

# 2-bases of quadruples

Zoltán Füredi<sup>1</sup> and Gyula O.H. Katona<sup>2</sup>

<sup>1</sup> Dept. of Mathematics, University of Illinois at Urbana-Champaign  
Urbana, IL61801, USA

<sup>1,2</sup> Rényi Institute of Mathematics of the Hungarian Academy  
Budapest, P. O. Box 127, Hungary-1364.

E-mail: z-furedi@math.uiuc.edu, furedi@renyi.hu and ohkatona@renyi.hu

## Abstract

Let  $\mathcal{B}(n, \leq 4)$  denote the subsets of  $[n] := \{1, 2, \dots, n\}$  of at most 4 elements. Suppose that  $\mathcal{F}$  is a set system with the property that every member of  $\mathcal{B}$  can be written as a union of (at most) two members of  $\mathcal{F}$ . (Then  $\mathcal{F}$  is called a 2-base of  $\mathcal{B}$ .) Here we answer a question of Erdős proving that

$$|\mathcal{F}| \geq 1 + n + \binom{n}{2} - \lfloor \frac{4}{3}n \rfloor,$$

and this bound is best possible for  $n \geq 8$ .

## 1. 2-bases

The  $n$ -element set  $\{1, 2, \dots, n\}$  is denoted by  $[n]$ . The family of all subsets of  $[n]$  is called the Boolean lattice and is denoted by  $\mathcal{B}(n)$ . Its  $k$ th level is  $\mathcal{B}(n, k) := \{B : B \subset [n] : |B| = k\}$ , and  $\mathcal{B}(n, \leq k) := \cup_{0 \leq i \leq k} \mathcal{B}(n, i)$ . The set system  $\mathcal{F}$  is called a **2-base** of  $\mathcal{A}$  if every member  $A \in \mathcal{A}$  can be obtained as a union of two members of  $\mathcal{F}$ , in other words  $A = F_1 \cup F_2$ ,  $F_1, F_2 \in \mathcal{F}$ . Note that we allow  $F_1 = F_2$  and we do not insist that the 2-base is a subset of the set system.

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The interest is in how small a base one can find. Let  $f(\mathcal{A}) := \min\{|\mathcal{F}| : \mathcal{F} \text{ is a 2-base of } \mathcal{A}\}$ . This is known exactly in very few cases, even when the set system is a natural one. For example, it is not known even for the power-set itself (the discrete cube). In 1993 Erdős [2] proposed the problem of determining  $f(\mathcal{B}(n))$  and also the problem of determining the minimum size of a 2-base of the small sets,  $f(\mathcal{B}(n, \leq k))$ . We also use  $f_k(n)$  for  $f(\mathcal{B}(n, \leq k))$ . Erdős conjectured that

$$f(\mathcal{B}(n)) = 2^{\lfloor n/2 \rfloor} + 2^{\lceil n/2 \rceil} - 1,$$

and that the extremal family consists of all subsets of  $V_1$  and  $V_2$  where  $V_1 \cup V_2 = [n]$  is a partition of  $[n]$  into two almost equal parts. A lower bound  $f(\mathcal{B}(n)) \geq (1 + o(1))2^{(n+1)/2}$  is obvious from the fact that

$$|\mathcal{A}| \leq \binom{|\mathcal{F}|}{2} + |\mathcal{F}|,$$

which holds for any 2-base  $\mathcal{F}$  of  $\mathcal{A}$ .

The aim in this paper is to answer this question for the family  $\mathcal{B}(n, \leq 4)$ . The question of the smallest base for  $\mathcal{B}(n, \leq k)$  is trivial for  $k \leq 2$ , and for  $k = 3$  it turns out to be a question about graphs whose answer follows immediately from Turán's theorem. So the case  $k = 4$  is the first non-trivial case. It boils down to an interesting question about 3-graphs (3-regular hypergraphs), and it might be somewhat surprising that it is possible to give an exact answer.

Let  $f_4(n) := 1 + n + \binom{n}{2} - h(n)$ . The main result of this paper can be summarized in the following table.

$n$	0	1	2	3	4	5	6	7	$n \geq 8$
$h(n)$	0	0	1	2	4	5	7	8	$\lfloor \frac{4}{3}n \rfloor$

**Theorem 1.1.** For  $n \geq 8$   $f_4(n) = 1 + n + \binom{n}{2} - \lfloor \frac{4}{3}n \rfloor$ .

In the following section we discuss  $f_k(n)$  in the (easy) case  $k \leq 3$ . Then give constructions for  $f_4(n)$  separating the cases  $n \leq 7$  and  $n \geq 8$  and thus providing lower bounds for  $h(n)$ . In Chapter 2 the structure of minimal bases of  $\mathcal{B}(n, \leq 4)$  is investigated, namely those with minimum deficiency with at least 2, and then (the upper bounds for) the values of  $h(n)$  in the above table is proved in Chapter 3.

