

THEOREM 2.4. *Let $P \subset \mathbb{R}^2$ be an independent point-set with $|P| = n$. Then*

$$n^2 - 0(n \log n) \leq f_2(P).$$

We have to remark here that G. Purdy [13] announced $f_2(n) = 0(n^2)$ without proof. H. Harborth [8] pointed out that $f_2(n) = n^2 - 5n + 7$ for $n = 3, 4, 5, 6, 7, 8, 9$ but not for $n = 10$ because $f_2(10) = 58$.

3. **Empty polygons on the plane.** More than 50 years ago Erdös and Szekeres [5] proved that for every integer $k \geqq 3$ there exists an integer $n(k)$ with the following property: If $P \subset R^2$, $|P| \geq n(k)$ and P is independent, then there exists a subset $A \subset P$ such that $|A| = k$ and conv A is a convex k-gon.

We call a k-subset A of P empty if conv A contains no point of P in its interior. Erdös [4] asked whether the following sharpening of the Erdös-Szekeres theorem is true. Is there an $N(k)$ such that if $|P| \geqq N(k)$. $P \subset R^2$ independent, then there exists an empty k-gon with vertex set $A \subset P$. He pointed out that $N(4) = 5$ $(= n(4))$ and [8] proved that $N(5) = 10$ (while $n(5) = 9$). A proof of the existence of $N(k)$ was presented at a combinatorial conference in 1978 but it turned out to be wrong. This is no wonder because Horton [9] proved that $N(7)$ does not exist. The question about the existence of $N(6)$ is still open; a recent example of Fabella and O'Rourke [6] shows twenty-two independent points in the plane without an empty hexagon.

EXAMPLE 3.1. (Horton [9]). *(This is a squashed version of the well-known van der Corput sequence.) We will define by induction a pointset $Q(n)$ where n is a power of 2. In $Q(n)$ each point has positive integer coordinates and the set of the first coordinates is just $\{1, 2, \ldots, n\}$. To start with let $Q(1) = \{(1, 1)\}$ and $Q(2) = \{(1, 1), (2, 2)\}$. When $Q(n)$ is defined, set*

$$Q(2n) = \{(2x - 1, y):(x, y) \in Q(n)\} \cup \{(2x, y + d_n):(x, y) \in Q(n)\}$$

where d_n is a large number, e.g., $d_n = 3^n$ will do.

Now denote by $f^k(P)$ the number of empty k-gons in P and let $f^k(n) = \min\{f^k(P)$: $P \subset R^2$ independent, $|P| = n\}$. So $f^3(n)$ is just $f_2(n)$ defined in the previous section. Though $f^k(P)$ can be as large as $\binom{n}{k}$, Example 3.1 shows the following estimations.

THEOREM 3.2. *When n is a power of 2, then*

$$f^3(n) \leqq 2n^2 \tag{3}$$

$$f^4(n) \leqq 3n^2 \tag{4}$$

$$f^5(n) \leqq 2n^2 \tag{5}$$

$$f^6(n) \leqq \frac{1}{2}n^2 \tag{6}$$

$$f^k(n) = 0 \quad \textit{for} \quad k \geqq 7. \tag{7}$$

We remark that the random example of Theorem 2.3 gives a quadratic upper bound on $f^k(n)$, too. The only lower bounds we can prove are

THEOREM 3.3.

$$f^4(n) \geqq \frac{1}{4}n^2 - 0(n), \qquad f^5(n) \geqq \left\lfloor \frac{n}{10} \right\rfloor. \tag{8}$$

The second inequality here is implied by $N(5) = 10$.

4. **The covering number of simplices**. Let P be an independent set of points in R^d. We say that $Q \subset R^d$ is a cover of the simplices of P if for every $(d + 1)$-tuple $\{p_1, \ldots, p_{d+1}\} \subset P$ there exists a $q \in Q$ with $q \in \operatorname{int} \operatorname{conv}\{p_1, \ldots, p_{d+1}\}$. Denote by $g(P)$ the minimum cardinality of a cover and let $g_d(n) = \max\{g(P):P \subset R^d, |P| = n\}$. Katchalsky and Meir [11] proved that $g_2(n) = 2n - 5$ and $g_3(n) \leqq (n - 1)^2$.

