**Exact Solitary Water Waves with Vorticity**

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*Communicated by J. M. Ball*

**Abstract**

The solitary water wave problem is to find steady free surface waves which approach a constant level of depth in the far field. The main result is the existence of a family of exact solitary waves of small amplitude for an arbitrary vorticity. Each solution has a supercritical parameter value and decays exponentially at infinity. The proof is based on a generalized implicit function theorem of the Nash–Moser type. The first approximation to the surface profile is given by the “KdV” equation. With a supercritical value of the surface tension coefficient, a family of small amplitude solitary waves of depression with subcritical parameter values is constructed for an arbitrary vorticity.

**1. Introduction**

The solitary water wave problem is to find steady free surface waves, which approach a flat surface at infinity. Surface tension is neglected. Such waves are steady solutions to the Euler equations with gravity acted on as an external force in the two-dimensional channel, bounded from above by a free surface where the pressure is constant, and bounded from below by a rigid horizontal bottom. The main result is the existence of small amplitude solutions for an arbitrary vorticity. The method involved is a generalized implicit function theorem of Nash–Moser type.

The first existence theory for exact solitary water waves in the case of zero vorticity is the construction by Lavrentiev [18] and Ter-Krikorov [29] of infinitesimal solitary waves as limits of periodic waves as the wavelength increases to infinity. A more direct proof is due to Friedrichs and Hyers [12], which is based on a series expansion method. This result was improved by Beale [4] who employed a generalized implicit function theorem of Nash–Moser type and showed the smooth dependence of the solution on the small amplitude parameter. Beale later
in [5] presented an alternative proof replacing the abstract implicit function theorem by a straightforward iteration method; the waves admit nonzero surface tension. While these results were limited to the small amplitude waves, Amick and Toland [2, 3], Benjamin, Bona and Bose [7] established the existence of nontrivial solitary waves of large amplitude.

The existence theory for rotational solitary water waves is, on the other hand, much less complete. The Korteweg–de Vries (“KdV”) theory of rotational solitary waves on the shallow stream was studied by Benjamin [6], Freeman and Johnson [11] and others. In the case where the vorticity is constant, large amplitude waves are numerically computed by Pullin and Grimshaw [24], Teles da Silva and Peregrine [28], Vanden-Broeck [30], and Sha and Vanden-Broeck [26]. To the best of our knowledge however there has been no existence theory of exact solitary water waves with vorticity until now, even for the case of small amplitude waves. The purpose of the present work is to obtain a result analogous to [4] with vorticity, namely the existence of exact rotational solitary waves of small amplitude. The main result states that there exists a family of small amplitude solitary waves for an arbitrary distribution of vorticity. Each solution has a supercritical value of parameter and decays exponentially at infinity. An important difference from [4] lies in the lack of the velocity potential due to the nontrivial vorticity, which renders the application of a Nash–Moser implicit function theorem more complicated.

In Section 2 the vorticity-stream formulation for the exact solitary wave problem with vorticity is carefully derived as is done in [8] for periodic waves. Solutions constructed in the present work, which run on the shear flow of relative flow speed $U(y) - c$, are given by

\[
\begin{align*}
  u(x, y) - c &= U(y) - c - \epsilon (D_2 w)(\sqrt{\epsilon}x, -\psi(x, y)) + O(\epsilon^2), \\
  \eta(x) &= \epsilon w(\sqrt{\epsilon}x, 0).
\end{align*}
\]

Here $u(x, y) - c$ and $\eta(x)$ mean respectively the relative horizontal velocity and the surface elevation from the asymptotic level of $y = 0$; $w$ decays to zero exponentially as $|x| \to \infty$ and $\epsilon > 0$ is small; $\psi(x, y)$ is the (relative) stream function; $D_2$ denotes differentiation in the second argument. The critical speed of wave propagation $c_0$ corresponding to $\epsilon = 0$ is the unique solution of

\[
\int_{-d}^{0} \frac{dy}{(c_0 - U(y))^2} = \frac{1}{g},
\]

where $d$ is the depth and $g$ is the gravitational constant of acceleration. The limiting value of $w$ at the critical point $c = c_0$ (and $\epsilon = 0$) is the well known “KdV” soliton of sech²-type with vorticity (see [11]).

In Section 3 the unknown fluid region is conveniently mapped onto a fixed infinite strip in the plane by the change of variables which exchanges the roles of the stream function and the vertical coordinate function. The vertical elevation from the fluid bottom is regarded as a solution of a quasi-linear elliptic partial differential equation with a nonlinear boundary condition. It is typical to scale the horizontal variable by stretching it by $\sqrt{\epsilon}$, where $\epsilon > 0$ is the small parameter. The unknown function is scaled in accordance with it. A similar procedure was used in [4] and
also in “KdV” theory. The problem is then formulated as an operator equation of the form \( F(w, \epsilon) = 0 \) (see (3.12)). The operator however, loses ellipticity as the parameter \( \epsilon \) decreases to zero, which causes a loss of regularity at \( \epsilon = 0 \) in the scaled horizontal variable, \( q \) for example. Hence, the standard implicit function theorem does not apply.

In the beginning of Section 4 the analysis of the null space and the range at \( \epsilon = 0 \) suggests the use of the projections of \( F \) onto the domain and the target spaces. This leads to a modification \( \Phi \) of \( F \), of which the order-1 degeneracy at \( \epsilon = 0 \) is less severe (see (4.9)). The “KdV” soliton of sech\(^2\)-type, denoted by \( w_0 \), arises as a solution of \( \Phi(w, 0) = 0 \) whilst an exact solitary wave is sought to solve \( \Phi(w, \epsilon) = 0 \) for \( \epsilon > 0 \) small. The regularized operator \( \Phi \) acts on a family of Banach spaces of analytic functions in the \( q \)-variable in complex strips with exponential decay at infinity. In this setting, regularity in the \( q \)-variable is measured in terms of the width of the complex strip, which facilitates a detailed analysis of the Nash–Moser implicit function theorem.

The investigation now turns to the construction of a right inverse of the linearization \( \Phi_w(w, \epsilon) \), where \( (w, \epsilon) \) is near \( (w_0, 0) \). First at \( (w_0, 0) \), the construction is fairly straightforward once \( \Phi_w(w_0, 0) \) is explicitly split into actions on various subspaces, which is presented in Section 5. In the process it is established that \( \Phi_w(w, \epsilon) \) restricted to the projected subspaces has a bounded inverse for \( (w, \epsilon) \) close to \( (w_0, 0) \).

Next is the construction for \( \epsilon > 0 \) of a right inverse of \( \Phi_w(w, \epsilon) \), where \( (w, \epsilon) \) is near \( (w_0, 0) \). We begin Section 6 by solving the boundary value problem \( F_w(w, \epsilon) \) about \( w = 0 \) to approximate \( \Phi_w(w, \epsilon) \) in the subspaces complementing the estimates on the projected subspaces; in this part a loss of regularity occurs. Finally, the right inverse is obtained for the linearization \( \Phi_w \) about \( (w, \epsilon) \) which is arbitrary but close to \( (w_0, 0) \) and the exact solution is found by a Nash–Moser implicit function theorem.

The same arguments are applied to the solitary wave problem in the presence of surface tension, which is discussed in Section 7. For large values of the surface tension coefficient a family of small amplitude exact solitary waves is constructed with an arbitrary vorticity. Each solitary wave has a depression, has a subcritical parameter value, and approaches the asymptotic depth exponentially at infinity. After the completion of the present manuscript, the author was informed that Groves and Wahlén employed the spatial dynamics approaches to construct, with the effects of surface tension added on, generalized rotational solitary waves of small amplitude.

The existence of the left inverse, which implies the uniqueness and the smooth dependence of the solution on the parameter, is open. The existence result of large amplitude solitary waves with vorticity is a big open problem.

A more straightforward proof of the result can be made by replacing the Nash–Moser implicit function theorem by a standard iteration method, as is done in [5].

2. Formulation and the main result

The classical water wave problem concerns the motion of an incompressible inviscid fluid with a free surface acted on by gravity. We consider a two-dimensional
flow which at time $t$ is contained in a channel in the $(x, y)$-plane bounded from above by a free surface $y = \eta(t; x)$ and from below by a rigid horizontal bottom of $y = -d$, where $0 < d < \infty$. In the fluid region $\{(x, y) : -d < y < \eta(t; x)\}$, the velocity field $(u(t; x, y), v(t; x, y))$ and the pressure $P(t; x, y)$ satisfy the Euler equations

$$
\begin{align*}
    u_t + uu_x + vv_y &= -P_x, \\
v_t + uv_x + vv_y &= -P_y - g, \\
u_x + v_y &= 0.
\end{align*}
$$

(2.1)

Here $g > 0$ denotes the gravitational constant of acceleration. The flow is supposed to be rotational and characterized by the vorticity $\omega = v_x - u_y$.

The dynamic and kinematic boundary conditions hold on the free surface $y = \eta(t; x)$:

$$
P = P_0 \quad \text{and} \quad v = \eta_t + u\eta_x,
$$

(2.2)

where $P_0$ is the constant atmospheric pressure. The impermeability condition at the bottom is

$$
v = 0 \quad \text{on} \quad y = -d.
$$

(2.3)

The boundary conditions at infinity

$$
\begin{aligned}
    \eta(x) &\to 0 \quad \text{as} \quad |x| \to \infty \\
v(x, y) &\to 0 \quad \text{as} \quad |x| \to \infty \quad \text{uniformly for} \quad y
\end{aligned}
$$

(2.4)

express that in the far field the wave profile approaches a constant level of depth and the flow is almost horizontal, respectively.

A solitary wave is referred to as a solution of (2.1)–(2.4) of a single hump traveling with a constant speed $c > 0$; that is, the wave profile, the velocity and the pressure have space-time dependence $(x - ct, y)$.

We further supplement the solitary-wave problem with the symmetry conditions

$$
\eta(-x) = \eta(x), \quad u(-x, y) = u(x, y), \quad -v(-x, y) = v(x, y).
$$

(2.5)

It is traditional in the theory of steady waves to define the (relative) stream function $\psi(x, y)$ by

$$
\psi_x = -v, \quad \psi_y = u - c.
$$

(2.6)

This reduces the original formulation to a stationary elliptic boundary value problem.

The Poisson equation

$$
-\Delta \psi = v_x - u_y = \omega
$$

follows at once from (2.6). The vorticity equation, $(u - c)\omega_x + v\omega_y = 0$, indicates that $\omega$ is a function of $\psi$ at least locally away from a stagnation point, a point where $u = c$ and $v = 0$. Further assumed in this work is that $\psi_y = u - c < 0$ throughout the fluid region, namely there are no critical layers. Experimental evidence [19]
suggests that for wave patterns which are not near the spilling or breaking state, the speed of wave propagation is in general considerably larger than the horizontal velocity of any water particle. The physically motivated stipulation of no critical layers guarantees that
\[ \omega = \gamma(\psi) \]
for some function \( \gamma \) throughout the fluid; see Section 3. The vorticity function \( \gamma \) measures the strength of the vorticity.

Note that \( \psi \) is uniquely determined up to a constant. We normalize \( \psi \) so that \( \psi = 0 \) on the free surface and \( \psi = -p_0 > 0 \) on the flat bottom. The relative mass flux \( p_0 \) is defined by
\[ p_0 = \int_{-d}^{\eta(x)} \psi_y(x, y) \, dy, \quad (2.7) \]
which is independent of \( x \). This uses the kinematic boundary conditions (2.2) and (2.3).

