

## Note

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# Random Ramsey graphs for the four-cycle

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### Abstract

It is shown that there is a graph  $\mathcal{G}$  with  $n$  vertices and at least  $n^{1.36}$  edges such that it contains neither  $\mathcal{C}_3$  nor  $\mathcal{K}_{2,3}$  but every subgraph with  $2n^{4/3}(\log n)^2$  edges contains a  $\mathcal{C}_4$ , ( $n > n_0$ ). Moreover, the chromatic number of  $\mathcal{G}$  is at least  $n^{0.1}$ .

## 1. Results, problems

A graph  $\mathcal{G}$  is *Ramsey with respect to  $\mathcal{H}$* ,  $\mathcal{G} \rightarrow \mathcal{H}$ , if every two-coloration of the edges of  $\mathcal{G}$  results in a monochromatic subgraph isomorphic to  $\mathcal{H}$ . It is easy to see that  $\mathcal{K}_6 \rightarrow \mathcal{C}_4$  and  $\mathcal{K}_{3,7} \rightarrow \mathcal{C}_4$ . Erdős and Faudree [2] asked to find a  $\mathcal{K}_{2,3}$ -free graph Ramsey with respect to  $\mathcal{C}_4$ . The graphs  $\mathcal{K}_6, \mathcal{K}_{3,7}$  are saturated by  $\mathcal{C}_4$ 's, so it could be not so surprising they avoid  $\mathcal{C}_4$ . What Erdős and Faudree asked was whether a graph  $\mathcal{G}$  exists with  $\mathcal{G} \rightarrow \mathcal{C}_4$ , such that any two four-cycles in  $\mathcal{G}$  are either (vertex) disjoint, or share a common vertex, or an edge. The aim of this note is to show that the random method implies the existence of such a graph.

We obtain that for  $E > E_0$  and for some  $c > 0$  there are  $\mathcal{K}_{2,3}$ -free graphs with  $E$  edges such that the largest  $\mathcal{C}_4$ -free subgraph has only  $E^{1-c}$  edges. Given this result Erdős asked for the best exponent. We have  $c \geq 1/51 - o(1)$  which can be easily

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improved to  $c \geq 1/21 - o(1)$ . On the other hand, obviously, every (connected) graph contains a  $\mathcal{C}_4$ -free subgraph of size  $|V(\mathcal{G})| - 1$  (namely, a spanning tree). This is at least  $E^{2/3}$ , as every  $\mathcal{K}_{2,3}$ -free graph has at most  $(1 + o(1))n^{3/2}$  edges. Probably, the best exponent is at least  $8/9$ . It seems interesting to consider other graph pairs  $(\mathcal{A}, \mathcal{B})$ , i.e.

**Problem 1.** Determine the minimum size of the largest  $\mathcal{A}$ -free graphs in a  $\mathcal{B}$ -free graph with  $E$  edges.

The most is known about the case  $\mathcal{A} = \mathcal{K}_3$ . Frankl and Rödl [3] proved, e.g., that there are  $\mathcal{K}_4$ -free graphs (even with  $\Omega(n^2)$  edges) in which every  $(\frac{1}{2} + \varepsilon)E$  edges contain a triangle.

Let  $f(\mathcal{A}, n)$  be the number of  $\mathcal{A}$ -free graphs on  $n$  vertices. The latest asymptotic results on  $f(\mathcal{A}, n)$  for several  $\mathcal{A}$ 's with  $\chi(\mathcal{A}) \geq 3$  were given by Prömel and Steger [8]. Here we need a generalization.

**Problem 2.** Find bounds on  $f(\mathcal{A}, n, E)$ , the number of labeled  $\mathcal{A}$ -free graphs on  $n$  vertices with  $E$  edges.

Having a good bound on  $f(\mathcal{A}, n, E)$ , we would be able to extend our Ramsey-result for graphs other than  $\mathcal{C}_4$ . A class of graphs  $\mathfrak{B}$  is *Ramsey* if for all  $\mathcal{G} \in \mathfrak{B}$  there exists an  $\mathcal{H} \in \mathfrak{B}$  with  $\mathcal{H} \rightarrow \mathcal{G}$ . So our result is a modest first step in proving that  $\text{Forb}(\mathcal{K}_{2,3})$  the class of  $\mathcal{K}_{2,3}$ -free graphs is Ramsey. Nešetřil and Rödl [6] proved that  $\text{Forb}(\mathcal{C}_4)$  is Ramsey, and announced that  $\text{Forb}(\mathcal{C}_i)$  for  $i = 5, 6, 7$  are Ramsey-classes, too. Recently, Nešetřil informed me that their method can be extended. They can decide for each  $\mathcal{G}$  whether  $\text{Forb}(\mathcal{G})$  is Ramsey or not, and they will return to this problem in a forthcoming book. (The bipartite case can be found in [7].) However, their method, using Hales-Jewett theorem and the partite lemma, generally is unable to yield density theorems.

Another recent density result is due to Łuczak [5], who obtained by counting the  $\mathcal{C}_{2k}$ -free graphs on  $n$  vertices that there exists a graph  $\mathcal{G}$  with at most  $2^{6l}(kl)^{100k^2}$  vertices of girth  $2k$  such that any  $|E(\mathcal{G})|/l$  edges of it contains a  $\mathcal{C}_{2k}$ .