Actually they proved

$$g_2(P) = 2|P| = (\# \text{ vertices of conv } P) - 2.$$

Though such an exact result seems to be elusive in higher dimensions, we can determine the asymptotic value of $g_d(n)$.

THEOREM 4.1.

$$g_d(n) = \begin{cases} 2\binom{n}{d/2} + 0(n^{d/2-1}) & \text{if} \quad d \text{ is even} \\ \binom{n}{[d/2]} + 0(n^{[d/2]}) & \text{if} \quad d \text{ is odd} \end{cases}$$

holds for any fixed d when $n \to \infty$.

COROLLARY 4.2. $g_3(n) = \binom{n}{2} + 0(n)$.
The constructions and proofs will be given in section 11.

The high value of $g_d(n)$ is a bit surprising (at least for the authors), because it was proved in [2] and [1] that there exists a positive constant $c(d)$ ($c(2) = 2/9$, $c(d) > d^{-d}$) with the following property. For any pointset $P \subset R^d$, $|P| = n$ there exists a point contained in at least $c(d)\binom{n}{d+1}$ simplices of P.

5. **The distribution of volumes of random simplices**. Consider a bounded convex set $A \subset R^d$ with $\mathrm{Vol}(A) > 0$. Choose randomly and independently the points $p_1, \ldots, p_{d+1}$ from A with uniform distribution.

LEMMA 5.1. *There exists a* $C = C(d) > 0$ *such that for every* $0 < v < 1$, $h > 0$

$$\mathrm{Prob}(v < \mathrm{Vol}(p_1, \ldots, p_{d+1})/\mathrm{Vol}(A) < v + h) < Ch$$

where $\mathrm{Vol}(p_1, \ldots, p_{d+1})$ *is a shorthand for* $\mathrm{Vol}(\mathrm{conv}\{p_1, \ldots, p_{d+1}\})$.

PROOF. A theorem of Fritz John [10] says that there exist two concentrical and homothetic ellipsoids E_1 and E_2 with $E_1 \subset A \subset E_2$ and $E_2 \subset dE_1$. As an affine transformation does not change the value of $\mathrm{Vol}(p_1, \ldots, p_{d+1})/\mathrm{Vol}(A)$ we may assume that E_1 and E_2 are balls of radius r_1 and r_2 and $r_2 \leq dr_1$. Define w_d to be the volume of the d-dimensional unit ball, i.e.,

$$w_d = \pi^{d/2}\left(\Gamma\left(\frac{d}{2} + 1\right)\right)^{-1}.$$

Let $0 < t < t + a$ and denote the Euclidean distance between $\mathrm{aff}(p_1, \ldots, p_i)$ and p_{i+1} by D_i. Then

$$\mathrm{Prob}(t < D_i < t + a) \leqq \frac{w_{i-1} r_2^{i-1}}{\mathrm{Vol}(A)}(w_{d+1-i}(t + a)^{d+1-i} - w_{d+1-i}t^{d+1-i})$$

holds for every $i = 1, \ldots, d$; the right hand side is the volume of the difference of two cylinders. Hence we have

$$\operatorname{Prob}(t < D_i < t + a) \leqq \frac{a}{r_2}\left(\frac{t}{r_2}\right)^{d-i} \frac{(d+1-i) w_{d+1-i} w_{i-1}}{w_d} \frac{w_d r_2^d}{\operatorname{vol}(A)}$$

$$+ 0\left(\left(\frac{a}{r_2}\right)^2\right) < \frac{a}{r_2}\left(\frac{t}{r_2}\right)^{d-i} 2^d d^{d+1}\left(1 + 0\left(\frac{a}{r_2}\right)\right).$$

The choice of p_i and p_j is independent so we have

(9) $\quad \operatorname{Prob}(t_i < D_i < t_i + a \quad \text{holds for} \quad i = 1, \ldots, d)$

$$\leqq \left(\frac{a}{r_2}\right)^d \left(\frac{t_1}{r_2}\right)^{d-1} \left(\frac{t_2}{r_2}\right)^{d-2} \cdot \ldots \cdot \left(\frac{t_{d-1}}{r_2}\right) 2^{d^2} d^{d^2+d}\left(1 + 0\left(\frac{a}{r_2}\right)\right).$$

Now $\operatorname{Vol}(p_1, \ldots, p_{d+1}) = (d!)^{-1} D_1 \cdot D_2 \cdot \ldots \cdot D_d$. Hence (9) yields