Let the function
\[ \Gamma(p) = \int_0^p \gamma(-s) \, ds \quad (2.8) \]
have the minimum value \( \Gamma_{\text{min}} \) on \( p_0 \leq p \leq 0 \). From the equations of motion follows Bernoulli’s law, which states that the quantity
\[ E = \frac{1}{2}(\psi_x^2 + \psi_y^2) + gy + P - \Gamma(-\psi) \]
is constant throughout the fluid. The sum of the first four terms in the expression of \( E \) is the total mechanical energy of the flow; \( \frac{1}{2}(\psi_x^2 + \psi_y^2) \) is the kinetic energy, \( gy \) is the gravitational potential energy, and \( P \) is the energy of fluid pressures. In view of Bernoulli’s law, the dynamic boundary condition (2.2) takes the form of
\[ \psi_x^2(x, \eta(x)) + \psi_y^2(x, \eta(x)) + 2g\eta(x) = \lambda \]
which is independent of \( x \). Since \( \eta(x) \to 0 \) and \( \psi_x(x, \eta(x)) \to 0 \) as \( |x| \to \infty \), the Bernoulli constant \( \lambda \) is the square of the relative upstream flow speed in the far field, that is
\[ (u(x, \eta(x)) - c)^2 \to \lambda \quad \text{as} \quad |x| \to \infty. \]

In summary, there results the following formulation of the solitary water wave problem, equivalent to the original one. For a given vorticity function \( \gamma \), there exists a parameter value \( \lambda > 0 \) and functions \( \gamma(x) \) and \( \psi(x, y) \) such that
\[
\begin{align*}
-\Delta \psi &= \gamma(\psi) \quad \text{in} \quad -d < y < \eta(x), \quad (2.9a) \\
\eta(x) &= \eta(-x), \quad \psi(x, y) = \psi(-x, y), \quad (2.9b) \\
\psi &= 0 \quad \text{on} \quad y = \eta(x), \quad \psi = -p_0 \quad \text{on} \quad y = -d, \quad (2.9c) \\
\psi_x^2(x, \eta(x)) + \psi_y^2(x, \eta(x)) + 2g\eta(x) &= \lambda \quad \text{for} \quad x \in \mathbb{R}, \quad (2.9d) \\
\eta(x) &\to 0 \quad \text{as} \quad |x| \to \infty. \quad (2.9e) \\
\psi_x(x, y) &\to 0 \quad \text{as} \quad |x| \to \infty \quad \text{uniformly for} \quad y. \quad (2.9f)
\end{align*}
\]
In this setting, $\lambda$ and $d$ are considered as parameters whose values form part of the solution. Once the relative mass flux $p_0$ is given, the depth $d$ will be determined in terms of $\lambda$ through

$$d = \int_{p_0}^{0} \frac{dp}{\sqrt{\lambda + 2G(p)}};$$  \hspace{1cm} (2.10)

see the proof of Lemma 1. It is typical in the gravity water wave problem to define the Froude number $Fr = \sqrt{\lambda} / gd$.

In the statement of the result below, solutions are obtained for supercritical values $\lambda > \lambda_0$ but close to $\lambda_0$. For this end, it is convenient to write $\lambda$ in the form

$$\lambda = \lambda_0 + \epsilon b,$$  \hspace{1cm} (2.11)

where a constant $b$ will be fixed in Section 4. The critical value $\lambda_0$ is the unique solution of

$$\int_{p_0}^{0} \frac{dp}{\sqrt{\lambda_0 + 2G(p)}} = \frac{1}{g}.$$  \hspace{1cm} (2.12)

The corresponding critical speed of wave propagation $c_0$ is given by

$$\int_{-d}^{0} \frac{dy}{(c_0 - U(y))^2} = \frac{1}{g},$$

where $U(y)$ is the horizontal velocity of the underlying shear flow at infinity, that is, $u(x, y) \to U(y)$ as $|x| \to \infty$. The same critical speed of wave propagation is obtained in the framework of the “KdV” approximation (see [11] for instance), which supports the contention that the “KdV” approximation theory gives a fair representation of the propagation of solitary waves.

Our main result is the following theorem on the existence of a family of nontrivial solutions to the solitary water wave problem (2.9) parametrized by $\epsilon$ with a prescribed vorticity.

**Theorem 1.** (Main theorem) Let the vorticity function $\gamma \in C([0, |p_0|])$ and the relative mass flux $p_0 < 0$ be given. For each $\epsilon > 0$, let $\lambda$ be determined by (2.11) and (2.12). Then for $\epsilon > 0$ sufficiently small, there exists a nontrivial solution pair, $\eta(x)$ and $\psi(x, y)$, to the solitary water wave problem (2.9) which satisfies $\psi_y < 0$ throughout the fluid region. Both $\eta(x)$ and $\psi(x, y)$ are (real) analytic in the $x$-variable, and $\psi(x, y)$ is of class $C^2$.

The symmetry condition (2.9b) is included in the formulation for convenience. Recently, it is proved that [14] which provided $\gamma \in C^{1+\alpha}([0, |p_0|])$ with $\alpha \in (0, 1)$, any supercritical solitary wave $\psi(x, y)$ of class $C^{2+\alpha}$, those with $\lambda > \lambda_0$, and of elevation only, i.e., $\eta(x) > 0$ for all $x$, has a priori symmetric profile which is monotone on either side of the crest.

A more precise statement, specifying the function spaces in which the Nash–Moser implicit function theorem is applied, appears in Theorem 3.
3. Reformulation: reduction to an operator equation

We now turn to establishing the existence of solutions to (2.9). On account of
the unknown surface, we use the coordinate transformation devised by Dubreil–
Jacotin[10] to reformulate the problem as a nonlinear elliptic boundary value pro-
blem in a fixed domain. A similar procedure was used in [8].

3.1. Change of variables

We begin with the observation that \( \psi \) is constant on the free surface and on
the flat bottom and that it decreases with \( y \), that is, the \( y \)-coordinate is a single-
valued function of \( \psi \) for each fixed \( x \). This suggests the introduction of the new
independent variables

\[
\hat{q} = x \quad \text{and} \quad \hat{p} = -\psi(x, y),
\]

which map the fluid region \( \{(x, y) \in \mathbb{R}^2 : -d < y < \eta(x)\} \) to an infinite strip
\( (-\infty, \infty) \times (p_0, 0) \) in the \( (\hat{q}, \hat{p}) \)-plane, and the free surface \( \{(x, \eta(x)) : x \in \mathbb{R}\} \)
to the horizontal line \( \hat{p} = 0 \). Let

\[
\Omega = \{ (\hat{q}, \hat{p}) \in (-\infty, \infty) \times (p_0, 0) \}.
\]

The vertical elevation measured from the flat bottom \( h(\hat{q}, \hat{p}) = y + d \) replaces
the dependent variables. An explicit calculation yields

\[
h_{\hat{q}} = -\frac{\psi_x}{\psi_y}, \quad h_{\hat{p}} = -\frac{1}{\psi_y}.
\] (3.1)

The assumption \( \psi_y < 0 \) throughout the fluid region guarantees that \( \omega \) is a single-
valued function of \( \hat{p} \). Indeed,

\[
\partial_{\hat{q}} \omega = \left( \partial_x + \frac{h_{\hat{q}}}{h_{\hat{p}}} \partial_{\hat{p}} \right) \omega = \left( \partial_x - \frac{\psi_x}{\psi_y} \partial_y \right) \omega = 0
\]

by the vorticity equation \( \psi_y \omega_x - \psi_x \omega_y = 0 \). We say \( \omega = \gamma(-\hat{p}) \).

The vorticity-stream formulation (2.9) is then reformulated as an elliptic boun-
dary value problem in a fixed domain

\[
(1 + h_{\hat{q}}^2)h_{\hat{p}} + 2h_{\hat{q}}h_{\hat{p}}h_{\hat{q}} + h_{\hat{p}}^2h_{\hat{q}}^2 = -\gamma(-\hat{p})h_{\hat{p}}^3 \quad \text{in} \ \Omega, (3.2a)
\]
\[
1 + h_{\hat{q}}^2 + (2gh - 2gd - \lambda)h_{\hat{p}}^2 = 0 \quad \text{on} \ \hat{p} = 0, (3.2b)
\]
\[
h = 0 \quad \text{on} \ \hat{p} = p_0, (3.2c)
\]
\[
h_{\hat{q}} \to 0 \quad \text{as} \ |\hat{q}| \to \infty \quad \text{uniformly for} \ \hat{p}. (3.2d)
\]

The above formulation is equivalent to (2.9). The proof is almost identical to
that of [8, Lemma 2.1], and thus is omitted.

The next lemma constructs the underlying shear currents which determine the
form of solitary waves at infinity. As in Section 2, let

\[
\Gamma(\hat{p}) = \int_0^\hat{p} \gamma(-s)ds, \quad \Gamma_{\min} = \min_{\hat{p} \in [p_0, 0]} \Gamma(\hat{p}) \leq 0.
\]
Lemma 1 (Primary shear currents). For each \( \lambda > -2\Gamma_{\text{min}} \geq 0 \) the system (3.2) has a solution

\[
H(\hat{p}) = H(\hat{p}; \lambda) = \int_{p_0}^{\hat{p}} \frac{ds}{\sqrt{\lambda + 2\Gamma(s)}},
\]

which corresponds to a parallel shear flow in the horizontal direction under the flat surface \( \eta \equiv 0 \).

Proof. These solutions do not depend on \( \hat{q} \), and thus (3.2a) reduces to

\[
H'' = -\gamma (-\hat{p})(H')^3.
\]

Here and elsewhere the prime denotes differentiation with respect to \( \hat{p} \). Solutions to this ordinary differential equation are

\[
H'(\hat{p}) = (\lambda + 2\Gamma(\hat{p}))^{-1/2},
\]

which are defined for \( \lambda > -2\Gamma_{\text{min}} \). The formula (3.3) then follows from the boundary condition at the bottom \( H(p_0) = 0 \).

The nonlinear boundary condition on top

\[
1 + (2gH(0) - 2gd - \lambda)(H'(0))^2 = 0
\]

reduces to \( H(0) = d \), which determines the asymptotic depth \( d \) in terms of the parameter \( \lambda \):

\[
d = \int_{p_0}^{0} \frac{d\hat{p}}{\sqrt{\lambda + 2\Gamma(\hat{p})}}.
\]

When the flow is irrotational, that is \( \gamma \equiv 0 \), the trivial laminar flow corresponding to (3.3) is the uniform current \( u \equiv 0 \) and the flow speed is the same at every vertical level. In contrast, if the vorticity is nontrivial the horizontal velocity varies with depth

\[
u(y) - u(0) = \sqrt{\lambda + 2\Gamma(\psi(y))} - \sqrt{\lambda}.
\]

Notation 1. Let us denote

\[
a(\lambda) = a(\hat{p}; \lambda) = \sqrt{\lambda + 2\Gamma(\hat{p})}.
\]

The derivatives of \( H \) can be expressed in terms of \( a \)

\[
H'(\hat{p}) = a^{-1}(\hat{p}; \lambda), \quad H''(\hat{p}) = -\gamma (-\hat{p})a^{-3}(\hat{p}; \lambda).
\]

Note that \( a(\lambda) \) is a \( C^1 \)-function of \( \lambda \) and that

\[
\lim_{\lambda \to -2\Gamma_{\text{min}}} \int_{p_0}^{0} a^{-3}(\hat{p}; \lambda)d\hat{p} = \infty,
\]

\[
\lim_{\lambda \to \infty} \int_{p_0}^{0} a^{-3}(\hat{p}; \lambda)d\hat{p} = 0.
\]

Therefore, by continuity, there exists a \( \lambda_0 > -2\Gamma_{\text{min}} \) such that

\[
\int_{p_0}^{0} a^{-3}(\hat{p}; \lambda_0)d\hat{p} = \frac{1}{g}.
\]
The critical value $\lambda_0$ is unique. Indeed, the mapping $\lambda \mapsto \int_{p_0}^0 a^{-3}(\hat{p}; \lambda) d\hat{p}$ is strictly decreasing. In case of zero vorticity, $\lambda_0 = (g|p_0|)^{2/3}$.

In order to establish the existence of nontrivial solutions to (3.2) via the implicit function theorem, we need to formulate the problem as an operator equation between Banach spaces. However, a set of functions with the upstream condition $h(\hat{q}, 0) \to d > 0$ as $|\hat{q}| \to \infty$ does not form a linear space. For this reason, we introduce the “nontrivial perturbation” $w(\hat{q}, \hat{p})$ of the elevation function $h(\hat{q}, \hat{p})$ from that of a trivial flow $H(\hat{p})$, which is defined as

$$ h(\hat{q}, \hat{p}) = H(\hat{p}) + w(\hat{q}, \hat{p}). $$

Therefore, problem (2.9) is ultimately formulated as follows

\begin{align}
(1 + w^2) \hat{q}^2 \hat{p} \hat{p} - 2(a^{-1}(\lambda) + \hat{q}) w \hat{q} \hat{p}^2 & + (a^{-1}(\lambda) + \hat{q})^2 w \hat{q} \hat{q} - \gamma(-p)a^{-3}(\lambda)(1 + w^2) \\
& + \gamma(-p)(a^{-1}(\lambda) + \hat{q})^3 = 0 \quad \text{in} \quad p_0 < \hat{p} < 0, \\
1 + w^2 + (2gw - \lambda)(\lambda^{-1/2} + w) = 0 & \quad \text{on} \quad \hat{p} = 0,
\end{align}

with the added conditions

\begin{align}
w &= 0 \quad \text{on} \quad \hat{p} = p_0, \quad (3.6c) \\
w, \nabla w, \nabla^2 w & \to 0 \quad \text{as} \quad |\hat{q}| \to \infty \text{ uniformly for } \hat{p}. \quad (3.6d)
\end{align}

**Remark 1.** It is proved in [14] that for an arbitrary vorticity, any solitary wave of positive elevation ($w > 0$) with a supercritical value of parameter ($\lambda > \lambda_0$) is symmetric about a single crest and its profile monotonically decreases on either side of the crest. The proof combines with an asymptotic description of solutions to (3.6) at infinity the method of moving planes ([9], for instance) for a nonlinear elliptic boundary value problem in an infinite strip.

