# Notes on a theorem of Jeff Cheeger

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## 1 Introduction and definitions

These notes are based on a course given by Juha Heinonen at the University of Michigan in the spring of 1999 and were compiled by Jeremy Tyson and Nageswari Shanmugalingam. The goal of the course was to read and understand a recent paper [4] of Jeff Cheeger ("Differentiability of Lipschitz functions on metric measure spaces"), extending the classical theorem of Rademacher on differentiability of Lipschitz functions to a certain general class of metric measure spaces. In their current form, these notes cover the first four sections of Cheeger's paper.

The first section of the notes provides some introductory material and definitions. In the second section of these notes, I state a theorem which combines the principal results of sections 4, 5 and 6 of Cheeger's paper [4] and go through a proof of those parts of the theorem which come from section 4. (I hope to include sections 5 and 6 in these notes at a later date.) As an aid to the reader, I have indicated throughout where each of the various lemmas and propositions can be found in [4]. Most of the proofs are expanded versions of the arguments given in [4].

A crucial role in Cheeger's proof is played by a certain notion of a Sobolev class of functions on a metric measure space. In these notes, the discussion has been framed in terms of a different (but equivalent) definition given by Shanmugalingam [15], [16]. Shanmugalingam's approach seems to me to be more geometric while Cheeger's definition is more functional analytic. The theory can be developed starting from either definition. The interested reader is invited to consult section 2 of [4], where the basic properties of these Sobolev functions are developed according to Cheeger's definition. At least one aspect of the proof of the main theorem (Theorem 2.1.1) is simplified if Cheeger's definition is used; see 2.3.6(3).

### 1.1 Upper gradients and pointwise Lipschitz constants

**Definition 1.1.1.** Let $X$ be a metric space and let $A \subset X$. Let $f : A \to \mathbb{R}$. Following Heinonen and Koskela [8], we say that a nonnegative Borel function $\rho : A \to [0, \infty]$ is an *upper gradient* of $f$ on $A$ if

$$|f(x) - f(y)| \le \int_\gamma \rho \, ds$$

for every rectifiable curve $\gamma$ joining $x$ to $y$ in $A$.

It is clear that if we take $X = \mathbb{R}^n$ and let $f$ be any smooth function on $\mathbb{R}^n$, then $|\nabla f|$ is an upper gradient for $f$. In fact, $|\nabla f|$ is the smallest upper gradient of $f$ in the following sense: if $\rho : \mathbb{R}^n \to \mathbb{R}$ is any other upper gradient of $f$, then $|\nabla f(x)| \le \rho(x)$ for a.e. $x \in \mathbb{R}^n$.

**Definition 1.1.2.** Let $f : X \to \mathbb{R}$ and let $x \in X$. Define the *upper and lower pointwise Lipschitz constants* of $f$ at $x$ to be

$$\operatorname{Lip} f(x) := \limsup_{r \to 0} \frac{L(x, f, r)}{r}$$

and

$$\operatorname{lip} f(x) := \liminf_{r \to 0} \frac{L(x, f, r)}{r}$$

where, for $r > 0$, we set $L(x, f, r) = \sup\{|f(x) - f(y)| : d(x, y) \leq r\}$.

Clearly $\operatorname{lip} f(x) \leq \operatorname{Lip} f(x)$ for all $x \in X$. Moreover,

$$\operatorname{Lip} f(x) = \limsup_{r \to 0} \sup_{\substack{y \\ d(x,y)=r}} \frac{|f(x) - f(y)|}{r} = \limsup_{y \to x} \frac{|f(x) - f(y)|}{d(x, y)}.$$

We leave the proof of this fact for the reader.

**Lemma 1.1.3.** *If $f$ is Lipschitz, then $\operatorname{lip} f$ is an upper gradient for $f$.*

*Proof.* Let $x, y \in X$ and let $\gamma$ be a rectifiable curve in $X$ joining $x$ to $y$. Assume that $\gamma : [0, L] \to X$ is parameterized by arc length. Then $f \circ \gamma : [0, L] \to \mathbb{R}$ is Lipschitz and hence differentiable at almost every point $t \in [0, L]$. Moreover,

$$|f(x) - f(y)| = |(f \circ \gamma)(0) - (f \circ \gamma)(L)| \leq \int_0^L |(f \circ \gamma)'(t)| \, dt.$$

An easy verification shows that $|(f \circ \gamma)'(t)| \leq \operatorname{lip} f(\gamma(t))$ at any point $t$ where $(f \circ \gamma)'(t)$ exists. □

**Definition 1.1.4.** Let $\mu$ be a nontrivial Borel measure on the metric space $X = (X, d)$. We say that $\mu$ is a *doubling measure* if there exists a constant $C = C(\mu) < \infty$ so that $\mu(B(x, 2r)) \leq C\mu(B(x, r))$ for all $x \in X$ and $r > 0$.