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### 1.1. The case $\mathcal{B}(n, \leq 3)$

For  $k \geq 1$  every 2-base of  $\mathcal{B}(n, k)$  must contain the  $\emptyset$  and all singletons. This easily leads to

$$f_0(n) = 1, \quad f_1(n) = 1 + n, \quad f_2(n) = 1 + n.$$

Suppose that  $\mathcal{F}$  is a 2-base of  $\mathcal{B}(n, \leq k)$ ,  $1 < k \leq n$ , such that  $|\mathcal{F}| = f_k(n)$  and  $\sum_{F \in \mathcal{F}} |F|$  is minimal. Such bases are called **minimal**. Then

- (i)  $\emptyset \in \mathcal{F}$ ,  $\mathcal{B}(n, 1) \subset \mathcal{F}$ ,
- (ii) for every  $F \in \mathcal{F}$  we have  $|F| \leq k - 1$ .

Indeed, one only need to observe that in case of  $F \in \mathcal{F}$ ,  $|F| = k$ ,  $x \in F$  one can replace  $F$  by  $F' := F \setminus \{x\}$ , i.e.,  $\mathcal{F} \setminus \{F\} \cup \{F'\}$  is also a 2-base.

**Construction 1.** Consider a 2-partition  $V_1 \cup V_2$  of  $[n]$  with  $\lfloor n/2 \rfloor \leq |V_1| \leq |V_2| \leq \lceil n/2 \rceil$  and let  $\mathcal{F}$  be all the subsets of  $V_1$  and  $V_2$  of size at most 2. Every triple from  $[n]$  meets a  $V_i$  in at least 2 elements so it also contains a 2-element member of  $\mathcal{F}$ . Hence  $\mathcal{F}$  is a 2-base of  $\mathcal{B}(n, \leq 3)$ .

**Claim 1.2.**  $f_3(n) = 1 + n + \binom{\lfloor n/2 \rfloor}{2} + \binom{\lceil n/2 \rceil}{2}$ .

*Proof of Claim 1.2.* Suppose that  $\mathcal{F}$  is a minimal 2-base of  $\mathcal{B}(n, \leq 3)$  satisfying the (i) and (ii). Split it into subfamilies according to the sizes of its members,  $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1 \cup \mathcal{F}_2$  where  $\mathcal{F}_i := \mathcal{F} \cap \mathcal{B}(n, i)$ . Then  $\mathcal{F}_2$  is a graph (i.e., a 2-graph) with the property that every triple contains an edge, so its complement  $\mathcal{H}_2$  is triangle-free. ( $\mathcal{H}_2 := \mathcal{B}(n, 2) \setminus \mathcal{F}_2$ .) Then Turán's theorem [7] implies that  $|\mathcal{H}_2| \leq \lfloor n^2/4 \rfloor$ , hence

$$|\mathcal{F}| = |\mathcal{F}_0| + |\mathcal{F}_1| + |\mathcal{F}_2| \geq 1 + n + \binom{n}{2} - \lfloor \frac{n^2}{4} \rfloor. \quad \square$$

### 1.2. Constructions for $\mathcal{B}(n, \leq 4)$ if $n \leq 7$

Let  $\mathcal{F}$  be a minimal 2-base of  $\mathcal{B}(n, \leq 4)$  satisfying (i) and (ii). Let  $\mathcal{F}_i := \mathcal{F} \cap \mathcal{B}(n, i)$ , then  $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$  where  $\mathcal{F}_0 = \{\emptyset\}$ ,  $\mathcal{F}_1 = \mathcal{B}(n, 1)$ . Use the notation  $\mathcal{H}_2 := \mathcal{B}(n, 2) \setminus \mathcal{F}_2$ . Then

$$|\mathcal{F}| = 1 + n + \binom{n}{2} - |\mathcal{H}_2| + |\mathcal{F}_3| := 1 + n + \binom{n}{2} - h(n).$$

Since  $\mathcal{B}(n, \leq 2)$  is a 2-base of  $\mathcal{B}(n, \leq 4)$  we have  $h(n) \geq 0$ .

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Let us summarize the properties of  $\mathcal{F}_2 \cup \mathcal{F}_3$ .

For every triple  $T \subset [n]$  either  $T$  contains a pair from  $\mathcal{F}_2$  (1)

or  $T \in \mathcal{F}_3$  (2)

For every quadruple  $F \subset [n]$  either  $F$  contains a triple from  $\mathcal{F}_3$  (3)

or  $F$  is a union of two edges from  $\mathcal{F}_2$ . (4)

**Construction 2.** For  $n \geq 4$  let  $\mathcal{H}_2$  be a Hamilton cycle,  $|\mathcal{F}_3| = 0$ .

It is easy to show that this family  $\mathcal{F}_2$  satisfies (1) and (4) so (together with  $\mathcal{B}(n, \leq 1)$ ) it is a 2-base. This construction shows that  $h(n) \geq n$  (for  $\geq 4$ ), and one can see that this is the best possible for  $n = 4$  and  $n = 5$ .