Solutions to (3.6) with $\lambda > \lambda_0$ decay at exponential rates as $|\hat{q}| \to \infty$ if they decay [14] (supercritical values of parameter are those for which zero does not lie in the continuous spectrum of the corresponding linearized problem), but the exponential rates depend explicitly on $\lambda$. In particular, the coefficient of the exponential rate tends to zero as $\lambda \to \lambda_0$ (zero is at the beginning of the continuous spectrum at $\lambda = \lambda_0$ of the corresponding linearized problem), that is to say, the decay rate is not uniform. The lack of compactness causes a severe difficulty in the existence theory “in-the-large” of solitary waves.

**Remark 2.** The global bifurcation problem which corresponds to finite-amplitude solitary waves is singular in the sense that the linearized problem of (3.6) about the trivial solution

\begin{align}
(a^3(\lambda)w) + a(\lambda)w \hat{q} \hat{q} &= 0 \quad \text{in} \quad p_0 < \hat{p} < 0, \\
gw = \lambda^{3/2}w & \quad \text{on} \quad \hat{p} = 0, \\
w &= 0 & \quad \text{on} \quad \hat{p} = p_0.
\end{align}
subject to the vanishing boundary conditions at infinity
\[ w, w_\hat{p}, w_\hat{q} \rightarrow 0 \quad \text{as} \quad |\hat{q}| \rightarrow \infty \quad \text{uniformly for} \quad \hat{p} \quad (3.8) \]

has no nontrivial solution. Amick and Toland [2] studied approximate regular problems in finite cylinders to obtain a global continuum of solutions to the *irrotational* solitary water wave problem. The method unfortunately does not generalize to the case with nontrivial vorticity, and an existence theory of finite-amplitude rotational solitary waves requires further refinements of the bifurcation theory currently available.

### 3.2. Scaling and the operator equation

For the construction of small amplitude solitary water waves, it is useful to use the scaling
\[ q = \sqrt{\epsilon} \hat{q} \quad \text{and} \quad p = \hat{p} \]
as in [4,12]. In accordance with this, the solution is required to be scaled as
\[ \lambda \mapsto \lambda_0 + \epsilon b, \quad w \mapsto \epsilon w. \quad (3.9) \]
Later in Section 4, \( b > 0 \) will be determined so that a “KdV” soliton of sech\(^2\)-type gives rise to the first approximation.

With the scalings of the independent variable and the dependent variable, (3.2) is formulated as the elliptic partial differential equation in the domain \( \Omega \)
\[
(1 + \epsilon^3 w_q^2) w_{pp} - 2\epsilon^2 (a^{-1}(\lambda) + \epsilon w_p) w_q w_{pq} + \epsilon (a^{-1}(\lambda) + \epsilon w_p)^2 w_{qq} \\
- \epsilon^2 \gamma (\gamma p) a^{-3}(\lambda) w_q^2 + \gamma (\gamma p) (3a^{-2}(\lambda) + 3\epsilon a^{-1}(\lambda) w_p + \epsilon^2 w_p^2) w_p = 0, \\
(3.10a)
\]
with the boundary condition on \( p = 0 \):
\[
\epsilon^2 w_q^2 + 2gw(\lambda^{-1/2} + \epsilon w_p)^2 - 2\lambda^{1/2} w_p - \epsilon \lambda w_p^2 = 0. \\
(3.10b)
\]
Here \( w \) is even in the \( q \)-variable and satisfies the boundary conditions
\[
w = 0 \quad \text{on} \quad p = p_0, \\
w, \nabla w \rightarrow 0 \quad \text{as} \quad |q| \rightarrow \infty \quad \text{uniformly for} \quad p. \\
(3.10c)
\]

The left side of (3.10a) forms an elliptic differential operator. This operator however becomes degenerate as \( \epsilon \) decreases to zero, and the Fréchet derivative at \( \epsilon = 0 \) is not expected to have a bounded inverse as it loses regularity in the \( q \)-variable. Thus, the standard implicit function theorem between Banach spaces does not apply.

One may perform a higher-order perturbation analysis to resolve the limiting form of \( w \) at \( \epsilon = 0 \), or equivalently to consider a regularized problem. As is done in [4], the system (3.10a)–(3.10b) is regarded as defining an operator equation
$F(w, \epsilon) = 0$ on an appropriate scale of Banach spaces. The boundary conditions (3.10c) and (3.10d) are built into the definitions of function spaces.

On account of the degeneracy as $\epsilon \to 0$, we define the regularized operator in the spaces of analytic functions in the $q$-variable in complex strips with exponential decay at infinity. The limiting value at $\epsilon = 0$ is $\lambda = \lambda_0$ and

$$w_0(q, p) = m_0(q)\phi_0(p),$$

where $\lambda_0$ is given in (2.12); $m_0(q)$ is a constant multiple of $\text{sech}^2 q$ and $\phi_0(p)$ is proportional to $\int_{p_0}^p a^{-3}(s; \lambda_0) ds$. This choice of $w_0$ and $\lambda_0$ yields the usual first approximation of the solitary wave.

A right partial inverse of the linearization of the modified operator is constructed, and a generalized implicit function theorem of Nash–Moser type applies right away to result in the existence of nontrivial solutions $w(\epsilon)$ for small $\epsilon > 0$.

We now introduce the function spaces. For $0 < \sigma \leq 1$, let $X_\sigma$ be the set of functions on $\{q \in \mathbb{C} : |\text{Im}\, q| < \sigma\}$, even and real for $q$ real, analytic on the open strip $|\text{Im}\, q| < \sigma$ and continuous on its closure $|\text{Im}\, q| \leq \sigma$, and finite in the norm

$$|w|_\sigma = \sup_{q} \exp(\alpha |\text{Re}\, q|) |w(q)|.$$

Here and elsewhere $0 < \alpha < 2$, and otherwise $\alpha$ is arbitrary but fixed. Let us denote by $X_{2, \sigma}$ the subspace of $X_\sigma$ which consists of functions $w(q)$ with its derivatives $\partial^j w$ for $0 \leq j \leq 2$ analytic on $|\text{Im}\, q| < \sigma$ and continuous on $|\text{Im}\, q| \leq \sigma$, and finite in the norm

$$|w|_{2, \sigma} = \sup_{q} \sup_{0 \leq j \leq 2} \exp(\alpha |\text{Re}\, q|) |\partial^j w(q)|.$$

For $\sigma = 0$, set $X_0 = C(\mathbb{R})$ and $X_{2, 0} = C^2(\mathbb{R})$ with the weighted norms

$$|w|_0 = \sup_{q} \exp(\alpha |q|) |w(q)|,$$

$$|w|_{2, 0} = \sup_{p} \sup_{0 \leq j \leq 2} \exp(\alpha |q|) |w(q)|.$$
Analogous definitions for \( Y_0 \) and \( Y_{2,0} \) can be made. Let \( Z_\sigma = Y_\sigma \times X_\sigma \) with the product topology. The norm of \( Z_\sigma \) is defined as

\[
\| (w_1, w_2) \|_\sigma = \| w_1 \|_\sigma + |w_2|_\sigma.
\]

The Cauchy integral formula states that \( \partial_j^j w \in X_\sigma' \) for \( w \in X_\sigma \) and \( \sigma' < \sigma \), \( j \geq 1 \) and that the estimate

\[
|\partial_j^j w|_{\sigma'} \leq \frac{C}{(\sigma - \sigma')^j} |w|_\sigma
\]

holds, where \( C > 0 \) is independent of \( \sigma \). An analogous estimate is valid for \( Y_\sigma \).

We define a nonlinear differential operator

\[
F(w, \epsilon) = (F_1(w, \epsilon), F_2(w, \epsilon)) : Y_{2,\sigma} \times [0, 1] \rightarrow Z_\sigma,
\]

where

\[
F_1(w, \epsilon) = (1 + \epsilon^3 w_{qq}^2)w_{pp} - 2\epsilon^2 (a^{-1}(\lambda) + \epsilon w_p)w_q w_{pq} + \epsilon (a^{-1}(\lambda) + \epsilon w_p)^2 w_{qq}
\]

\[
- \epsilon^2 \gamma (-p)a^{-3}(\lambda)w_{q}^2 + \gamma (-p) (3a^{-2}(\lambda) + 3\epsilon a^{-1}(\lambda)w_p + \epsilon^2 w_p^2)w_p,
\]

\[
F_2(w, \epsilon) = \epsilon^2 w_{q}^2 + 2gw(\lambda^{-1/2} + \epsilon w_p) - 2\lambda^{1/2}w_p - \epsilon \lambda w_p^2|_{p=0},
\]

and \( \lambda = \lambda_0 + b\epsilon \).

The operator form of the solitary water wave problem of small amplitude is to find a nontrivial solution to \( F(w, \epsilon) = 0 \) for \( \epsilon > 0 \) small. Evidently, \( F \in C^\infty \).

Recorded below is a generalized implicit function theorem of Nash–Moser type in the form formulated by ZEHNDER [33]. Similar statements appeared in MOSER [22] and NIRENBERG [23].

**Theorem 2** (A Nash–Moser implicit function theorem). *One-parameter families of Banach spaces \((W_\sigma, \| \cdot \|_{W_\sigma})\) and \((Z_\sigma, \| \cdot \|_{Z_\sigma})\) with \( 0 \leq \sigma \leq 1 \) are considered such that, for \( 0 \leq \sigma' \leq \sigma \leq 1 \):

\[
W_\sigma \subseteq W_{\sigma'} \text{ and } \| w \|_{W_{\sigma'}} \leq \| w \|_{W_\sigma},
\]

where \( w \in W_\sigma \). Let analogous properties hold for \( Z_\sigma \).

A continuous mapping \( \Phi : W_0 \times [0, 1] \rightarrow Z_0 \) satisfies \( \Phi(w_0, 0) = 0 \) for some \( w_0 \in W_1 \). The introduction of the open ball follows with

\[
U_{r,\sigma} = \{(w, \epsilon) \in W_\sigma \times [0, 1] : \| w - w_0 \|_{W_\sigma} < r, 0 \leq \epsilon < 1 \},
\]

where \( r > 0 \) is a constant. Suppose that for each \( 0 < \sigma \leq 1 \), the mapping \( \Phi : U_{r,\sigma} \rightarrow Z_\sigma \) is of \( C^2 \) with all derivatives bounded in \( U_{r,\sigma} \) independently of \( \sigma \). Suppose furthermore that for each \( 0 < \sigma' < \sigma \leq 1 \) and \((w, \epsilon) \in U_{r,\sigma} \) there is a linear operator \( R(w, \epsilon) : Z_\sigma \rightarrow W_{\sigma'} \) such that

\[
\Phi_w(w, \epsilon) \circ R(w, \epsilon)[z] = z \quad \text{for any } z \in Z_\sigma
\]

and that

\[
\| R(w, \epsilon)[z] \|_{W_{\sigma'}} \leq \frac{C}{(\sigma - \sigma')^\beta} \| z \|_{Z_\sigma};
\]

with positive constants \( \beta \) and \( C \) are independent of \( \sigma \).

Then for \( \epsilon > 0 \) sufficiently small, there is a solution \( (w(\epsilon), \epsilon) \in U_{r,1/2} \) to \( \Phi(w, \epsilon) = 0 \).
4. Approximate “KdV” soliton

In this section, we modify the mapping $F$ so that the “KdV” soliton will arise as the first approximation to the solitary wave. In the process, $b > 0$ is determined. This approach is reminiscent of the Lyapunov–Schmidt procedure for bifurcation.

