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The Maximum Number of Balancing Sets

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Abstract. Let $a_1, \ldots, a_n$ be a sequence of nonzero real numbers with sum zero. A subset B of $\{1,2,\ldots,n\}$ is called a balancing set if $\sum a_b = 0$ $(b \in B)$. S. Nabeya showed that the number of balancing sets is bounded above by $\binom{n}{n/2}$ and this bound achieved for n even with $a_j = (-1)^j$. Here his conjecture is verified, showing a tight upper bound $2\binom{2k}{k-1}$ when $n = 2k + 1$. The essentially unique extremal configuration is: $a_1 = 2, a_2 = \cdots = a_k = 1, a_{k+1} = \cdots = a_{2k+1} = -1$.

1. Introduction, Results

Let $[n]$ denote the set $\{1,2,\ldots,n\}$, $2^{[n]}$ is its power-set. Let $a_1, \ldots, a_n$ be a sequence of nonzero real numbers, and suppose that $\sum a_i = 0$. A subset $B \subset [n]$ is called a *balancing set* if $\sum \{a_b : b \in B\} = 0$. Denote the set of balancing sets by $\mathscr{B}(a_1, \ldots, a_n)$. By definition $\varnothing \in \mathscr{B}$. In this note we determine the maximum number of balancing sets. Let

$$f(n) = \max\left\{|\mathscr{B}(a_1,\ldots,a_n)| : \sum a_i = 0, a_i \neq 0\right\}.$$

Example 1.1. Suppose n is even and define $a_1 = \cdots = a_{n/2} = 1$, $a_{n/2+1} = \cdots = a_n = -1$. Then $|\mathscr{B}| = \sum_i \binom{n/2}{i}^2 = \binom{n}{n/2}$.

Example 1.2. Suppose n is odd, $n = 2k + 1$ (≥ 3). Define $a_1 = 2, a_2 = \cdots = a_k = 1$, $a_{k+1} = \cdots = a_{2k+1} = -1$. Then

$$|\mathscr{B}| = \sum_i \binom{k-1}{i}\binom{k+1}{i+2} + \sum_i \binom{k-1}{i}\binom{k+1}{i} = 2\binom{2k}{k-1}.$$

We call two sequences of reals $(a_1, \ldots, a_n)$ and $(a_1', \ldots, a_n')$ to be *isomorphic* if there exists a permutation $\pi: [n] \to [n]$ and a nonzero real α such that $a_i = \alpha a_{\pi(i)}$.

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Theorem 1.3.

$$f(n) = \begin{cases} \binom{n}{n/2} & \text{if } n \text{ is even,} \\ 2\binom{2k}{k-1} & \text{if } n = 2k+1. \end{cases}$$

Moreover $|\mathscr{B}(a_1, \ldots, a_n)| = f(n)$ *holds if and only if* $(a_1, \ldots, a_n)$ *is isomorphic to one of the sequences defined by the above Examples.*

This was conjectured by S. Nabeya [5], who proved the case n even. He used generator functions and calculus with complex variables. Our aim is to give an elementary proof, showing the strength and applicability of the results of the theory of finite sets.

2. The Proof of the Upper Bound

A family of finite sets $\mathscr{F}$ is called a *Sperner* family if $F \not\subset F'$ holds for any two $F, F' \in \mathscr{F}$. The family $\mathscr{F}$ is *intersecting* if $F \cap F' \neq \emptyset$ holds for any two members of $\mathscr{F}$. The following theorems are probably the most frequently cited results of extremal set-theory.

Theorem 2.1 (Sperner [9]). *Suppose* $\mathscr{F} \subset 2^{[n]}$ *is a Sperner family, then* $|\mathscr{F}| \leq \binom{n}{\lfloor n/2 \rfloor}$. *Here equality holds if and only if* $\mathscr{F}$ *consists of either all subsets of size* $\lfloor n/2 \rfloor$ *or all subsets of size* $\lceil n/2 \rceil$.

Theorem 2.2 (Erdös, Ko and Rado [3]). *Suppose* $\mathscr{F} \subset 2^{[n]}$ *is an intersecting Sperner family and suppose that every* $F \in \mathscr{F}$ *has at most k elements,* $k \leq n/2$. *Then* $|\mathscr{F}| \leq \binom{n-1}{k-1}$. *Here for* $n > 2k$ *equality holds if and only if* $\mathscr{F}$ *is taken to be all the k-subsets of* $[n]$ *containing a common element.*