(10) $\quad \operatorname{Prob}(v < \operatorname{Vol}(p_1, \ldots, p_{d+1})/\operatorname{Vol}(A) < v + h)$

$$\leqq \int_{x_1=0}^{2} \cdots \int_{x_d=0}^{2} x_1^{d-1} x_2^{d-2} \ldots x_{d-1} 2^{d^2} d^{d^2+d} dx_1 dx_2 \ldots dx_d$$

where the integration is taken for $(x_1, \ldots, x_d)$ with

$$v \cdot \operatorname{Vol}(A) < r_2^d x_1 \ldots x_d (d!)^{-1} < (v + h)\operatorname{Vol}(A).$$

Because

$$0 \leq x_d - d! v r_2^{-d} \cdot \operatorname{Vol} A/(x_1 \ldots x_{d-1}) \leq h d! (\operatorname{Vol} A/r_2^d)/(x_1 \ldots x_{d-1})$$

we have

$$\int dx_d = h d! (\operatorname{Vol} A/r_2^d)/(x_1 \ldots x_{d-1}).$$

Hence the right-hand-side of (10) equals

$$\left[(2^{d^2} d^{d^2+d}) d! \frac{\operatorname{Vol} A}{r_2^d}\right] h \int_{0<x_1<2} \cdots \int_{0<x_{d-1}<2} x_1^{d-2} \ldots x_{d-2}^1 dx_1 \ldots dx_{d-1}$$

$$= (2^{\binom{d}{2}}/(d-1)!) \cdot C_0 h < (2d)^{2d^2} h,$$

where C_0 is the coefficient in square brackets.

6. **Proof of Theorem 2.1**. For given $p_1, \ldots, p_{d+1}$ choose the points $p_{d+2}, \ldots, p_n$ randomly. Define $\mu(v) = \operatorname{Prob}(\operatorname{Vol}(p_1, \ldots, p_{d+1}) < v)$. Obviously we have

$$\operatorname{Prob}(p_1, \ldots, p_{d+1} \text{ is empty}) = \int_{0 \leq v \leq 1} (1 - v)^{n-d-1} d\mu(v)$$

$$\leq \int_{0 \leq v \leq 1} (1 - v)^{n-d-1} C dv = C/(n - d).$$

Hence

$$E(f(P)) \leq \binom{n}{d+1} \frac{C}{n-d} = \frac{C}{d+1}\binom{n}{d}.$$

7. **Proof of Theorem 2.3**. Consider the points $A = (i, x)$, $B = (i + a, y)$, and $C = (i + k, z)$ where $k = a + b \geqq 3$. Let $m = |y - x + (a/k)(z - x)|$, i.e., the distance between B and $l_{i+a} \cap [AC]$. Choose randomly a point p_j on l_j, $(i < j < i + k,\ j \neq i + a)$. Then

$$\begin{aligned}
&\operatorname{Prob}(ABC \text{ is an empty triangle})\\
&= \left(1 - \frac{m}{a}\right)\left(1 - 2\frac{m}{a}\right)\dots\left(1 - (a-1)\frac{m}{a}\right)\left(1 - (b-1)\frac{m}{b}\right)\dots\left(1 - \frac{m}{b}\right)\\
&\leq \exp\left[-\frac{m}{a} - 2\frac{m}{a} - \dots - (a-1)\frac{m}{a} - (b-1)\frac{m}{b} - \dots - 2\frac{m}{b} - \frac{m}{b}\right]\\
&= \exp\left(-\binom{a}{2}\frac{m}{a} - \binom{b}{2}\frac{m}{b}\right) = \exp(-(k-2)m/2).
\end{aligned}$$

Now choose the points p_i $(1 \leq i \leq n)$ randomly. We obtain

$$\begin{aligned}
\operatorname{Prob}(p_i p_{i+a} p_{i+k} \text{ is empty}) &\leqq \int_{0<x<1}\int_{0<y<1}\int_{0<z<1} \exp(-(k-2)m/2)\,dx\,dy\,dz\\
&\leq 2\int_{0<m<1/2} \exp(-(k-2)m/2)\,dm \leq 4/(k-2).
\end{aligned}$$