## 2. The properties of $\mathcal{G}(n, 3n^{-0.64})$

To prove the theorem stated in the abstract we use standard probabilistic methods. Consider the random graph  $\mathcal{G}(n, p)$  (with  $p = 3/n^{0.64}$ ) where the edges are chosen independently with probability  $p$ , and suppose that  $n$  is sufficiently large,  $n \geq n_0$ . With probability  $1 - o(1)$   $\mathcal{G}(n, p)$  has

$$(1 + o(1)) \binom{n}{2} p \sim \Theta(n^{1.36})$$

edges, and it contains

$$(1 + o(1)) \binom{n}{3} p^3 \sim \Theta(n^{1.08})$$

triangles and

$$(1 + o(1))10\binom{n}{5}p^6 \sim \Theta(n^{1.16})$$

copies of the complete bipartite graph  $\mathcal{K}_{2,3}$ . Delete the edges of the appearing triangles and  $\mathcal{K}_{2,3}$ 's. The obtained graph,  $\mathcal{G}^n$ , still has more than  $n^{1.36}$  edges.

The expected number of  $\mathcal{C}_4$ -free subgraphs with  $T := 2n^{4/3}(\log n)^2$  edges in  $\mathcal{G}(n, p)$  is exactly  $f(\mathcal{C}_4, n, T)p^T$ , where, as above,  $f(\mathcal{C}_4, n, T)$  denotes the number of distinct  $\mathcal{C}_4$ -free graphs over  $n$  (labeled) vertices with  $T$  edges. This expected number is  $o(1)$ , as shown by the next lemma.

**Lemma.** For all  $T \geq 2n^{4/3}(\log n)^2$

$$f(\mathcal{C}_4, n, T) < \left(4 \frac{n^3}{T^2}\right)^T. \quad (1)$$

The proof of (1) is the only nontrivial part of this note, and is postponed to the next section.

Finally, to prove  $\chi(\mathcal{G}^n) > n^{0.1}$  we use the fact that (with probability  $1 - o(1)$ ) for  $v > 20(\log n)/p$  every  $v$ -subset of the vertices of  $\mathcal{G}(n, p)$  contains at least  $\frac{1}{2}\binom{v}{2}p$  edges. (This is a consequence of the Chernoff inequality, see [1]). It follows that every subset of size  $n^{0.9}$  contains more edges in  $\mathcal{G}(n, p)$  than we have deleted, so  $\alpha(\mathcal{G}^n) < n^{0.9}$  yielding  $\chi(\mathcal{G}^n) > n^{0.1}$ .

### 3. The number of $\mathcal{C}_4$ -free graphs

Here we prove (1). We extend the ideas of Kleitman and Winston [4] who established

$$f(\mathcal{C}_4, n) < (2.15 \dots)^{n^{3/2}}.$$

Their key lemma is as follows. If  $\mathcal{G}$  is a  $\mathcal{C}_4$ -free graph on  $n-1$  vertices with minimum degree at least  $d-1$ , then there are at most

$$n \binom{n}{z} \binom{x}{d-z} \quad (2)$$

ways to extend it to a  $\mathcal{C}_4$ -free graph by adding a new vertex of degree  $d$ . In (2)  $x$  is defined as

$$x = \left\lceil n \left(1 - \frac{d^2}{n+3d}\right)^z + \frac{n}{d} \right\rceil, \quad (3)$$

and  $z$  can be any integer,  $0 \leq z \leq d$ . As a  $\mathcal{C}_4$ -free graph has at most  $\frac{1}{2}n(\sqrt{n}+1)$  edges we have  $d \leq \sqrt{n}+1$ . (In [4] there is an unimportant error. Instead of  $d^2/(n+3d)$  they simply write  $d^2/n$ .) One can build a  $\mathcal{C}_4$ -free graph on  $n$  vertices with  $T$  edges by

starting with a single point (step one) and adding new points of minimum degree  $d_i$  (in step  $i$ ,  $2 \leq i \leq n$ ). For  $d_i \leq n^{1/3} \log n$  we set  $z=0$  and get

$$n \binom{n}{d_i} < \exp(n^{1/3} (\log n)^2) \quad \text{for } n > n_0.$$

For the other terms set  $z = \lfloor n^{1/3} \rfloor$ . In (3) we get  $x = \lfloor n/d \rfloor$ , hence the last binomial coefficient in (2) is not more than  $(en/d^2)^d$ . Collecting all factors we get

$$f \leq \exp \left( n^{4/3} (\log n)^2 + \sum d_i + (\log n) \sum d_i - 2 \sum_{i=2}^n d_i \log d_i \right). \quad (4)$$

Here  $\sum_{2 \leq i \leq n} d_i = T$ , the function  $x \log x$  is convex, so we get

$$\sum d_i \log d_i \geq n(T/n) \log(T/n)$$

by Jensen's inequality. Then (4) gives the desired upper bound.  $\square$

For  $T = \lfloor 2n^{4/3} (\log n)^2 \rfloor$  (1) gives the upper bound

$$f(\mathcal{C}_4, n, T) < n^{T(1+o(1))/3}$$

while considering all  $T$ -subsets of a  $\mathcal{C}_4$ -free graph with  $\frac{1}{2}n^{3/2}(1+o(1))$  edges we get

$$f(\mathcal{C}_4, n, T) > n^{T(1-o(1))/6}.$$

Having no other reasonable example one can think that the  $1/6$  is the correct exponent.

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