At $\epsilon = 0$ (so that $\lambda = \lambda_0$), $F$ reduces to the linear operator

$$F(w, 0) = \left( w_{pp} + 3y(-p)a^{-2}(\lambda_0)w_p, [2g\lambda_0^{-1}w - 2\lambda_0^{1/2}w_p]_{p=0} \right) \left( a^{-3}(\lambda_0)(a^3(\lambda_0)w_p)_p, [2g\lambda_0^{-1}w - 2\lambda_0^{1/2}w_p]_{p=0} \right).$$

Accordingly, $F(w, 0) = 0$ becomes the following boundary value problem

$$\begin{cases}
(a^3(\lambda_0)w_p)_p = 0 & \text{for } p \in (p_0, 0), \\
gw = \lambda_0^{3/2}w_p & \text{on } p = 0, \\
w = 0 & \text{on } p = p_0.
\end{cases} \quad (4.1)$$

A straightforward calculation shows that

$$\phi_0(p) := g \int_{p_0}^{p} a^{-3}(s; \lambda_0)ds = g \int_{p_0}^{p} \frac{ds}{\sqrt{\lambda_0 + 2\Gamma(s)^3}} \quad (4.2)$$

solves (4.1). Note that $\phi_0(0) = 1$. The null space of $F(\cdot, 0)$ is infinite-dimensional which contains the functions of the form $m(q)\phi_0(p)$ with $m \in X_{2,\sigma}$.

More generally, let us consider the eigenvalue problem

$$\begin{cases}
-(a^3(\lambda_0)w')' = \mu a(\lambda_0)w & \text{for } p \in (p_0, 0), \\
gw(0) = \lambda_0^{3/2}w'(0), \\
w(p_0) = 0.
\end{cases} \quad (4.3)$$

The following lemma summarizes the properties of solutions to (4.3), which will be useful later in Lemma 5.

**Lemma 2.** The system (4.3) has a sequence of eigenvalues $\mu = \mu_n$ and corresponding eigenfunctions $\phi_n$, where $n \geq 0$ is an integer. The smallest eigenvalue is $\mu_0 = 0$ and a corresponding eigenfunction $\phi_0$ is given in (4.2). Moreover, $\mu_n$ increases of order $n^2$:

$$\mu_n = \left( \int_{p_0}^{p} a^{-1}(p; \lambda_0)dp \right)^{-1} \left( n + \frac{1}{2} \right)^2 \pi^2 + O(1) \quad (4.4)$$

for $n \geq 1$. The eigenfunctions $\phi_n$ are orthogonal

$$\int_{p_0}^{p} a(\lambda_0)\phi_n(p)\phi_m(p) \, dp = 0 \text{ if } n \neq m.$$ 

We choose $\phi$ so that $|\phi_n(p)|, |\phi_n'(p)|$ are bounded uniformly for $n$. 

Proof. The Rayleigh principle states that the smallest eigenvalue of (4.3) is characterized by

$$\mu_0 = \inf \{ R(v) : v \in H^1((p_0, 0)), v(p_0) = 0 \text{ and } v \not\equiv 0 \},$$

where

$$R(v) = -gv^2(0) + \int_{p_0}^0 a^3(\lambda_0)(v')^2 dp \int_{p_0}^0 a(\lambda_0)v^2 dp.$$

Our aim is to show that $\mu_0 = 0$ when $\lambda = \lambda_0$.

First, for every $v \in H^1((p_0, 0))$ with $v(p_0) = 0$ the following inequality holds

$$v^2(0) = \left( \int_{p_0}^0 v'(p) dp \right)^2 \leq \left( \int_{p_0}^0 a^{-3}(\lambda_0) dp \right) \left( \int_{p_0}^0 a^3(\lambda_0)(v')^2 dp \right) = \frac{1}{g} \int_{p_0}^0 a^3(\lambda_0)(v')^2 dp.$$

The last equality uses (2.12). This implies $R(v) \geq 0$, and subsequently $\mu_0 \geq 0$.

Next, an integration by parts shows that $R(\phi_0) = 0$. Therefore, $\mu_0 = R(\phi_0) = 0$ and $\phi_0 > 0$ is a corresponding eigenfunction.

Other assertions then follow from the standard theory of Sturm–Liouville problems. The asymptotic behavior (7.5) is a well-known property of the eigenvalues of Sturm–Liouville problems (see [13, Theorem 8.3.4], for instance).

Since $F(\cdot, 0)$ is linear, the null space of $F(\cdot, 0) : Y_{2, \sigma} \to Z_{\sigma}$ consists of all functions of the form $w(q, p) = m(q)\phi_0(p)$, where $m \in X_{2, \sigma}$ is arbitrary. It is natural to introduce a projection $\Xi$ of $Y_{2, \sigma}$ onto $\text{ker } F(\cdot, 0)$ by

$$(\Xi w)(q, p) = w(q, 0)\phi_0(p);$$

thus $Y_{2, \sigma}$ can be considered as a direct sum $\Xi Y_{2, \sigma} \oplus (I - \Xi)Y_{2, \sigma}$.

The modification of the mapping $F$ takes account of a condition on the range as $\epsilon \to 0$. Let $(f_1, f_2) \in \text{range } F(\cdot, 0)$, that is,

$$f_1 = a^{-3}(\lambda_0)(a^3(\lambda_0)v_p)_p,$$

$$f_2 = 2g\lambda_0^{-1}v - 2\lambda_0^{1/2}v_p|_{p=0}$$

for some $v \in Y_{2, \sigma}$. An integration by parts shows that

$$\int_{p_0}^0 a^3(\lambda_0) f_1(\cdot, p)\phi_0(p) dp = \int_{p_0}^0 (a^3(\lambda_0)v_p)_p \phi_0(p) dp$$

$$= \int_{p_0}^0 (a^3(\lambda_0)\phi_0')' v dp + \left[ a^3(\lambda_0)(v_p\phi_0 - v\phi_0') \right]_{p=0}$$

$$= -\frac{\lambda_0}{2} f_2.$$
This defines a projection $\Pi$ on $Z_{\sigma}$ by

$$\Pi(f_1, f_2) = \left(0, f_2 + \frac{2}{\lambda_0} \int_{p_0}^{0} a^3(\lambda_0) f_1(\cdot, p) \phi_0(p) dp \right). \quad (4.8)$$

This permits the decomposition $Z_{\sigma} = \Pi Z_{\sigma} \oplus (I - \Pi) Z_{\sigma}$. It is immediate to see that $\Pi F(w, 0) = 0$ for any $w \in Y_{2,\sigma}$, that is, $Z_{\sigma}$ is a projection onto a subspace transversal to the range of $F(\cdot, 0)$.

We define a modified operator of $F$ by

$$\Phi(w, \epsilon) = \begin{cases} \epsilon^{-1} \Pi F(w, \epsilon) + (I - \Pi) F(w, \epsilon) & \text{for } \epsilon > 0, \\ \Pi F_\epsilon(w, 0) + (I - \Pi) F(w, 0) & \text{for } \epsilon = 0. \end{cases} \quad (4.9)$$

Here $F_\epsilon(w, 0)$ denotes the Frechet derivative of $F$ in its second argument at $\epsilon = 0$. Note that for $\epsilon > 0$ the zeros of $\Phi$ coincide with those of $F$.

Since $\Pi F(w, 0) = 0$, the modified operator $\Phi(w, \epsilon)$ is further written as

$$\Phi(w, \epsilon) = \Pi F^{(1)}(w, \epsilon) + (I - \Pi) F(w, \epsilon) \quad \text{for } \epsilon \geq 0, \quad (4.10)$$

where $F^{(1)}(w, \epsilon) = \epsilon^{-1} (F(w, \epsilon) - F(w, 0))$. An explicit calculation yields

$$F^{(1)}(w, \epsilon) = (F^{(1)}_1(w, \epsilon), F^{(1)}_2(w, \epsilon)),$$

where

$$F^{(1)}_1(w, \epsilon) = \epsilon^2 w_q^2 w_{pp} - 2\epsilon (a^{-1}(\lambda) + \epsilon w_p) w_q w_{pq} + (a^{-1}(\lambda) + \epsilon w_p)^2 w_{qq} + \epsilon \gamma(-p) a^{-3}(\lambda) w_q^2 + 3\gamma(-p) \epsilon^{-1}(a^{-2}(\lambda) - a^{-2}(\lambda_0)) w_p + 3\gamma(-p) a^{-1}(\lambda) w_p^2 + \epsilon \gamma(-p) w_p^3,$$

$$F^{(1)}_2(w, \epsilon) = \left[ \epsilon w_q^2 + 2gw w_p (2\lambda^{-1/2} + \epsilon w_p) - \lambda w_p + 2\epsilon \gamma(-1) - \lambda_0^{-1}) w - 2\epsilon (\lambda^{1/2} - \lambda_0^{1/2}) w \right]_{p=0}.$$

For $(w, \epsilon)$ in a bounded subset of $Y_{2,\sigma} \times [0, 1]$, the operator $F^{(1)}(w, \epsilon)$ and its Frechet derivative $F^{(1)}_w(w, \epsilon)$ as operators from $Y_{2,\sigma}$ to $Z_{\sigma}$ are uniformly bounded with bounds independently of $\epsilon$. Indeed, $\lambda - \lambda_0 = O(\epsilon)$. For any $0 \leq \sigma \leq 1$ the operator $\Phi(w, \epsilon) : Y_{2,\sigma} \to Z_{\sigma}$ is well defined and of $C^\infty$.

Our task now is to identify the first approximation to the solitary wave by seeking a root of $\Phi(w, \epsilon)$ at $\epsilon = 0$. In view of (4.10), it is equivalent to solve

$$\Pi F^{(1)}(w, 0) = 0 \quad \text{and} \quad (I - \Pi) F(w, 0) = 0.$$

The latter equation suggests the selection of $w$ in the null space of $F(\cdot, 0)$, that is, $w(q, p) = m(q) \phi_0(p)$ for some $m \in X_{2,\sigma}$. An explicit calculation yields

$$F^{(1)}_1(m(q) \phi_0(p), 0) = a^{-2}(\lambda_0) \phi_0 m - 3b \gamma(-p) a^{-4}(\lambda_0) \phi_0^2 m + 3\gamma(-p) a^{-1}(\lambda_0) (\phi_0')^2 m^2,$$

$$F^{(1)}_2(m(q) \phi_0(p), 0) = -3gb \lambda_0^{-2} m + 3g^2 \lambda_0^{-2} m^2 \big|_{p=0}.$$
This uses $\lambda = \lambda_0 + \epsilon b$. Here and in the sequel a dot above a variable denotes differentiation with respect to $q$. Therefore $\Pi F^{(1)}(m(q)\phi_0(p), 0) = 0$ reduces to the ordinary differential equation on the real line

$$\ddot{m} - \frac{3}{2} b \beta m + \frac{3}{2} g \beta m^2 = 0$$

with evenness and the decay condition at infinity. Here

$$\beta = \frac{g^2}{2} \left| \int_{p_0}^{0} a^{-5}(p; \lambda_0) dp \right| > 0.$$  

The decay condition at infinity requires $b > 0$. One may express the solution $m_0(q)$ to (4.11) of the form $A \text{sech}^2(\alpha q)$, where $A$ and $\alpha$ are constants. The constant $\alpha$ may then be normalized to 1 provided with the choice $b = \frac{8}{3 \beta}$. A simple substitution of this expression in (4.11) yields that

$$m_0(q) = \frac{4}{g \beta} \text{sech}^2 q.$$  

This is the unique decaying solution satisfying the evenness. Note that the function $m_0$ is analytic in the $q$-variable in the complex strip $|\text{Im} q| < \pi / 2$ and decays exponentially at infinity: $m_0 \in O(\exp(-2|q|))$ as $|q| \to \infty$.

The next lemma summarizes these results.

**Lemma 3.** The function

$$w_0(q, p) = m_0(q)\phi_0(p) = \frac{4}{g \beta} \text{sech}^2 q \cdot \phi_0(p)$$

satisfies $\Phi(w_0, 0) = 0$ and $m_0 \in X_{2, \sigma}$ for $0 < \sigma \leq 1$.

For the remainder of this work, Theorem 2 applies to $\Phi(w, \epsilon) : Y_{2, \sigma} \times [0, 1] \to Z_\sigma$ and a nontrivial solution to $\Phi(w, \epsilon) = 0$ is found for $(w, \epsilon)$ near $(w_0, 0)$. To do so, a partial inverse of the Fréchet derivative $\Phi_w(w, \epsilon) : Y_{2, \sigma} \to Z_\sigma$ is constructed for $(w, \epsilon)$ near $(w_0, 0)$.