Hence we have

$$\begin{aligned}
E(f(P)) &\leq n - 1 + \sum_{1\leq i\leq n}\ \sum_{3\leq k\leq n-i}\ \sum_{1<a<k} 4/(k-2)\\
&= n - 1 + \sum_{3\leq k\leq n} (n-k+1)\frac{4(k-1)}{k-2}\\
&= n - 1 + \sum_{3\leq k\leq n} (n-k+1)4/(k-2) + 4\sum_{3\leq k\leq n}(n-k+1)\\
&= 0(n \log n) + 2n^2.
\end{aligned}$$

8. A lemma on graphs.

Lemma 8.1. *Let G be a graph on the vertices $\{1, 2, \dots, n\}$. Suppose that there exist no four vertices $i < j < k < \ell$ such that (i, k), (i, ℓ), and $(j, \ell) \in E(G)$. Then*

$$|E(G)| \leq 3n\lceil \log_2 n \rceil. \tag{11}$$

Proof. Let $E(G) = E(G_1) \cup \dots E(G_i) \cup \dots$ where $1 \leq i \leq \lceil \log_2 n \rceil$ and $E(G_i) = \{(u, v) : 1 \leq u \leq v \leq n,\ 2^{i-1} \leq v - u < 2^i,\ (u, v) \in E(G)\}$. Split $E(G_i)$ into three parts U, D and T:

$$\begin{aligned}
U = \{(u, v) : (u, v) \in E(G_i) \quad &\text{and} \quad \exists w \quad \text{such that} \quad u < w < v\\
&\text{and} \quad (w, v) \in E(G_i)\}\\
D = \{(u, v) : (u, v) \in E(G_i) \quad &\text{and} \quad \exists w \quad \text{such that} \quad u < w < v\\
&\text{and} \quad (u, w) \in E(G_i)\}
\end{aligned}$$

and $T = E(G_i) - U - D$.

Clearly $U \cap D = \phi$, U, D and T do not contain a circuit. Hence their cardinality is at most $n - 1$.

We note that (11) can be improved to $\lfloor n \log_2 n \rfloor$, and there exists a graph G'' with $|E(G)| \geq n(\log_2 n - 2)$ which fulfills the constraints of Lemma 8.1.

9. **Proof of Theorem 2.4**. Consider the points $p_1, \ldots, p_n \in \mathbb{R}^2$ and an arbitrary line $e \subset \mathbb{R}^2$. Let q_i be the projection of p_i on e. We can choose e such that $q_i \neq q_j$. We can suppose that q_i lays between q_{i-1} and q_{i+1} (eventually reordering the indices).

Let G_u and G_d be two graphs on vertices $\{q_1, \ldots, q_n\}$ such that

$E(G_u) = \{q_iq_j$: every p_k for $i < k < j$ is below the $[p_ip_j]$ and only (at most) one $p_ip_kp_j$ triangle is empty$\}$

$E(G_d) = \{(q_iq_j)$: every p_k for $i < k < j$ is above the $[p_ip_j]$ and only (at most) one of the triangles $p_ip_kp_j$ is empty$\}$.

It is easy to see that G_u and G_d fulfills the constraints of Lemma 8.1. Indeed, suppose on contrary $(q_iq_k), (q_iq_\ell), (q_jq_\ell) \in E(G_u)$. Then one can find an j', $i < j' \leq j$ and a k', $k \leq k' < \ell$ such that the triangles p_ip_j, p_ℓ and p_ip_k, p_ℓ are empty, contradicting $p_ip_\ell \in E(G_u)$. Hence

$$f(P) = \sum_{1 \leq i < j \leq n} \#(\text{empty triangles with vertices } p_ip_kp_j,\ i < k < j)$$

$$\geq 2\binom{n}{2} - |E(G_u)| - |E(G_d)| = n^2 - 0(n \log n).$$

10. **Proof of 3.2.** Let P be a pointset in the plane, consider $u_1, u_2 \in P$ with $u_1 = (x_1, y_1)$, $u_2 = (x_2, y_2)$. We say that the line segment $[u_1, u_2]$ connecting u_1 and u_2 is empty from below if the interior of the "infinite triangle" with vertices u_1, u_2, $(\frac{x_1+x_2}{2}, -\infty)$ contains no point of P. Emptiness from above is defined analogously. Denote by $h_2^-(P)$ and $h_2^+(P)$, respectively the number of segments in P empty from below and above.