**Claim 1.3.**  $h(0) = h(1) = 0$ ,  $h(2) = 1$ ,  $h(3) = 2$ ,  $h(4) = 4$  and  $h(5) = 5$ .

The proof of this (and the following two claims concerning  $n = 6$  and  $7$ ) is a short, finite process. For completeness we sketch them in Section 3.

**Construction 3.** For  $n = 6$  let  $\mathcal{F}_3$  be two disjoint triples  $F_1, F_2$  and let  $\mathcal{F}_2$  be the six pairs contained in either of  $F_1$  or  $F_2$ .

**Claim 1.4.**  $h(6) = 7$ .

**Construction 4.** For  $n = 7$  label the seven elements by two coordinates,  $V := \{v(1, 1), v(1, 2), v(1, 3), v(2, 1), v(2, 2), v(3, 1)\}$ . Let  $\mathcal{F}_2$  be the twelve pairs  $v(\alpha, \beta)v(\alpha', \beta')$  with  $\alpha \neq \alpha'$  and  $\beta \neq \beta'$ , and let  $\mathcal{F}_3$  be formed by the three triples having a constant coordinate i.e.,  $\{v(1, 1), v(1, 2), v(1, 3)\}$ ,  $\{v(2, 1), v(2, 2), v(2, 3)\}$  and  $\{v(1, 1), v(2, 1), v(3, 1)\}$ . (This is a truncated version of Construction 6 for  $n = 9$ .)

**Claim 1.5.**  $h(7) = 8$ .

**Construction 5.** Let  $n_1, n_2$  be nonnegative integers,  $V^1 \cup V^2$  a partition of  $[n]$  with  $|V^i| = n_i$ ,  $\mathcal{F}^i$  a minimal 2-base on  $V_i$ . Define  $\mathcal{F}$  as  $\mathcal{F}^1 \cup \mathcal{F}^2$  together with all pairs joining  $V^1$  and  $V^2$ .

It is easy to see that this construction satisfies (1)–(4), it is a 2-base. Indeed, it is sufficient to check the triples  $T$  and quadruples  $F$  meeting both  $V_1$  and  $V_2$ . Each such  $T$  contains a pair joining  $V^1$  and  $V^2$  thus  $T$  satisfies (1). If  $|F \cap V^1| = |F \cap V^2| = 2$ , then it is a union of two crossing pairs. Finally, if  $F = \{a, b, c, d\}$  and  $F \cap V^1 = \{a, b, c\}$ , then since  $\mathcal{F}^1$  is a 2-base,  $F \cap V^1$  satisfies either (1) or (2). In the first case  $F \cap V^1$  it contains a pair, say  $ab$  from  $\mathcal{F}^1$ , then  $\{a, b\} \cup \{c, d\}$  is a partition of  $F$  satisfying (4). In the second case  $F \cap V^1 \in \mathcal{F}^1$ , so  $F$  satisfies (3). We obtained:

**Claim 1.6.** For  $n_1, n_2$  nonnegative integers  $h(n_1 + n_2) \geq h(n_1) + h(n_2)$ .  $\square$

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### 1.3. Constructions for $n \geq 8$

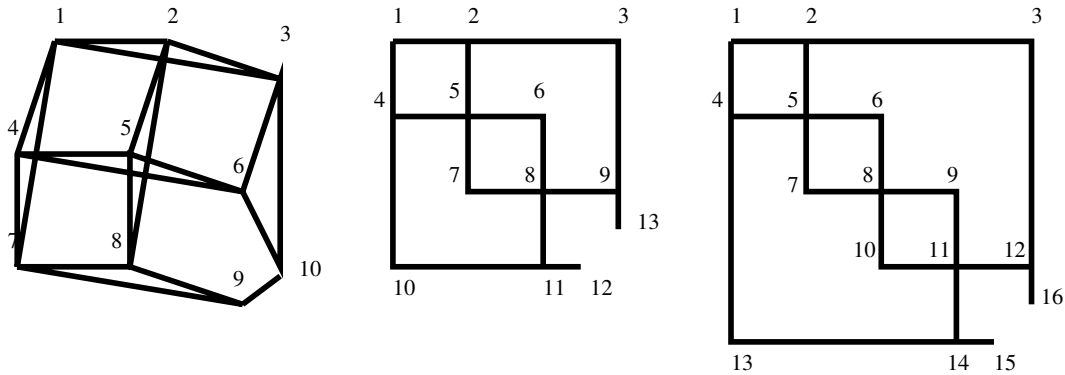
**Construction 6.** Suppose that  $\mathcal{F}_3$  is a triple system on  $[n]$  of *girth at least 4*, i.e.,  $|F' \cap F''| \leq 1$  for  $F', F'' \in \mathcal{F}_3$  and  $F_1, F_2, F_3 \in \mathcal{F}_3$  and  $F_1 \cap F_2 \neq \emptyset, F_1 \cap F_3 \neq \emptyset, F_2 \cap F_3 \neq \emptyset$  imply  $F_1 \cap F_2 \cap F_3 \neq \emptyset$ . Suppose further that every *degree* of  $\mathcal{F}_3$  is at most two, i.e., every singleton is contained in at most two triples. Define  $\mathcal{H}_2$  as the pairs covered by the members of  $\mathcal{F}_3$ .