In general, for a given vorticity function $\gamma$ one can numerically calculate the critical value $\lambda_0$. In case of the uniform vorticity, the calculations are considerably simpler and the dispersion relation is explicit, as we now show.

**Example 1.** If the flow is irrotational, $\gamma \equiv 0$, the trivial solution (3.3) for the parameter value $\lambda$ reduces to the uniform horizontal stream with the relative flow speed $\sqrt{\lambda}$. The eigenfunction of (4.3) corresponding to $\mu = 0$ (the smallest eigenvalue) is

$$\phi_0(p) = g \lambda_0^{-3/2} (p - p_0)$$

with the dispersion relation

$$\lambda_0 = gd.$$  

In other words, the critical value of the Froude number $Fr = \sqrt{\lambda_0 / gd}$ is 1, which is consistent with the results in [5, 12]. McLeod [20] proved that solitary water waves of elevation occur for the supercritical speed of $Fr > 1$.  

Example 2. In case of constant vorticity $\gamma \equiv \gamma_0$, where $\gamma_0$ is a constant, the primary current for the parameter value $\lambda$ is the shear flow whose relative flow speed is

$$a(p; \lambda) = H'(p)^{-1} = (\lambda + 2\gamma_0 p)^{1/2}.$$  

A straightforward calculation shows that the eigenfunction of (4.3) corresponding to $\mu = 0$ is

$$\phi_0(p) = \frac{g}{\gamma_0} \left((\lambda_0 + 2\gamma_0 p_0)^{-1/2} - (\lambda_0 + 2\gamma_0 p)^{-1/2}\right).$$  

By the normalization condition $\phi_0(0) = 1$ follows

$$1 = \frac{g}{\gamma_0} \left((\lambda_0 + 2\gamma_0 p_0)^{-1/2} - \lambda_0^{-1/2}\right), \quad (4.12)$$  

and thus (2.10) reduces to

$$d = \frac{1}{\gamma_0} \left(\sqrt{\lambda_0} - \sqrt{\lambda_0 + 2\gamma_0 p_0}\right). \quad (4.13)$$  

Solving (4.12) and (4.13) for $\sqrt{\lambda_0}$ gives the dispersion relation

$$\sqrt{\lambda_0} = \frac{d\gamma_0 + \sqrt{(d\gamma_0)^2 + 4dg}}{2}.$$  

Accordingly, the critical value of the Froude number is given by

$$Fr^2 = 1 + \frac{1}{2} \left(\frac{d\gamma_0^2}{g} + \frac{\gamma_0 g}{g} \sqrt{(d\gamma_0)^2 + 4dg}\right).$$  

This recovers the previous example of the zero-vorticity case if $\gamma_0$ is taken to be zero.

5. Solution to linear approximation

The purpose of this section is to find a right inverse for the linearization $\Phi_w$ of the modified mapping $\Phi$ at $\epsilon = 0$ about the “KdV” soliton $w_0$.

With the projections $\Xi$ and $\Pi$ defined in the previous section, $\Phi_w(w, \epsilon) : Y_{2,\sigma} \rightarrow Z_\sigma$ may be decomposed as a $2 \times 2$ matrix of linear operators

$$\Phi_w(w, \epsilon) = \left( \begin{array}{cc} \Pi \Phi_w(w, \epsilon) \Xi & \Pi \Phi_w(w, \epsilon)(I - \Xi) \\ (I - \Pi) \Phi_w(w, \epsilon) \Xi & (I - \Pi) \Phi_w(w, \epsilon)(I - \Xi) \end{array} \right). \quad (5.1)$$  

For arbitrary $(w, \epsilon) \in Y_{2,\sigma} \times [0, 1]$, let $A(w, \epsilon)$ be the restriction to $\Xi Y_{2,\sigma}$ of $\Pi \Phi_w(w, \epsilon) : Y_{2,\sigma} \rightarrow \Pi Z_\sigma$, the upper left corner of the matrix (5.1). Our first task is to show that $A(w_0, 0)$ is invertible.

Lemma 4. The operator $A(w_0, 0) : \Xi Y_{2,\sigma} \rightarrow \Pi Z_\sigma$ has a bounded inverse.
Proof. The proof is similar to that of [4, Lemma 1]. A straightforward calculation shows that for \( v \in Y_{2,\sigma} \)

\[
F_w^{(1)}(w_0, 0)[v] = \left( a^{-2}(\lambda_0) v_{qq} - 3b\gamma(-p) a^{-4}(\lambda_0) v_p + 6\gamma(-p) a^{-1}(\lambda_0) (\phi_0')^2 w_0 p v_p, \right.
\]

\[
\left. \left[ 4g \lambda_0^{-1/2} w_0 p v + 4g \lambda_0^{-1/2} w_0 v_p - 2\lambda_0 w_0 v_p - 2b g \lambda_0^{-2} v - b \lambda_0^{-1/2} v_p \right]_{p=0} \right),
\]

and thus for \( v(q, p) = m(q) \phi_0(p) \in \Pi Z_{\sigma} \)

\[
A(w_0, 0)[m(q) \phi_0(p)] = \Pi F_w^{(1)}(w_0, 0)[m(q) \phi_0(p)] = \left( 0, \tilde{m} - \frac{3}{2} b \beta m + 3 g \beta m_0 m \right).
\]

It is natural to identify \( \Sigma Y_{2,\sigma} \) with \( X_{2,\sigma} \) and \( \Pi Z_{\sigma} \) with \( X_\sigma \). Our goal is to show that for a given \( f \in X_\sigma \) there is a unique solution \( m \in X_{2,\sigma} \) to

\[
\tilde{m} - \frac{3}{2} b \beta m + 3 g \beta m_0 m = f. \tag{5.2}
\]

The unique solvability of (5.2) uses the Green’s function. It is immediately seen that \( G_1(q) = \tilde{m}_0(q) \) is an odd solution of the homogeneous equation, (5.2) with \( f = 0 \), since \( m_0(q) \) solves (4.11). Let us denote by \( G_2(q) \) an even solution of the homogeneous equation. Note that \( G_1 \) and \( G_2 \) are analytic on the complex strip \(|Im q| < \pi/2 \) and satisfy

\[
|G_1(q)| \leq C \exp(-2|Re q|) \quad \text{and} \quad |G_2(q)| \leq C \exp(2|Re q|) \tag{5.3}
\]

for some \( C > 0 \). Since the only solutions to the homogeneous equation decaying at infinity are odd, the unique even solution to (5.2) decaying at infinity is given by

\[
m(q) = G_2(q) \int_q^\infty G_1(\tilde{q}) f(\tilde{q}) d\tilde{q} + G_1(q) \int_0^q G_2(\tilde{q}) f(\tilde{q}) d\tilde{q}. \tag{5.4}
\]

Next we examine analyticity. It is convenient to write (5.4) for \( q > 0 \) as

\[
\exp(\alpha|Re q|) m(q) = \int_0^\infty K(q, \tilde{q}) \exp(\alpha|Re \tilde{q}|) f(\tilde{q}) d\tilde{q},
\]

where

\[
K(q, \tilde{q}) = \begin{cases} 
\exp(\alpha|Re q| - \alpha|Re \tilde{q}|) G_1(\tilde{q}) G_2(q) & \text{if } q \leq \tilde{q} \\
\exp(\alpha|Re q| - \alpha|Re \tilde{q}|) G_1(q) G_2(\tilde{q}) & \text{if } q \geq \tilde{q}.
\end{cases}
\]

From the asymptotic behavior (5.3) it is evident that

\[
\int_0^\infty \partial_q^j K(q, \tilde{q}) d\tilde{q} \leq \tilde{C} \quad \text{for all } q \text{ and } j \leq 2.
\]
As a consequence, the estimate

$$|m|_\sigma \leq C|f|_\sigma$$

holds, where $C > 0$ is independent of $\sigma$. Differentiating (5.4) together with the above observations on $G_1$ and $G_2$ then yields that $m \in X_{2,\sigma}$ and that the estimate

$$|m|_{\sigma,2} \leq C|f|_\sigma$$  (5.5)

holds, where $C > 0$ is independent of $\sigma$. This completes the proof. □

By a perturbation argument we deduce the following result, which will be useful later in Section 6. Note that $\Phi$ and $\Phi_w$ are of $C^\infty$.

**Corollary 1.** For $r_1, \epsilon_1 > 0$ sufficiently small, $A(w, \epsilon) : \mathcal{E}Y_{2,\sigma} \to \Pi Z_\sigma$ has a bounded inverse which is independent of $\sigma$, provided that $\|w - w_0\|_\sigma < r_1$ and $0 \leq \epsilon < \epsilon_1$.

Returning to the case of $\epsilon = 0$, we need to carry out the analogous analysis for other components of (5.1) to complete the construction of the right inverse of $\Phi_w(w_0, 0)$. Since $(I - \Pi)F(\cdot, 0)\mathcal{E} \equiv 0$ it follows that $(I - \Pi)\Phi_w(w_0, 0)\mathcal{E} = 0$, that is, the lower left corner of (5.1) at $(w_0, 0)$ is zero.

It remains to verify that $(I - \Pi)\Phi(w_0, 0)(I - \mathcal{E})$, the lower right corner of (5.1) at $(w_0, 0)$, is invertible. Since $(I - \Pi)F(\cdot, 0)$ is linear, this reduces to finding for a given $f_1 \in Y_\sigma$ a solution $v$ to the Dirichlet problem

$$\begin{cases}
a^{-3}(\lambda_0)(a^3(\lambda_0)v_p)_p = f_1 & \text{in } \Omega, \\
v(q, 0) = 0 = v(q, p_0). & \end{cases}$$  (5.6)

We remark that $f_2$ is determined by the equation $\Pi(f_1, f_2) = 0$ under the added condition $(f_1, f_2) \in (I - \Pi)Z_\sigma$.

It is standard to show that there exists a unique solution $v$ to (5.6) with $\partial^k_p v \in Y_\sigma$ for $k \leq 2$. The differential operator in (5.6) is however degenerate as an elliptic operator in both $q$ and $p$ variables, and $\partial^k_q v \in Y_{2,\sigma}$ is not known. We then employ the Cauchy integral formula (3.11) to assert that $v \in Y_{2,\sigma'}$ for $\sigma' < \sigma$. To interpret, a loss of regularity in the $q$-variable occurs. Note that any solution $v$ to (5.6) belongs to the subspace $(I - \mathcal{E})Y_{2,\sigma'}$ since $v(q, 0) = 0$. Accordingly, we define a bounded operator

$$E_0 : (I - \Pi)Z_\sigma \to (I - \mathcal{E})Y_{2,\sigma'}$$

by mapping $(f_1, f_2) \in (I - \Pi)Z_\sigma$ to a solution $v \in (I - \mathcal{E})Y_{2,\sigma'}$ of (5.6), where $0 < \sigma' < \sigma \leq 1$. It is straightforward to see that $E_0$ is a right inverse of $(I - \Pi)\Phi_w(w_0, 0)(I - \mathcal{E})$ in the sense described in Theorem 2.

To summarize, $\Phi_w(w_0, 0)$ has a right inverse with decrease in $\sigma$. 

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**Exact Solitary Water Waves with Vorticity**

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By a perturbation argument we deduce the following result, which will be useful later in Section 6. Note that $\Phi$ and $\Phi_w$ are of $C^\infty$.

**Corollary 1.** For $r_1, \epsilon_1 > 0$ sufficiently small, $A(w, \epsilon) : \mathcal{E}Y_{2,\sigma} \to \Pi Z_\sigma$ has a bounded inverse which is independent of $\sigma$, provided that $\|w - w_0\|_\sigma < r_1$ and $0 \leq \epsilon < \epsilon_1$.
6. Proof of Theorem 1

This section is devoted to the proof of the main theorem, Theorem 1. First, Theorem 2 applies to \( \Phi(w, \epsilon) : Y_{2,\sigma} \times [0, 1] \to Z_{\sigma} \) and a nontrivial solution to \( \Phi(w, \epsilon) = 0 \), which is also a solution to \( F(w, \epsilon) = 0 \), is found for \( \epsilon > 0 \) small. The transformations in Section 2 and Section 3 back to the physical variables then ensure the existence of an exact solution to the solitary wave problem (2.9).

The basic existence theorem is the following.