Consider $Q(2n)$ from Example 3.1. $Q(2n)$ splits in a natural way into two parts: $Q^+(n)$ and $Q^-(n)$ where $Q^+(n) = \{(2x, y + d_n):(x, y) \in Q(n)\}$ and $Q^-(n) = \{(2x - 1, y); (x, y) \in Q(n)\}$. The next two statements are obvious.

(12) If $u_1, u_2 \in Q(2n)$ and $[u_1, u_2]$ is empty from below in $Q(2n)$ then either $u_1, u_2 \in Q^-(n)$ or $u_1 \in Q^-(n)$ and $u_2 \in Q^+(n)$ and $|x_1 - x_2| = 1$ or $u_1 \in Q^+(n)$ and $u_2 \in Q^-(n)$ and $|x_1 - x_2| = 1$.

(13) $$h_2^-(Q(2n)) = h_2^-(Q^-(n)) + 2n - 1.$$

Using induction (13) implies that

(14) $$h_2^-(Q(n)) < 2n.$$

$Q(n)$ is centrally symmetric and so

$$h_2^+(Q(n)) < 2n. \tag{15}$$

Now call a triple $(u_1, u_2, u_3) \in Q(n)$ empty from below if all the three line segments $[u_1u_2]$, $[u_1u_3]$, $[u_2u_3]$ are empty from below and denote by $h_3^-(Q(n))$ the number of triples of $Q(n)$, that are empty from below. Clearly,

$$h_3^-(Q(2n)) = h_3^-(Q^-(n)) + n - 1$$

hence by induction

$$h_3(Q(n)) < n.$$

To prove (3), (4), . . . , (7) we can use induction and the facts established about h_2^+, h_2^-, h_3^+ and h_3^-. For instance, we can estimate $f^4(Q(2n))$ in the following way:

$$f^4(Q(2n)) = f^4(Q^+(n)) + h_3^+(Q^+(n))n + h_2^-(Q^+(n))h_2^+(Q^-(n)) + nh_3^+(Q^-(n)) + f^4(Q^-(n)) < 2f^4(Q(n)) + 6n^2.$$

which shows that $f^4(Q(2n)) \leqq 12n^2$.

The proofs of (3), (5), (6) are similar.

11. **Proof of 3.3.** Consider an arbitrary n-element set P in the plane, and assume no three points of P are on a line.

LEMMA 11.1. *Suppose u, v, a, $b \in P$ and the segments $[uv]$ and $[ab]$ intersect (in an interior point). Then there exist a', $b' \in P$ such that $uva'b'$ is an empty quadrilateral with diagonal $[uv]$.*

PROOF. Trivial: if the uva triangle is empty then take $a' = a$ if not let $a' \in P$ be the nearest to $[uv]$ point from the interior of the triangle uva.

Now define a graph G with vertex set P. A pair $\{u, v\} \subset P$ is an edge of G if $[uv]$ is *not* a diagonal of any convex empty quadrilateral of P. By the above Lemma G must be a planar graph hence the number of its edges is at most $3n - 6$. All other pairs are contained in an empty quadrilateral hence $f^4(P) \geq \frac{1}{2}(\binom{n}{2} - (3n - 6))$.

12. **Proof of 4.1**. First we give the upper bound. Our main tool is Radon's theorem [3] which we need in the following form.

LEMMA 12.1. *Let $x_1, \ldots, x_{d+1} \in R^d$ be the vertices of a simplex S and let L be a line not parallel to any one of the facets of S. Then there exists a line L' parallel to L such that $L' \cap S = [ab]$ and $a \in \operatorname{relint} F_a$ and $b \in \operatorname{relint} F_b$ with F_a and F_b disjoint faces of S.*

PROOF. Consider the projection of $x_1, \ldots, x_{d+1}$ onto the subspace orthogonal to L and apply Radon's theorem in that subspace.

We use the lemma in the following way. Pick a line L not parallel to any affine subspace spanned by at most d points of P. Choose $\epsilon > 0$ small enough and let v be a vector parallel to L and $\|v\| = \epsilon$. We define a covering system Q as follows:

$$Q = \left\{v + \frac{1}{t}\sum_{x \in X} x : t \leqq \frac{d+1}{2}, X \subset P, |X| = t\right\}$$

when d is odd, and

$$Q = \left\{\delta v + \frac{1}{t}\sum_{x \in X} x : \delta = \pm 1,\ t \leqq \frac{d}{2}, X \subset P, |X| = t\right\}.$$

when d is even.