Proof of the upper bound in Theorem 1.3. Let $a_1, \ldots, a_n$ be nonzero reals with sum zero. Let $P = \{i: a_i > 0\}$, $N = \{j: a_j < 0\}$. For a set $B \in \mathscr{B}$ define B^+ as $B \cap P$ and $B^- = B \cap N$. Moreover let

$$S(B) = B^+ \cup (N - B^-),$$

$\mathscr{S}(\mathscr{B}) = \{S(B): B \in \mathscr{B}\}$. For a set $C \subset [n]$, $\bar{C}$ denotes its complement, $\bar{C} = [n] - C$. As $B \in \mathscr{B}$ implies $\bar{B} \in \mathscr{B}$ we have

$$S \in \mathscr{S}(\mathscr{B}) \quad \text{imples } \bar{S} \in \mathscr{S}(\mathscr{B}). \tag{2.1}$$

Moreover we claim

$$\mathscr{S}(\mathscr{B}) \text{ is a Sperner family.} \tag{2.2}$$

Indeed, $S(B) \subset S(B')$ implies $B \cap P \subset B' \cap P$ and $B \cap N \supset B' \cap N$. Hence

$$\sum_{i \in B \cap P} a_i \leq \sum_{i \in B' \cap P} a_i = - \sum_{i \in B' \cap N} a_i \leq - \sum_{i \in B \cap N} a_i.$$

Then equality is forced, implying $B \cap P = B' \cap P$, $B \cap N = B' \cap N$, a contradiction.

Now in the case n even, (2.2) and Theorem 2.1 implies $|\mathscr{B}| = |\mathscr{S}(\mathscr{B})| \leq \binom{n}{n/2}$ with equality holding if $\mathscr{S}(\mathscr{B})$ consists of all $n/2$-element sets.

In the case $n = 2k + 1$ we have $\binom{n}{\lfloor n/2 \rfloor} > 2\binom{2k}{k-1}$, so we need more preparations. Define $\mathscr{F}$ as the set of members of $\mathscr{S}(\mathscr{B})$ with at most k elements. Then (2.1) imples $|\mathscr{F}| = \frac{1}{2}|\mathscr{S}(\mathscr{B})|$, and Theorem 2.2 implies $|\mathscr{F}| \leq \binom{2k}{k-1}$, as desired.

3. The Case of Equality

We are going to use the following claim

$$\text{If } S(B) - S(B') = \{i\}, \quad S(B') - S(B) = \{j\} \quad \text{then } |a_i| = |a_j|. \tag{3.1}$$

Proof. By definition $\sum_{t \in P \cap B} a_t + \sum_{t \in N - B} a_t = 0$, and the same holds for B'. These imply

$$\sum_{t \in P \cap (B \setminus B')} a_t - \sum_{t \in P \cap (B' \setminus B)} a_t + \sum_{t \in N \cap (B' \setminus B)} a_t - \sum_{t \in N \cap (B \cap B')} a_t = 0. \tag{3.2}$$

Now we have four possibilities: $i, j \in N$ or $i, j \in P$ imply $a_i = a_j$, and in the cases $i \in P$, $j \in N$ or $i \in N, j \in P$ (3.2) implies $a_i + a_j = 0$.

Returning to the proof of Theorem 1.3 we have obtained in the previous section that $|\mathscr{S}(\mathscr{B})| = f(n)$ implies (both in the cases n even and odd) the following: There exists an element of $[n]$, say 1, such that $\mathscr{S}(\mathscr{B})$ contains all the $\lfloor n/2 \rfloor$-element subsets of $[n]$ through the element 1. So (3.1) implies that $|a_i| = |a_j|$ holds for all $2 \leq i < j \leq n$. This implies easily that $(a_1, \ldots, a_n)$ is isomorphic to one of the Examples 1.1-2.

4. Remarks

Instead of the 50 years old Theorem 2.1 and 2.2 we can use a stronger result about intersecting and union-free Sperner families $\left(\text{if } \mathscr{F} \subset 2^{[n]} \text{ is a Sperner family, intersecting, and } F \cup F' \neq [n] \text{ for any two } F, F' \in \mathscr{F} \text{ then } |\mathscr{F}| \leq \binom{n-1}{\lfloor n/2 \rfloor - 1}\right)$. This was proved independently by a few authors, (see Brace and Daykin [2], Kleitman and Spencer [4], Schönheim [8]). Even our special case $\left(\text{if } \mathscr{F} \subset 2^{[n]} \text{ is Sperner and } F \in \mathscr{F} \text{ implies } \bar{F} \in \mathscr{F} \text{ then } |\mathscr{F}| \leq \binom{n-1}{\lfloor n/2 \rfloor - 1}\right)$ was investigated and proved by Bollobás [1] and Purdy [7]. But here I wanted to emphasize the simplicity of the proof using the most basic theorems from extremal hypergraph theory.

The much more difficult question, what is the number of balancing sets if all the a_i's are different, was essentially solved by Stanley [10] (also see [6]). He verified the conjecture of Erdös and Moser proving that the best sequence is either

$$\{-(n-1)/2, \ldots, -1, 0, 1, \ldots, (n-1)/2\}, \quad \text{or } \{-n/2, \ldots, -2, -1, 1, 2, \ldots, n/2\}.$$

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