This construction (together with  $\mathcal{B}(n, \leq 1)$ ) form a 2-base. Indeed, if a triple  $T \subset [n]$  contains no edge from  $\mathcal{F}_2$ , then it belongs to  $\mathcal{F}_3$ , so either (1) or (2) holds. Moreover, if  $F = \{a, b, c, d\} \subset [n]$  is a quadruple and contains no triple from  $\mathcal{F}_3$ , then the induced graph  $\mathcal{H}_2|_F$  contains no triangle. So  $\mathcal{F}_2|_F$  contains two disjoint edges (and thus fulfills (4)) unless  $\mathcal{H}_2|_F$  has a vertex of degree 3, say,  $ab, ac, ad \in \mathcal{H}_2$ . Since the degree of  $\mathcal{F}_3$  at the vertex  $a$  is at most two and the edges of  $\mathcal{H}_2$  are obtained from the triples of  $\mathcal{F}_3$  we get that there exists a triple  $T \in \mathcal{F}_3$  with  $a \in T \subset F$ . We obtained that Construction 6 indeed defines a 2-base.

For  $n = 3k, k \geq 3$  we obtain  $h(3k) \geq 4k$  as follows. Let  $[n] = \{a_1, a_2, \dots, a_k\} \cup \{b_1, b_2, \dots, b_k\} \cup \{c_1, c_2, \dots, c_k\}$ . Define  $\mathcal{F}_3$  as all triples of the form  $a_i b_i c_i$  and  $a_i b_{i+1} c_{i+2}$  (indices are taken modulo  $k$ ). This satisfies the constraint of Construction 6. Since  $|\mathcal{H}_2| = 3|\mathcal{F}_3|$ , we get  $h(n) \geq 2|\mathcal{F}_3| = 4k$ .

If we leave out from the above construction the 2 triples of  $\mathcal{F}_3$  and the 4 pairs of  $\mathcal{H}_2$  containing the element  $3k$  we obtain that  $h(3k - 1) \geq 4k - 2$ . Thus we already have the cases  $n = 3k$  and  $n = 3k - 1$  in the following

**Claim 1.7.**  $h(n) \geq \lfloor \frac{4}{3}n \rfloor$  for  $n \geq 8$ .



*Proof.* We only need a construction for  $n = 3k + 1, k \geq 3$  to show  $h(3k + 1) \geq 4k + 1$ . It is enough to show  $h(10) \geq 13, h(13) \geq 17$  and  $h(16) \geq 21$ , then the general case follows from  $h(9) \geq 12$  using Claim 1.6.

Define the six triples of  $\mathcal{F}_3$  as  $\{1, 2, 3\}$ ,  $\{4, 5, 6\}$ ,  $\{7, 8, 9\}$ ,  $\{1, 4, 7\}$ ,  $\{2, 5, 8\}$  and  $\{3, 6, 10\}$  and  $\mathcal{H}_2$  as the 18 pairs covered by these triples and  $\{9, 10\}$ . The graph  $\mathcal{H}_2$  has only these 6 triangles, so (1)-(2) hold, and it is not difficult to check the four-tuples, too.

The other cases are similar: for  $n = 13$  we can define  $\mathcal{F}_3 := \{1, 2, 3\}$ ,  $\{4, 5, 6\}$ ,  $\{7, 8, 9\}$ ,  $\{10, 11, 12\}$  and  $\{1, 4, 10\}$ ,  $\{2, 5, 7\}$ ,  $\{6, 8, 11\}$ ,  $\{3, 9, 13\}$  and  $\mathcal{H}_2$  consists of these triangles and the pair  $\{12, 13\}$ .

Finally, for  $n = 16$  we define  $\mathcal{F}_3$  as  $\{1, 2, 3\}$ ,  $\{4, 5, 6\}$ ,  $\{7, 8, 9\}$ ,  $\{10, 11, 12\}$ ,  $\{13, 14, 15\}$  and  $\{1, 4, 13\}$ ,  $\{2, 5, 7\}$ ,  $\{6, 8, 10\}$ ,  $\{9, 11, 14\}$ , and  $\{3, 12, 16\}$ . Again  $\mathcal{H}_2$  consists of the triangles obtained from  $\mathcal{F}_3$  and the edge  $\{15, 16\}$ .  $\square$

## 2. Bases with deficiency at least 2

The aim of this paper is to prove Theorem 1.1 so suppose that  $\mathcal{F}$  is a minimal 2-base of  $\mathcal{B}(n, \leq 4)$  and that  $\mathcal{F}_2 \cup \mathcal{F}_3$  satisfies (1)-(4).