Now we give a construction for the lower bound. Let $p(i) = (i, i^2, \ldots, i^d) \in R^d$, $i = 1, \ldots, n$ and set $P = \{p(i) : i = 1, \ldots, n\}$. P is the set of vertices of the cyclic polytope [7, 12]. We will use certain properties of the cyclic polytope without explanation. Consider first the case when d is odd. Define

$$\mathcal{F} = \Big\{\{i_1, \ldots, i_{d+1}\} \subset \{1, \ldots, n\}\ i_\alpha < i_{\alpha+1} \quad \text{for} \quad 1 \leqq \alpha \leqq d \quad \text{and}$$

$$i_{2\beta} = i_{2\beta-1} + 1 \quad \text{for} \quad 1 \leqq \beta \leqq \frac{d+1}{2}\Big\}$$

So the members of the family $\mathcal{F}$ are unions of segments of $\{1, 2, \ldots, n\}$ of even length. Clearly

$$|\mathcal{F}| = \binom{n}{\frac{d+1}{2}} - 0(n^{(d-1)/2}).$$

We claim that the simplices $\operatorname{conv}\{p(i) : i \in F\}$, $F \in \mathcal{F}$ are pairwise disjoint. Let $F_1, F_2 \in \mathcal{F}$ with $F_1 = \{i_1, \ldots, i_{d+1}\}$, $F_2 = \{j_1, \ldots, j_{d+1}\}$ and let k be the minimal element of the symmetric difference $F_1 \Delta F_2$, $k \in F_1$, say. Clearly $k = i_{2\alpha-1}$, i.e., its order in F_1 is odd. Consider the hyperplane H passing through the vertices $\{p(i) : i \in F_1 - \{k\}\}$. We claim that H separates conv F_1 and conv F_2. The equation of H is

$$H(x_1, x_2, \ldots, x_d) = \det \begin{vmatrix} 1 & x_1 & x_d \\ 1 & i_1 & i_1^d \\ & \cdot & \\ 1 & \cdot & \\ & \cdot & \\ 1 & i_{d+1} \cdots & i_{d+1}^d \end{vmatrix} = 0$$

where the row corresponding to k is missing. Set $f(t) = H(t, t^2, \ldots, t^d)$, this is a polynomial in t of degree d. Then $f(i_s) = 0$ for $i_s \neq k$, i.e., its roots are exactly $\{i_1, \ldots, i_{d+1}\} \backslash \{k\}$. Let, say $f(k) > 0$. Then the sign of $f(t)$ is negative for every integer $t > k$ except for those with $t = i_s$. So $H(x) \geq 0$ for $x \in \{p(i) : i \in F_1\}$ and $H(x) \leqq 0$ for $x \in \{p(i) : i \in F_2\}$. Thus we obtained $|\mathcal{F}|$ pairwise disjoint simplices. To cover them requires at least that many points so $g_d(n) \geq |\mathcal{F}|$.

The case d is even is similar. We define

$$Q = \{p(i): i = 1, 2, \ldots, n - 2\} \cup \{v, -v\}$$

where v is in general position with respect to $p(i)$ and $\|v\|$ is large enough. This means that each facet of $\pi = \operatorname{conv}\{p(i): i = 1, \ldots, n - 2\}$ is visible from either v or $-v$. As it is well-known [7, 12], π has $\binom{n}{d/2} + 0(n^{d/2-1})$ facets $F_1, \ldots, F_s$. Now in the following set of simplices no two have a common interior point:

$$\begin{aligned} &\{\operatorname{conv}(F_i \cup \{v\}): F_i \text{ is visible from } v\} \\ &\cup \{\operatorname{conv}(F_i \cup \{v\}): F_i \text{ is visible from } -v\} \\ &\cup \{\operatorname{conv}\{p(i_1), \ldots, p(i_{d+1})\}: 1 \leqq i_1 < i_2 < \ldots < i_d < i_{d+1} = n - 2, \\ &\quad i_{2\beta} = i_{2\beta-1} + 1 \quad \text{for} \quad \beta = 1, \ldots, d/2\}. \end{aligned}$$

This set of simplices shows that the simplices of Q cannot be covered by less than $2\binom{n}{d/2} + 0(n^{d/2-1})$ points. Details are left to the reader.

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