**Theorem 3.** For \( \epsilon \geq 0 \) sufficiently small, \( \Phi(w, \epsilon) = 0 \) has a family of nontrivial solutions \( w(\epsilon) \in Y_{2,1/2} \) and \( \lambda = \lambda_0 + \epsilon b \) such that \( w(0) = w_0; b = 8(3 \beta)^{-1} > 0 \) is given in the discussion of Lemma 3.

The idea of the proof is to construct a right inverse of \( \Phi_w(w, \epsilon) \) for \( (w, \epsilon) \) close to \( (w_0, 0) \). We differentiate (4.9) to obtain

\[
\Phi_w(w, \epsilon) = \Pi \Phi_w(w, \epsilon) + (I - \Pi)\Phi_w(w, \epsilon)
\]

for \( \epsilon > 0 \). Corollary 1 constructs a right inverse of \( \Pi \Phi_w(w, \epsilon) \). Our task is to find an inverse for \( F_w(w, \epsilon) \) with inhomogeneous terms in the subspace \( (I - \Pi)Z_\sigma \), which complements Corollary 1.

A straightforward calculation yields that for \( (w, \epsilon) \in Y_{2,\sigma} \times [0, 1] \)

\[
F_w(w, \epsilon) = (F_{1w}(w, \epsilon), F_{2w}(w, \epsilon)) : Y_{2,\sigma} \to Z_\sigma,
\]

where for \( v \in Y_{2,\sigma} \)

\[
F_{1w}(w, \epsilon)[v] = (1 + \epsilon^3 w_0^2) v_{pp} - 2\epsilon^2 (a^{-1}(\lambda) + \epsilon w_p) w_q v_{pq} + \epsilon (a^{-1}(\lambda) + \epsilon w_p)^2 v_{qq}
\]

\[
+ 2\epsilon^2 v_q \left[ \epsilon w_q v_{pp} - (a^{-1}(\lambda) + \epsilon w_p) w_{pq} - \gamma(-p) a^{-3}(\lambda) w_q \right]
\]

\[
+ v_p \left[ - 2\epsilon^3 w_q w_{pq} + 2\epsilon^2 (a^{-1}(\lambda) + \epsilon w_p) w_{qq}
\right.
\]

\[
+ 3\gamma(-p) (a^{-2}(\lambda) + 2\epsilon a^{-1}(\lambda) w_p + \epsilon^2 w_p^2) \right],
\]

\[
F_{2w}(w, \epsilon)[v] = 2g(\lambda^{-1/2} + \epsilon w_p)^2 v + 2\epsilon^2 w q v_q
\]

\[
+ v_p \left[ 4\epsilon g w (\lambda^{-1/2} + \epsilon w_p)^2 - 2\lambda^{1/2} - 2\epsilon \lambda w_p \right] \bigg|_{p=0}.
\]

Let us define approximate operators \( L(\epsilon) \) and \( B(\epsilon) \) to \( F_w(w, \epsilon) \) by

\[
L(\epsilon)[v] = (v_{pp} + 3\gamma(-p) a^{-2}(\lambda_0) v_p + \epsilon a^{-2}(\lambda_0) v_{qq} \bigg|_{p=0}) \bigg|_{\lambda_0}
\]

\[
= (a^{-2}(\lambda_0)(a^2(\lambda_0) v_p) p + \epsilon a^{-2}(\lambda_0) v_{qq} \bigg|_{p=0}) ,
\]

\[
(6.2)
\]
and

\[
B(w, \epsilon)[v] = (v_{pp} + 3\gamma(-p)a^{-2}(\lambda_0)v_p + \epsilon a^{-2}(\lambda_0)v_{qq} + e^2 w_q v_{pp} - 2e^2(a^{-1}(\lambda) + \epsilon w_p)w_q v_{pq} + e(\epsilon(\lambda) - a^{-2}(\lambda_0))v_{qq} + e^2(2a^{-1}(\lambda) + \epsilon w_p)w_p v_{qq} + 2e^2 v_q (\epsilon w_q w_{pp} - (a^{-1} + \epsilon w_p)w_q w_{pq} - \gamma(-p)a^{-3}(\lambda)w_q) + e^2 v_p (-2e^2 w_q w_{pq} + 2(a^{-1} + \epsilon w_p)w_q w_{qq} + w_p^2), [2g\lambda_0^{-1}v - 2\lambda_0^{1/2}v_p + 2e^2 g w_p^2 v + 2e^2 w_q v_q + 4e^2 g(2\lambda_0^{-1/2}w_p + w_p^2)w_{pp}]_{\epsilon = 0}).
\]

(6.3)

Note that \( L(\epsilon) \) is the linearization of \( F(w, \epsilon) \) about the trivial flow \( w = 0 \) and \( \lambda = \lambda_0 \) and that \( B(w, \epsilon) = L(\epsilon) + O(\epsilon^2) \). Note that

\[
F_w(w, \epsilon)[v] = B(w, \epsilon)[v] + (2\epsilon a^{-1}(\lambda)w_p v_p + 3\gamma(-p)(a^{-2}(\lambda) - a^{-2}(\lambda_0))v_p) + [2g(\lambda_0^{-1} - \lambda_0^{-1})v + 4eg\lambda_0^{-1/2}w_p v] + 4eg\lambda_0^{-1}w_p v_p - 2\epsilon\lambda w_p v_p - 2(\lambda_0^{1/2} - \lambda_0^{1/2})v_p\]_{\epsilon = 0}.
\]

The next lemma constructs an inverse of \( L(\epsilon) \) on the subspace \((I - \Pi)Z_\sigma\).

**Lemma 5.** For any \( 0 < \epsilon \leq 1 \) and \( 0 < \sigma' < \sigma \leq 1 \), the linear operator \( L(\epsilon) \) has a bounded inverse \( G(\epsilon) : (I - \Pi)Z_\sigma \rightarrow Y_\sigma \cap Y_{2,\sigma'} \), i.e., \( G(\epsilon)(f_1, f_2) = v \) solves \( L(\epsilon)[v] = (f_1, f_2) \). Moreover, the estimates

\[
||v||_\sigma \leq C ||(f_1, f_2)||_\sigma
\]

(6.4)

\[
||v_p||_\sigma \leq C ||(f_1, f_2)||_\sigma
\]

(6.5)

and

\[
||v||_{2,\sigma'} \leq \frac{C}{(\sigma - \sigma')}^2 ||(f_1, f_2)||_\sigma
\]

(6.6)

hold, where \( C > 0 \) is independent of \( \epsilon \) as well as \( \sigma \).

**Proof.** Our goal is to find for \((f_1, f_2) \in (I - \Pi)Z_\sigma\) a solution \( v \in Y_\sigma \cap Y_{2,\sigma'} \) to the boundary value problem

\[
a^{-3}(\lambda_0)(a^3(\lambda_0)v_p)_p + \epsilon a^{-2}(\lambda_0)v_{qq} = f_1 \quad \text{in} \quad p_0 < p < 0,
\]

\[
2g\lambda_0^{-1}v - 2\lambda_0^{1/2}v_p = f_2 \quad \text{on} \quad p = 0,
\]

\[
v = 0 \quad \text{on} \quad p = p_0.
\]

(6.7)

Observe that this problem becomes degenerate as \( \epsilon \rightarrow 0 \), which will cause the loss of regularity in the estimate (6.6).

Our first task is to show that with \( q \) real there exists a unique solution to (6.7) in a weak sense. Let \( H \) be the usual Sobolev space \( H^1(\Omega) \) of functions \( \phi(q, p) \in \)
$L^2(\Omega)$ with its first derivatives in $L^2(\Omega)$ and $\phi(q, p_0) = 0$. Recall that $\Omega = (-\infty, \infty) \times (p_0, 0)$ is a domain in the $(q, p)$-plane with $q$ real. Given $(f_1, f_2) \in L^2(\Omega) \times L^2(\mathbb{R})$ a function $v \in H$ is said to be a weak solution to (6.7) if

$$
\iint_{\Omega} (a^3(\lambda_0)v_p\phi_p + \epsilon a(\lambda_0)v_q\phi_q) \, dq dp - g \int_{p=0}^{\infty} v \phi \, dq = - \iint_{\Omega} a^3(\lambda_0)f_1\phi \, dq dp - \frac{\lambda_0}{2} \int_{p=0}^{\infty} f_2\phi \, dq
$$

for all $\phi \in H$. A $C^2$-weak solution is called a classical solution.

It is easy to see that the zero function is the only solution to the homogeneous equation, (6.8) with $(f_1, f_2) = (0, 0)$. Indeed, from the variational consideration of $\lambda_0$ in Lemma 2 follows the inequality

$$
g \int_{p=0}^{\infty} v^2 \, dq \leq \iint_{\Omega} a^3(\lambda_0)v_p^2 \, dq dp
$$

for any $v \in H$.

Now let $H_0$ be the subspace of $H$ of functions $\phi(q, p)$ which are orthogonal to $\phi_0$ in the sense that

$$
\int_{p_0}^{\infty} a(\lambda_0)\phi(q,p)\phi_0(p) \, dp = 0.
$$

Here $\phi_0$ is defined in (4.2) and is an eigenfunction of (4.3) corresponding to the eigenvalue $\mu = 0$.

Note that the left side of (6.8) defines a positive definite inner product of $H_0$, which is equivalent to the $H$-inner product. Given $(f_1, f_2)$ therefore there exists $v \in H_0$ which satisfies (6.8) for any $\phi \in H_0$. Such a $v$ is a weak solution to (6.7) provided that (6.8) holds for $\phi$ of the form $m(q)\phi_0(p)$ since any smooth function in $H$ can be written as a sum of a function in $H_0$ and a function of the form $m(q)\phi_0(p)$. It is straightforward to see that (6.8) holds for such $\phi$ if $\Pi(f_1, f_2) = 0$. Therefore, there exists a unique weak solution $v \in H_0$ to (6.7).

Our next task is to extend this solution to the complex strip $|Im\, q| < \sigma$. Let us recall the Sturm–Liouville problem (4.3):

\[
\begin{align*}
-(a^3(\lambda_0)w')' &= \mu a(\lambda_0)w \quad \text{for } p \in (p_0, 0), \\
gw(0) &= \lambda_0^{3/2}w'(0), \\
w(p_0) &= 0.
\end{align*}
\]

The results of Lemma 2 show that there is a sequence of eigenvalues $\mu_n$ for $n \geq 0$ with $\mu_0 = 0$ and that $\mu_n$ increases of order $n^2$:

$$
\mu_n = \left( \int_{p_0}^{\infty} a^{-1}(p; \lambda_0) \, dp \right)^{-2} \left( n + \frac{1}{2} \right)^2 \pi^2 + O(1)
$$
for $n \geq 1$, and that the corresponding eigenfunctions $\phi_n$ are orthogonal in the sense that

$$\int_{p_0}^{p} a(\lambda_0) \phi_n \phi_m dp = 0 \quad \text{if } n \neq m.$$ 

Appropriate choices of $\phi_n(0)$ ensure that $|\phi_n(p)| \leq M/n^2$, $|\phi'_n(p)| \leq M/n$, where $M > 0$ is independent of $n$.

To make use of the eigenfunctions, let

$$v_n(q) = \int_{p_0}^{p} a(\lambda_0) v(q, p) \phi_n(p) \, dp$$

and

$$f_{1n}(q) = \int_{p_0}^{p} a^3(\lambda_0) f_1(q, p) \phi_n(p) \, dp,$$

where $q$ takes values in the complex strip $|\text{Im} \, q| < \sigma$. Substituting $\phi$ by $m(q) \phi_n(p)$ in (6.8) leads to the equation

$$\epsilon \ddot{v}_n - \mu_n v_n = f_{1n} - \frac{\lambda_0}{2} \phi_n(0) f_2 \equiv g_n.$$ (6.9)

It follows by construction that $v_0 = 0$. The solution $v_n$ to (6.9) for each $n \geq 1$ can be written as a convolution (using the Fourier transform, for instance)

$$v_n = (\pi / \sqrt{\epsilon \mu_n}) \exp(-2\pi \sqrt{\mu_n / \epsilon \cdot | |}) \ast g_n.$$ 

We remark that for $g_n \in X_\sigma$ its Fourier transform $\hat{g}_n(\xi)$ is analytic in the strip $|\text{Im} \, \xi| < \sigma$. It is straightforward to show that $v_n \in X_\sigma$ and

$$|v_n|_\sigma \leq \tilde{C} n^{-2} |g_n|_\sigma \leq C n^{-2} \|(f_1, f_2)\|_\sigma,$$

where $C > 0$ is independent of $\epsilon$ and the index $n$. Since $|\phi_n(p)| \leq M/n^2$, therefore, $\sum_{n=1}^{\infty} v_n(q) \phi_n(p)$ converges to a function $v \in Y_\sigma$ and

$$\|v\|_\sigma \leq C \|(f_1, f_2)\|_\sigma$$

for some constant $C > 0$. That is, the operator norm of the mapping of $(f_1, f_2) \in Z_\sigma$ to the solution $v \in Y_\sigma$ of (6.7) is bounded independently of $\epsilon$ as well as $\sigma$.