**Lemma 2.1.** *If  $\{a, b, c\} \in \mathcal{F}_3$ , then either  $\{ab, bc, ca\} \subset \mathcal{F}_2$  or  $\{ab, bc, ca\} \subset \mathcal{H}_2$ .*

*Proof.* Suppose, on the contrary, that  $ab \in \mathcal{F}_2$ ,  $ac \notin \mathcal{F}_2$ . Replace  $abc$  by  $ac$  in  $\mathcal{F}$ . Since  $\sum_{F \in \mathcal{F}} |F|$  is minimal the family  $\mathcal{F}' := \mathcal{F} \setminus \{abc\} \cup \{ac\}$  is not a 2-base. What can go wrong? Since we added a new pair, conditions (1) and (2) still hold. The only condition we can violate is (3)-(4). We removed  $abc$ , so there exists an  $F = abcd$  not a union of two members of  $\mathcal{F}'$ . So  $abcd$  does not contain any triple from  $\mathcal{F}'$  and also  $bd$ ,  $cd \notin \mathcal{F}'$ . Consider  $bcd$ . We have  $bcd \notin \mathcal{F}$  so (1) implies that  $bc \in \mathcal{F}_2$ . Consider  $acd$ . Since  $ac, cd$ , and  $acd \notin \mathcal{F}$  again (1) implies that  $ad \in \mathcal{F}_2$ . However, then  $F = ad \cup bc$ , a contradiction.  $\square$

Use the notation  $\deg_2^-(x)$  for the degree of the vertex  $x$  in the graph  $\mathcal{H}_2$  and  $\deg_3(x)$  for the degree of  $x$  in  $\mathcal{F}_3$ . The difference  $\deg_2^-(x) - \deg_3(x)$  is called the **deficiency** of the vertex  $x \in V$ . From now on in this Section we suppose that

$$\deg_2^-(x) - \deg_3(x) \geq 2 \text{ for every } x \in [n]. \quad (5)$$

Let  $N(x)$  denote the neighborhood of  $x$  in  $\mathcal{H}_2$ ,  $N(x) := \{y : xy \in \mathcal{H}_2\}$ ,  $\deg_2^-(x) = |N(x)|$ . Let  $\mathcal{T}(x)$  denote the set of triples  $T$  from  $\mathcal{F}_3$  with  $x \in T \subset N(x) \cup \{x\}$ , and let  $t(x) := |\mathcal{T}(x)|$ . Suppose that  $D = \max_{x \in [n]} \deg_2^-(x)$ , and  $a$  has maximum degree in  $\mathcal{H}_2$ . Consider  $A = \{a\} \cup N(a)$ ,  $|A| = D + 1$ , let  $t := t(a)$ . Then (5) implies  $t, t(x) \leq D - 2$ .

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## 2.1. Eliminating the case $D \geq 5$

**Claim 2.2.** (5) implies that  $D \leq 4$ .

*Proof.* Consider the  $\binom{D}{3}$  four-tuples of  $A$  containing  $x$ , let  $\mathcal{B} := \{F : a \in F \subset A, |F| = 4\}$ . Note that none of these can satisfy (4) so each of them contains a member of  $\mathcal{F}_3$ . Classify them into two groups as follows:

$$\mathcal{B}_1 := \{abcd : b, c, d \in A \text{ and there exists a } T \in \mathcal{F}_3 \text{ with } a \in T \subset \{a, b, c, d\}\},$$

$$\mathcal{B}_2 := \{abcd : abcd \subset A, abc, abd, acd \notin \mathcal{F}_3\}.$$

Each  $F \in \mathcal{B}_2$  contains a member of  $\mathcal{F}_3|N(a)$ , hence

$$|\mathcal{B}_2| \leq |\mathcal{F}_3|N(a)|.$$

Each member of  $\mathcal{T}(a)$  is contained in  $D - 2$  four-tuples from  $\mathcal{B}_1$ , hence

$$|\mathcal{B}_1| \leq t(D - 2). \quad (6)$$

Here the sum of the left hand sides is  $\binom{D}{3}$ . The sum of right hand sides can be estimated by the degrees of  $\mathcal{F}_3$  on  $A$ . Using  $\deg_3(x) \leq D - 2$  we obtain

$$\begin{aligned} \binom{D}{3} &= |\mathcal{B}_1| + |\mathcal{B}_2| \leq t(D - 2) + |\mathcal{F}_3|N(x)| = t(D - 3) + |\mathcal{F}_3|A| \\ &\leq t(D - 3) + \frac{1}{3} \sum_{x \in A} \deg_3(x) \leq t(D - 3) + \frac{1}{3}(t + D(D - 2)). \end{aligned} \quad (7)$$

Hence

$$\frac{1}{6}D(D - 2)(D - 3) \leq t \frac{3D - 8}{3}. \quad (8)$$

Since  $t \leq D - 2$  we get  $D \leq 6$ . In case of  $t \leq D - 3$  (8) implies  $D \leq 4$ . So two cases left in the proof of the Claim, namely  $(D, t) = (6, 4)$  and  $(5, 3)$ .

In case of  $D = 6, t = 4$  the right hand side of (6) can be improved by 2, there are at least 2 coincidences when we estimated the cardinality of  $\mathcal{B}_1$ . So  $|\mathcal{B}_1| \leq 14$ , and we can decrease the right hand sides of (7) and (8) by 2, and that leads to a contradiction  $12 \leq 4 \times \frac{10}{3} - 2$ .

In case of  $D = 5, t = 3$  we use two things. The first one is implied by Lemma 2.1 and (1):

(C1) If  $abc \in \mathcal{T}(a)$  then  $bc \in \mathcal{H}_2$ ; if  $abc \notin \mathcal{T}(a)$  and  $b, c \in N(a)$  then  $bc \in \mathcal{F}_2$ .

Thus  $\mathcal{F}_2|N(a)$  has  $\binom{D}{2} - t$  edges.

(C2) If  $\deg_3(x) \geq 3$ , then  $t(x) = 3$ . Indeed, (5) implies  $\deg_2^-(x) \geq \deg_3(x) + 2 \geq 5$ ,

so equality holds. But then  $\deg_2^-(x) = 5 = D$ ,  $x$  has maximum degree,  $D$ , and then by our condition only  $t(x) = 3$  is possible.

We will show that  $(D, t) = (5, 3)$  is impossible. Suppose, on the contrary, and let  $N(a) = \{b, c, d, e, f\}$ . Consider the 3-edge graph  $G := \{xy : axy \in \mathcal{F}_3\}$ . There are 4 non-isomorphic possibilities for  $G$ .

- ( $\alpha$ )  $G$  is a triangle,  $\{bc, cd, bd\}$ ,
- ( $\beta$ )  $G$  is a path of length 3,  $\{bc, cd, de\}$ ,
- ( $\gamma$ )  $G$  is a star,  $\{bc, bd, be\}$ ,
- ( $\delta$ )  $G$  has 2 components,  $\{bc, cd, ef\}$ .

In each case we will find one or more  $x \in N(a)$  with  $t(x) = 3$ . Then the triples containing  $x$  cover no pair from  $\mathcal{F}_2$ , contradicting to condition (C1).

In case of ( $\alpha$ ) we have  $bef, cef, def \in \mathcal{F}_3$ , hence  $\deg_3(f) \geq 3$ . Then (C2) implies that  $\{b, c, d, e\} \subset N(f)$ ,  $ef \notin \mathcal{F}_2$ , contradicting to (C1). In case of ( $\beta$ ) we have  $bdf, bef, cef \in \mathcal{F}_3$ , hence  $\deg_3(f) \geq 3$ . Then  $t(f) = 3$  and  $\{b, c, d, e\} \subset N(f)$ ,  $ef \notin \mathcal{F}_2$ . In case of ( $\gamma$ ) we have  $cdf, cef, def \in \mathcal{F}_3$ , hence  $\deg_3(f) \geq 3$ . Then  $t(f) = 3$  and  $\{c, d, e\} \subset N(f)$ ,  $ef \notin \mathcal{F}_2$ . In case of ( $\delta$ ) we have  $bde, bdf \in \mathcal{F}_3$ , hence  $\deg_3(b) \geq 3$ . Then  $t(b) = 3$  and  $\{c, d, e, f\} \subset N(b)$ ,  $bf \notin \mathcal{F}_2$ . This final contradiction completes the proof of the case  $(D, t) = (5, 3)$  and Claim 2.2.  $\square$

## 2.2. The case $D \leq 4$

From now on in this Chapter we suppose that  $D \leq 4$ .

**Claim 2.3.** (5) and  $\deg_2^-(a) = 4$  imply that  $t(a) = \deg_3(a) = 2$  and the two triples containing the element  $a$  meet only in  $a$ , e.g.,  $N(a) = bcde$  and  $\mathcal{T}(a) = \{abc, ade\}$ .

*Proof.* Suppose, first, that  $t(a) = 0$ . Then all the four triples of the form  $xyz$ ,  $x, y, z \in N(a)$  belong to  $\mathcal{F}_3$ . Hence  $\deg_3(b) \geq 3$ , a contradiction. If  $t(a) = 1$ , say  $abc \in \mathcal{T}(a)$ , then  $bde, cde \in \mathcal{F}_3$  is implied by (3). Hence  $\deg_3(e) \geq 2$ , so  $\deg_2^-(e) = 4$ . Since  $be, ce, de \in \mathcal{F}_2$  we get that  $N(e) \cap \{b, c, d\} = \emptyset$ , so  $t(e) = 0$ . However, we have seen that  $\deg_2^-(e) = D = 4$  implies  $t(e) > 0$ .