From $v_p(q, p) = \sum_{n=1}^{\infty} v_n(q) \phi'_n(p)$ and $|\phi'_n(p)| < M/n$ it follows that

$$\|v_p\|_\sigma \leq C \|(f_1, f_2)\|_\sigma.$$ 

It remains to obtain the estimate (6.6). By virtue of the Cauchy integral formula (3.11) it follows that $\partial_q^j v \in Y_{\sigma''}$ for $\sigma'' < \sigma$ and $j \leq 2$ and that

$$\|\partial_q^j v\|_{\sigma''} \leq \frac{C}{(\sigma - \sigma'')^2} \|v\|_\sigma,$$
where $C > 0$ is independent of $\sigma$ as well as $\sigma''$. The boundary value problem (6.7) may be written as

$$(a^3(\lambda_0)v_p)_p = f_1 - \epsilon a(\lambda_0)v_{qq}$$

and thus $v_{pp}$ exists in the sense analogous to (6.8) and is in $Y_{\sigma''}$. This observation together with the boundary conditions

$$v_p(\cdot, 0) = g_{\lambda_0}^{-3/2}v(\cdot, 0) - f_2 \quad \text{on} \quad p = 0$$

and $v = 0$ on $p = p_0$ yields that $v_{pp} \in Y_{\sigma''}$ in the classical sense. Again from the Cauchy integral formula it follows that $v_{pq} \in Y_{\sigma'}$ for $\sigma' < \sigma''$ and that the estimate

$$\|v_{pq}\|_{\sigma'} \leq \frac{C}{(\sigma'' - \sigma')}\|v_{pp}\|_{\sigma''} \leq \frac{C}{(\sigma - \sigma')^3}\|v\|_{\sigma}$$

holds, where $C > 0$ is independent of $\sigma$ as well as $\sigma'$. Therefore, $v \in Y_{2,\sigma'}$ and the estimate (6.6) holds.

Finally, the proof is complete if we define $G(\epsilon) : (I - \Pi)Z_\sigma \rightarrow Y_\sigma \cap Y_{2,\sigma'}$ by mapping $(f_1, f_2)$ to the solution $v \in Y_\sigma \cap Y_{2,\sigma'}$ of (6.7). \(\square\)

From a perturbation argument we infer that $B(w, \epsilon)$ is invertible on the subspace $(I - \Pi)Z_\sigma$. Recall that $B(w, \epsilon) = L(\epsilon) + O(\epsilon^2)$.

**Corollary 2.** For $\epsilon_2 > 0$ sufficiently small and for any $0 < \sigma' < \sigma \leq 1$, the linear operator $B(w, \epsilon)$ has a bounded inverse $G(w, \epsilon) : (I - \Pi)Z_\sigma \rightarrow Y_\sigma \cap Y_{2,\sigma'}$ provided that $\|w - w_0\|_\sigma < \epsilon_2$ and $0 < \epsilon < \epsilon_2$. That is, $G(w, \epsilon)[z] = v$ satisfies $B(w, \epsilon)[v] = z$. The operator norm of $G(w, \epsilon) : (I - \Pi)Z_\sigma \rightarrow Y_\sigma$ is bounded independently of $(w, \epsilon)$ as well as $\sigma$, that is,

$$\|v\|_{\sigma} \leq C\|z\|_{\sigma}, \quad (6.10)$$

while the estimates

$$\|v_p\|_{\sigma} \leq C\|z\|_{\sigma}, \quad (6.11)$$

$$\|G(w, \epsilon)[z]\|_{2,\sigma'} \leq \frac{C}{(\sigma - \sigma')^3}\|z\|_{\sigma} \quad (6.12)$$

hold, where $C > 0$ is independent of $(w, \epsilon)$ as well as $\sigma$.

We are now in a position to construct the right inverse for the Fréchet derivative $\Phi_w(w, \epsilon) : Y_{2,\sigma} \rightarrow Z_\sigma$ and obtain the estimates.

**Proof.** (Proof of Theorem 3) The proof uses the generalized implicit function theorem, Theorem 2 for the setting $\Phi(w, \epsilon) : Y_{2,\sigma} \times [0, 1] \rightarrow Z_\sigma$. That is, $W_\sigma$ in Theorem 2 is taken to be $Y_{2,\sigma} \times [0, 1]$. Our goal is to construct a right inverse of the Fréchet derivative $\Phi_w(w, \epsilon) : Y_{2,\sigma} \rightarrow Z_\sigma$, where $(w, \epsilon)$ is close to $(w_0, 0)$. Specifically, let $\|w - w_0\| < \min(r_1, r_2)$ and $\epsilon < \min(\epsilon_1, \epsilon_2)$.

Suppose that

$$z = z^{(1)} + z^{(2)} \in Z_\sigma$$
An explicit calculation yields
\[ v^{(1)} = A^{-1}(w, \epsilon)[z^{(1)}] \quad \text{and} \quad v^{(2)} = G(w, \epsilon)[z^{(2)}]. \]

The approximate inverse \( S(w, \epsilon) : Z_\sigma \to Y_{2,\sigma} \) of \( \Phi_w(w, \epsilon) \) is then defined by
\[ S(w, \epsilon)[z] = v^{(1)} + v^{(2)} = v. \quad (6.13) \]

The idea of the proof is to estimate the errors in \( z \) from \( \Phi_w(w, \epsilon)[v] \) and obtain an exact inverse.

Our first task is to estimate the deviation of \( z^{(1)} \) from \( \Phi_w(w, \epsilon)[v^{(1)}] \). Let us write
\[ \Phi_w(w, \epsilon)[v^{(1)}] = z^{(1)} + r^{(1)}, \quad (6.14) \]
where the principal part is \( z^{(1)} = A(w, \epsilon)[v^{(1)}] = \Pi \Phi_w(w, \epsilon)[v^{(1)}] \) and the remainder is \( r^{(1)} = (r_1^{(1)}, r_2^{(1)}) \in (I - \Pi)Z_\sigma \). With the projections \( \Sigma \) and \( \Pi \), this can be written as
\[ \Phi_w(w, \epsilon)[v^{(1)}] = \begin{pmatrix} z^{(1)} \\ r^{(1)} \end{pmatrix}. \]

An explicit calculation yields
\[
\begin{align*}
   r_1^{(1)} &= \varepsilon^3 w^2_q v_{pp}^{(1)} - 2\varepsilon^2 (a^{-1}(\lambda) + \epsilon w_p) w_q v_{pq}^{(1)} + \varepsilon (a^{-1}(\lambda) + \epsilon w_p)^2 v_{qq}^{(1)} \\
   &\quad + 2\varepsilon^2 v_q^{(1)} [\epsilon w_q w_{pp} - (a^{-1}(\lambda) + w_p) w_{pq} - \gamma(-p)a^{-3}(\lambda) w_q] \\
   &\quad + \varepsilon v_p^{(1)} [-2\varepsilon^2 w_q w_{pq} + 2\varepsilon (a^{-1}(\lambda) + w_p) w_{qq} + 3\varepsilon^2 \gamma(-p) w_p^2] \\
   &\quad + 3\varepsilon v_p^{(1)} \gamma(-p)(a^{-2}(\lambda) - a^{-2}(\lambda_0)).
\end{align*}
\]

Then,
\[ r_2^{(1)} = -\frac{2}{\lambda_0} \int_{p_0}^0 a^3(\lambda_0) r_1^{(1)}(\cdot, p) \phi_0(p) \, dp \]
so that \( \Pi(r_1^{(1)}, r_2^{(1)}) = 0 \); see the definition (4.8). For any \( \varepsilon > 0 \) in a bounded neighborhood of 0, we have
\[ |a^{-2}(\lambda) - a^{-2}(\lambda_0)| \leq b\varepsilon \max_{[p_0, 0]} |a^{-4}(\lambda_0)| \quad (6.15) \]

since \( \lambda - \lambda_0 = O(\varepsilon) \). Hence the estimate
\[ \|r^{(1)}\|_\sigma \leq C\varepsilon \|v^{(1)}\|_{2,\sigma} \leq C_1 \varepsilon \|z^{(1)}\|_\sigma \quad (6.16) \]
holds, where \( C_1 > 0 \) is independent of \( \varepsilon \). The last inequality uses the boundedness of \( A(w, \epsilon) \).
Next is the comparison of $F_w(w, \epsilon)$ with $B(w, \epsilon)$. By the definition (6.3) it follows that

$$
(F_w(w, \epsilon) - B(w, \epsilon))[v^{(2)}] = (2\epsilon a^{-1}(\lambda)w_p v^{(2)}_p + 3\gamma(-p)(a^{-2}(\lambda) - a^{-2}(\lambda_0))v^{(2)}_p,
\]
$$

$$
[2g(\lambda^{-1} - \lambda_0^{-1})v^{(2)} + 4\epsilon g \lambda^{-1/2}w_p v^{(2)} + 4\epsilon g \lambda^{-1}w v^{(2)} - 2\epsilon \lambda w_p v^{(2)} - 2(\lambda^{1/2} - \lambda_0^{1/2})v^{(2)}_p]_{p=0}.
$$

Hence the estimate

$$
\| (F_w(w, \epsilon) - B(w, \epsilon))[v^{(2)}]\|_\sigma \leq \epsilon \tilde{C} \|v_p\|_\sigma \leq \epsilon C \|z^{(2)}\|_\sigma \quad (6.17)
$$

holds, where $C > 0$ is independent of $\epsilon$. As in (6.15), the last inequality uses the estimate (6.10) and that $\lambda - \lambda_0 = O(\epsilon)$ for $\epsilon$ in any bounded neighborhood of 0.

We now use

$$
\Phi_w(w, \epsilon) = \epsilon^{-1}(I - \Pi)F_w(w, \epsilon)
$$

to write

$$
\Phi_w(w, \epsilon)[v^{(2)}] = r^{(2)} + z^{(2)} + r^{(3)}, \quad (6.18)
$$

where the remainders are

$$
\begin{align*}
r^{(2)} &= \epsilon^{-1}(I - \Pi)F_w(w, \epsilon)[v^{(2)}] = \Pi F_w^{(1)}(w, \epsilon)[v^{(2)}], \\
r^{(3)} &= (I - \Pi)(F_w(w, \epsilon) - B(w, \epsilon))[v^{(2)}].
\end{align*}
$$

As is done above for $z^{(1)}$ and $r^{(1)}$, this can be written as

$$
\Phi_w(w, \epsilon)[v^{(2)}] = \begin{pmatrix} 0 & r^{(2)} \\ r^{(3)} & z^{(2)} \end{pmatrix}.
$$

From the boundedness of $F_w^{(1)}(w, \epsilon)$ and the boundedness of $G(w, \epsilon)$ (estimate (6.11)) it follows that

$$
\| r^{(2)} \|_\sigma \leq C_2 \| z^{(2)} \|_\sigma, \quad (6.19)
$$

and from the estimate (6.17) it follows

$$
\| r^{(3)} \|_\sigma \leq \epsilon C_3 \| z^{(2)} \|_\sigma, \quad (6.20)
$$

where $C_2, C_3 > 0$ are independent of $\epsilon$.