So we get  $t(a) \geq 2$ , i.e.,  $t(a) = 2$ . The only case left to exclude is when the triples in  $\mathcal{T}(a)$  meet in two elements, say  $\mathcal{T}(a) = \{abc, acd\}$ . Then  $bde \in \mathcal{F}_3$ , so  $\deg_3(b) \geq 2$ . Hence we get  $\deg_2^-(b) = 4$ ,  $t(b) = 2$  and  $\{c, d, e\} \subset N(b)$ . However,  $be \in \mathcal{F}_2$  and not to  $\mathcal{H}_2$ , a contradiction.  $\square$

**Claim 2.4.** (5) and  $\deg_2^-(x) = 3$  imply that  $\deg_3(x) = 1$ .

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*Proof.* Suppose, on the contrary, that  $\deg_3(x) = 0$ . Consider  $N(x) = abc$ , we have  $ab, bc, ca \in \mathcal{F}_2$  by (1) and  $abc \in \mathcal{F}_3$  by (3). Hence  $abc \notin \mathcal{T}(a)$ , so Claim 2.3 gives that  $\deg_2^-(a) \neq 4$ . Since  $\deg_3(a) \geq 1$  we get that  $\deg_2^-(a) = 3$ . Consider  $N(a) = xyz$ . Note that  $y, z \notin \{x, a, b, c, d\}$ . Then  $xyz \in \mathcal{F}_3$  by (3). This contradicts  $\deg_3(x) = 0$ . Inequality (5) implies  $\deg_3(x) \leq 1$ , so we get it must be 1.  $\square$

**Claim 2.5.** (5) implies that  $h(\mathcal{F}) \leq \frac{4}{3}n$ .

*Proof.* For  $x \in [n]$  define  $\varphi(x) := \frac{1}{2} \deg_2^-(x) - \frac{1}{3} \deg_3(x)$ . We are going to prove that  $\varphi(x) \leq 4/3$  for every  $x$ . This implies the Claim as follows

$$h(\mathcal{F}) = |\mathcal{H}_2| - |\mathcal{F}_2| = \sum_{x \in [n]} \varphi(x) \leq \frac{4}{3}n. \quad (9)$$

Using the previous three Claims split  $[n]$  into three parts,  $[n] = P \cup Q \cup R$ , where  $P := \{x : \deg_2^-(x) = 4, \deg_3(x) = 2\}$ ,  $Q := \{x : \deg_2^-(x) = 3, \deg_3(x) = 1\}$ , and  $R := \{x : \deg_2^-(x) \leq 2\}$ .

For the vertices of  $P$  we have  $\varphi = 4/3$ , for  $Q$  we have  $\varphi = 7/6$  and for  $x \in R$  we have  $\varphi(x) \leq \frac{1}{2} \deg_2^-(x) \leq 1$ .  $\square$

Note that  $h(\mathcal{F}) = \frac{4}{3}n$  in Claim 2.5 is only possible for Construction 6, especially

$$P = [n] \text{ and } Q = R = \emptyset. \quad (10)$$

### 3. Proof of the main result

Let  $\mathcal{F}$  be a minimal 2-base for  $\mathcal{B}(n, \leq 4)$ . Then

$$\begin{aligned} 1 + n + \binom{n}{2} - h(n) &= |\mathcal{F}| = |\mathcal{F}|([n] \setminus \{x\}) + 1 + (n - 1 - \deg_2^-(x)) + \deg_3(x) \\ &\geq 1 + n + \binom{n}{2} - h(n-1) - (\deg_2^-(x) - \deg_3(x)) \end{aligned} \quad (11)$$

gives that the deficiency of every vertex is at least  $h(n) - h(n-1)$ .

**Proof of Theorem 1.1.** We use induction on  $n$  to show that  $h(n) \leq \frac{4}{3}n$ . This is certainly true for  $n \leq 2$ . Suppose that  $h(n-1) \leq \frac{4}{3}(n-1)$  and consider  $h(n)$ . If

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$h(n) \leq h(n-1) + 1$ , then we are done. If  $h(n) \geq h(n-1) + 2$ , then, as we have seen in (11) there exists a minimal 2-base  $\mathcal{F}$  on  $[n]$  with deficiency at least 2. Then Claim 2.5 gives  $h(n) = h(\mathcal{F}) \leq \frac{4}{3}n$ .  $\square$

**Proofs of Claims 1.3–1.5.** The case  $n \leq 4$  is trivial. Suppose that  $5 \leq n \leq 7$  and let  $\mathcal{F}$  be a minimal 2-base on  $n$  vertices.

The case  $n = 5$  is easy.  $h(\mathcal{F}) \geq 6$  implies  $|\mathcal{F}_2| + |\mathcal{F}_3| \leq 4$ . If  $|\mathcal{F}_2| = 4$ , then there is a unique way to satisfy (1) (namely,  $\mathcal{F}_2$  is a union of an edge and a triangle) and then (4) is violated. If  $|\mathcal{F}_2| = 3$ , then there are at least 2 triples not containing any member of  $\mathcal{F}_2$ , so (2) gives  $|\mathcal{F}_3| \geq 2$ . If  $|\mathcal{F}_2| \leq 2$ , then they satisfy (1) with at most  $3|\mathcal{F}_2|$  triples. Hence, (2) gives  $|\mathcal{F}_3| \geq 10 - 3|\mathcal{F}_2|$ . Then  $|\mathcal{F}_2| + |\mathcal{F}_3|$  exceeds 4, a final contradiction.

If the minimum deficiency of  $\mathcal{F}$  is (at most) 1 then (11) gives  $h(n) \leq h(n-1) + 1$  and we are done. From now on suppose that the deficiency of  $\mathcal{F}$  is at least 2, i.e. (5) holds.

For  $n = 6$  Claim 2.5 gives that  $h(\mathcal{F}) \leq \frac{4}{3} \times 6 = 8$ . By (10)  $h(\mathcal{F}) = 8$  is only possible, if  $P = [n]$ , i.e.,  $\mathcal{H}_2$  is a 4-regular graph, and  $\mathcal{F}_3$  consists of four triples. Then  $\mathcal{F}_2$  is a matching, say,  $\mathcal{F}_2 = \{a_1a_2, b_1b_2, c_1c_2\}$ . Then (2) implies that all the eight triples of the form  $a_ib_jc_k$  should belong to  $\mathcal{F}_3$ , a contradiction. We have obtained  $h(\mathcal{F}) = h(6) \leq 7$ .

For  $n = 7$  Theorem 1.1 implies  $h(\mathcal{F}) \leq \lfloor 7 \times \frac{4}{3} \rfloor = 9$ . We claim that  $h(7) = 8$ . Suppose, on the contrary, that  $h(\mathcal{F}) = 9$ . Consider the partition of  $[n] = P \cup Q \cup R$  defined in the proof of Claim 2.5. For  $R \neq \emptyset$  (9) gives  $|R| = 1$ ,  $|P| = 6$ ,  $Q = \emptyset$ . Then  $\mathcal{H}_2|P$  is a 4-regular graph, not joined to  $R$  so  $\deg_2^-(R) = 2$  is impossible. Finally, if  $R = \emptyset$ ,  $|Q| = 2$  and  $|P| = 5$  then we get  $|\mathcal{F}_3| = 4$ . The four members of  $\mathcal{F}_3$  can pairwise meet in at most 1 vertex (by Claims 2.3 and 2.4) and have girth 4. But such an  $\mathcal{F}_3$  does not exist on 7 vertices.

So we have obtained the exact value of  $h(n)$  for every  $n$ .  $\square$

## 4. More hypergraphs

Let  $T(n, k, r)$  denote the minimum size of a hypergraph  $\mathcal{F} \subseteq \mathcal{B}(n, r)$  such that every  $k$ -subset of  $[n]$  contains a member of  $\mathcal{F}$ . The determination of  $T(n, k, r)$  is proposed by Turán [8] who solved the case  $r = 2$  (the case of graphs, see [7]) and has a longstanding conjecture  $T(n, 4, 3) = (\frac{4}{9} + o(1))\binom{n}{3}$ . For a survey on this see Sidorenko [6].

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One can prove for every odd integer  $k$  that our  $f_k(n)$  equals to  $(1+o(1))T(n, k, (k+1)/2)$ , but the even case is more involved and apparently leads to a new Turán type problem. The authors intend to return to this topic in a future work.

## References

- [1] B. BOLLOBÁS, *Extremal Graph Theory*, Academic Press, London, (1978).
- [2] PAUL ERDŐS, personal communication.
- [3] Z. FÜREDI, Turán type problems, pp. 253–300, in “Surveys in Combinatorics, 1991”, edited by A. D. Keedwell, Cambridge University Press 1991.
- [4] KATONA, NEMETZ, SIMONOVITS A new proof of a theorem of P. Turán and some remarks on a generalization of it, (in Hungarian), *Mat. Lapok* **15** (1964), 228-238.
- [5] D. MUBAYI AND V. RÖDL, On the Turán number of triple systems, *J. Combin. Theory, Ser. B* **100** (2002), 135-152.
- [6] A. F. SIDORENKO, What we know and what we do not know about Turán numbers, *Graphs Combin.* **11** (1995), 179–199.
- [7] P. TURÁN, On an extremal problem in graph theory, (in Hungarian), *Math. Fiz. Lapok* **48** (1941), 436–452.
- [8] P. TURÁN, Research problems, *MTA Mat. Kutató Int. Közl.* **6** (1961), 417-423.

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