The result of (6.14) and (6.18) is that

$$
\Phi_w(w, \epsilon)[v] = (z^{(1)} + r^{(2)}) + (r^{(1)} + z^{(2)} + r^{(3)}),
$$

where $z^{(1)} + r^{(2)} \in \Pi Z_\sigma$ and $r^{(1)} + z^{(2)} + r^{(3)} \in (I - \Pi)Z_\sigma$; in matrix form it takes

$$
\Phi_w(w, \epsilon)[v] = \begin{pmatrix} z^{(1)} \\ r^{(1)} + r^{(3)} \end{pmatrix}.
$$

It is easy to see that

$$
\| r^{(1)} + r^{(3)} \|_\sigma \leq \epsilon(C_1 \| z^{(1)} \|_\sigma + C_3 \| z^{(2)} \|_\sigma) \leq \epsilon(C_1 + C_3) \| z \|_\sigma \quad (6.21)
$$
and
\[ \|r^{(2)}\|_\sigma \leq C_2 \|z^{(2)}\|_\sigma \leq C_2 \|z\|_\sigma. \] (6.22)

A linear operator \( M : Z_\sigma \to Z_\sigma \) is defined by
\[ M(w, \epsilon)[z] = \Phi_w(w, \epsilon)S(w, \epsilon)[z] = \Phi_w(w, \epsilon)[v]. \] (6.23)

As a \( 2 \times 2 \) matrix of operators on \( \Pi Z_\sigma \oplus (I - \Pi)Z_\sigma \), it takes the form
\[ M = \begin{pmatrix} I & M_2 \\ \epsilon(M_1 + M_3) & I \end{pmatrix}, \]
where each \( M_i \) (\( i = 1, 2, 3 \)) is a bounded operator with \( \|M_i\| \leq C_i \) uniformly in the bounded open ball of \( \|w - w_0\| < \min(r_1, r_2) \) and \( 0 < \epsilon < \min(\epsilon_1, \epsilon_2) \). It is clear that \( M \) is invertible on \( Z_\sigma \) for \( \epsilon < ((C_1 + C_3)C_2)^{-1} \).

Finally, \( R(w, \epsilon) : Z_\sigma \to Y_{2,\sigma'} \) is defined by
\[ R(w, \epsilon) = S(w, \epsilon) \circ M^{-1}(w, \epsilon), \] (6.24)
which is a right inverse of \( \Phi_w(w, \epsilon) \) in the following sense
\[ \Phi_w(w, \epsilon)R(w, \epsilon)[z] = z \quad \text{for all } z \in Z_\sigma. \]

From Corollary 1 and Corollary 2, the estimates (6.21) and (6.22) follow the estimate
\[ \|R(w, \epsilon)[z]\|_{2,\sigma'} \leq \frac{C}{(\sigma - \sigma')^3} \|z\|_\sigma, \]
where \( C > 0 \) is independent of \( \epsilon \).

Therefore, we deduce from Theorem 2 that for \( \epsilon \) sufficiently small there exists \( w(\epsilon) \in Y_{1/2,2} \) near \( w_0 \) such that \( \Phi(w(\epsilon), \epsilon) = 0 \).

Subsequently, there exists a one-parameter family of solutions \( w(\epsilon) \) for \( \epsilon > 0 \) of \( F(w(\epsilon), \epsilon) = 0 \). We perform the transformations back to the physical variables to obtain solutions to the original formulation (2.9). \( \square \)

**Proof.** (Proof of Theorem 1) The solution \( w(\epsilon) \) to \( F(w, \epsilon) = 0 \) from Theorem 3 restricted to \( q \) real, solves (3.10) for a parameter \( \epsilon \). Clearly, \( w(\epsilon) \) is of class \( C^2 \) and decays exponentially as \( |q| \to \infty \):
\[ \exp(\alpha|q|)|w(q, p)| \leq C, \]
where \( \alpha \) is the same as in the definition of \( Y_{2,\sigma} \).

The results presented in Section 3 allow us to associate the solution \( w(\epsilon) \) to a solution of the solitary wave problem (2.9). In terms of the unscaled variables \( \hat{q} \) and \( \hat{p} \), the elevation function is written as
\[ h(\hat{q}, \hat{p}) = H(\hat{p}) + \epsilon w(\sqrt{\epsilon} \hat{q}, \hat{p}). \]
The stream function $\psi(x, y)$ can be recovered using (3.1). It is easy to see that $\psi(x, y)$ is of class $C^2$. Subsequently, the velocity field is given by

$$u(x, y) - c = \frac{\epsilon D_1 w(\sqrt{\epsilon}x, -\psi(x, y))}{DH(-\psi(x, y)) + \epsilon D_2 w(\sqrt{\epsilon}x, -\psi(x, y))},$$

$$v(x, y) = \frac{1}{DH(-\psi(x, y)) + \epsilon D_2 w(\sqrt{\epsilon}x, -\psi(x, y))},$$

where $D$ is differentiation, and $D_1$ and $D_2$ denote respectively differentiation in the first argument and in the second argument. The equation of the wave profile is given by

$$\eta(x) = \epsilon w(\sqrt{\epsilon}x, 0).$$

For $\epsilon > 0$ small, a Taylor expansion of the above expression gives an order-$\epsilon$ approximation of the solution from the trivial current $U(y) - c$:

$$u(x, y) - c = U(y) - c - \epsilon (D_2 w)(\sqrt{\epsilon}x, -\psi(x, y)) + O(\epsilon^2).$$

The limiting value of $w$ at $\epsilon = 0$ is the “KdV” soliton of sech$^2$-type.

7. Remark on the effect of surface tension

Adding the effects of surface tension in the solitary wave problem for (2.1)–(2.4) introduces higher-order derivatives into the boundary condition. The dynamic boundary condition states that the pressure is constantly atmospheric above the fluid and the jump in pressure across the free surface is proportional to the curvature [21,32]:

$$P = P_0 - \tau \frac{\eta_{xx}}{(1 + \eta_y^2(x))^{3/2}} \text{ on } y = \eta(t; x),$$

where $\tau > 0$ is the coefficient of surface tension.

Summarized in this section is a straightforward existence theory for small amplitude solitary waves when the coefficient of surface tension is large. Since the arguments are nearly the same as those in the absence of surface tension (Section 2 through Section 6) we shall only indicate how to modify the analysis as adding the higher-order derivative due to surface tension.

Bernoulli’s law [21] states that

$$\psi_x^2(x, \eta(x)) + \psi_y^2(x, \eta(x)) + 2g\eta(x) - 2\tau \frac{\eta_{xx}(x)}{(1 + \eta_y^2(x))^{3/2}} = \lambda$$

is independent of $x$. The Bernoulli constant $\lambda$ is the square of the relative upstream flow speed in the far field, that is,

$$(u(x, \eta(x)) - c)^2 \to \lambda \quad \text{as } |x| \to \infty,$$
provided that $\eta_{xx}(x) \to 0$ as $|x| \to \infty$. Tacitly assumed is that $\eta$ is sufficiently smooth; more specifically, $\eta \in C^2(\mathbb{R})$. Indeed, small amplitude solutions constructed via an analytic ($C^\omega$) Nash–Moser implicit function theorem will be analytic in the $x$-variable and will decay exponentially at infinity

$$|\eta(x)|, |\psi_x(x, y)| \leq \text{const.} \exp(-\alpha|x|),$$

where the constant $\alpha > 0$ depends on $\psi(x, y)$.

The solitary wave problem with surface tension is (2.9) with (2.9d) replaced by (7.2). In this setting, $\tau > 0$ is held to be fixed while $\lambda > 0$ is the (unknown) parameter. The critical value $\lambda_0$ of $\lambda$ is the unique solution of

$$\int_0^p \frac{dp}{\sqrt{\lambda_0 + 2\Gamma(p)}} = \frac{1}{g},$$

which is the same as (2.12) in the zero surface tension case. The critical value $\tau_0$ of $\tau$ is defined as

$$\tau_0 = g^2 \int_0^{p_0} \frac{ds}{\sqrt{\lambda_0 + 2\Gamma(s)}} \int_{p_0}^{p} \frac{dp}{\sqrt{\lambda_0 + 2\Gamma(s)}}\,. \tag{7.3}$$

In case of zero vorticity, $\tau_0 = (gp_0)^2/2\lambda_0$.

The eigenvalue problem corresponding to (4.3) takes the form

$$\begin{cases}
-(a^3(\lambda_0)w')' = \mu a(\lambda_0)w & \text{for } p \in (p_0, 0), \\
gw(0) - \lambda_0^{3/2}w'(0) = \mu(\lambda_0)\tau w(0), \\
w(p_0) = 0.
\end{cases} \tag{7.4}$$

If $\tau > \tau_0$ then (7.4) has a sequence of discrete eigenvalues $\mu = \mu_n$ and corresponding eigenfunctions $\phi_n$, where $n \geq 0$ is an integer [31]. The smallest eigenvalue is $\mu_0 = 0$ and a corresponding eigenfunction is given by

$$\phi_0(p) = g \int_{p_0}^{p} \frac{ds}{\sqrt{\lambda + 2\Gamma(s)}},$$

same as which is the in the absence of surface tension, (4.2). Moreover, $\mu_n$ increases of order $n^2$:

$$\mu_n = \left( \int_{p_0}^{0} a^{-1}(p; \lambda_0)dp \right)^{-2} \left( n + \frac{1}{2} \right)^2 \pi^2 + O(1) \tag{7.5}$$

for $n \geq 1$. The eigenfunctions $\phi_n$, after normalization, satisfy

$$\int_{p_0}^{0} a(\lambda_0)\phi_n(p)\phi_m(p) \ dp - \tau \phi_n(0)\phi_m(0) = 0 \quad \text{if } n \neq m.$$

We choose $|\phi_n(p)|, |\phi'_n(p)|$ are bounded uniformly for $n$. If $\tau < \tau_0$ then the eigenvalue problem (7.4) has one negative eigenvalue.
The “KdV” equation corresponding to (4.11) is

\[- (\tau - \tau_0)\ddot{m} - \frac{3}{2} b \beta m + \frac{3}{2} g \beta m^2 = 0\]  

(7.6)

with evenness and the decay condition at infinity. Here

\[\beta = \frac{g^2}{2} \int_{p_0}^{0} a^{-5}(p; \lambda_0) dp > 0.\]

For \(\tau > \tau_0\) the coefficient \(b\) must be negative, that is \(\lambda < \lambda_0\), and the effect of the solution \(m_0\) to (7.6) is to depress the free surface from the constant level. For \(\tau < \tau_0\), the constant \(b\) must be positive as in the case of zero surface tension, \(\tau = 0\). If \(\tau > \tau_0\), the coefficient \(b\) may be made to simplify the form of the solution \(m_0\). A straightforward calculation yields that \(b = -\frac{8}{3\beta}\) and that

\[m_0(q) = -\frac{4}{g\beta} \text{sech}^2 q\]

is the unique decaying solution to (7.6) satisfying the evenness.

We extend the analysis in the previous sections to obtain the following theorem of the existence on small amplitude solitary waves for strong surface tension.

**Theorem 4.** Let the vorticity function \(\gamma \in C([0, |p_0|])\) be given and the relative mass flux \(p_0 < 0\). Let a supercritical value \(\tau > \tau_0\) of the coefficient of surface tension be held fixed. For \(\epsilon > 0\) sufficiently small, there is a one-parameter family of solution pair, \(\eta(x)\) and \(\psi(x, y)\) to the solitary wave problem with surface tension, (2.9) with (2.9d) replaced by (7.2).

For each \(\epsilon > 0\), the parameter is set as \(\lambda = \lambda_0 + \epsilon b\), where \(b = -\frac{8}{3\beta}\), and each solution has a subcritical value of parameter \(\lambda < \lambda_0\). The first approximation of the wave profile is the “KdV” soliton of depression.

In the irrotational setting, \(\omega \equiv 0\) and \(\gamma \equiv 0\), the critical value of the surface tension coefficient \(\tau_0\) corresponds to the dimensionless Bond number \(1/3\). For Bond number bigger than \(1/3\), AMICK and KIRCHGÄSSNER [1] proved the existence of small amplitude solitary waves for \(b < 0\) and non-existence for \(b > 0\). SACHS [25] employed a Nash–Moser implicit function theorem as is explored in [4] to prove the existence of irrotational solitary waves of depression \((b < 0)\), which our proof is related to.

For \(\tau < \tau_0\), the method breaks down as the justification of the Lyapunov–Schmidt reduction which leads to the “KdV” approximation is no longer valid. An analysis by separation of variables shows the existence of a component with oscillation in the \(x\)-variable rather than exponential decay. In the irrotational setting, for the Bond number smaller but close to \(1/3\) BEALE [5] and SUN [27] established that small amplitude solitary waves with ripples at infinity exist. Approaches by the spatial dynamics, [16,17] for instance, prove a similar result. In the presence of surface tension, GROVES and WAHLÉN [15] employed the spatial dynamics approach to construct various generalized solitary waves of small amplitude with vorticity.
Acknowledgements. Vera Mikyoung Hur would like to thank Walter Strauss for his interest and many helpful conversations and to Erik Wahlén for comments, in particular, on Section 7.

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(Received May 17, 2005 / Accepted October 21, 2006)

*Published online January 31, 2008 – © Springer-Verlag (2008)*