We recall some well-known facts about doubling measures. First, there exist constants $\kappa > 0$ and $c > 0$ so that

$$\frac{\mu(B(x, r))}{\mu(B(x_0, r_0))} \geq c \left( \frac{r}{r_0} \right)^{\kappa} \tag{1.1.5}$$

whenever $x \in B(x_0, r_0)$ and $r \leq r_0$. Second, any doubling metric measure space is a *Vitali space*, that is, the conclusion of Vitali's covering theorem is valid: if $\mathcal{B}$ is any covering of a set $A$ by closed balls with $\inf\{r : B(x, r) \in \mathcal{B}\} = 0$ for each $x \in A$, then there exist disjoint balls $B_1, B_2, \ldots$ in $\mathcal{B}$ with $\mu(A \setminus \cup_i B_i) = 0$. For a proof of this fact, see e.g. [7, Theorem 1.6].

**Lemma 1.1.6.** *Let $X = (X, d, \mu)$ be a doubling space and let $f : X \to \mathbb{R}$ be Lipschitz. For $c \in \mathbb{R}$, let $E_c = \{x \in X : f(x) = c\}$. Then $\operatorname{Lip} f(x) = 0$ for a.e. $x \in E_c$.*

*Proof.* We will prove that $\operatorname{Lip} f(x_0) = 0$ at every point of density $x_0$ for the set $E_c$; recall that the fact that $(X, \mu)$ is a Vitali space implies that almost every point of $E_c$ is such a point. It suffices to show the following: if $x_j$ is a sequence of points in $X$ converging to $x_0$, then

$$\lim_{j\to\infty} \frac{|f(x_j) - f(x_0)|}{d(x_0, x_j)} = 0.$$

Let $s_j$ be the supremum of those $s > 0$ for which $B(x_j, s) \cap E_c$ is empty. Then $0 \le s_j \le r_j := d(x_0, x_j)$. By (1.1.5),

$$\frac{\mu(B(x_j, s_j))}{\mu(B(x_0, r_j))} \ge c \left(\frac{s_j}{r_j}\right)^\kappa$$

for all $j$. Moreover, $B(x_j, s_j) \subset B(x_0, 2r_j) \setminus E_c$. Since $x_0$ is a point of density of $E_c$, we conclude that $s_j/r_j \to 0$.

Choose $y_j \in \overline{B}(x_j, s_j) \cap E_c$. Then

$$\frac{|f(x_j) - f(x_0)|}{d(x_0, x_j)} \le \frac{|f(x_j) - f(y_j)|}{r_j} \le L\frac{s_j}{r_j} \to 0,$$

where $L$ is a Lipschitz constant for the map $f$. □

## 1.2 $D$-structures on metric measure spaces

Throughout this section $X = (X, d, \mu)$ will be a fixed metric (Borel) measure space such that $0 < \mu(B) < \infty$ for each ball $B \subset X$. We do **not** in general assume that $\mu$ is doubling.

**Definition 1.2.1.** A *weak $D$-structure* on $X$ is a triple $(X, \mathbf{F}, d)$, where

- $\mathbf{F} = \{F_x\}_{x\in X}$ is a measurable bundle of Banach spaces on $X$, i.e. to a.e. point $x \in X$, there corresponds a Banach space $F_x$ (we denote by $\|\cdot\|_x$ the norm on $F_x$);
- $d : \mathcal{L}ip_{\mathrm{loc}}(X) \to \Gamma(\mathbf{F})$ is a linear map from the algebra of locally Lipschitz functions on $X$ to the sections of $\mathbf{F}$, that is, for $u \in \mathcal{L}ip_{\mathrm{loc}}(X)$ and a.e. $x \in X$,

$$du(x) \in F_x.$$

  We assume that the mapping $x \mapsto |du(x)| := \|du(x)\|_x$ is measurable.

We make two further assumptions on the triple $(X, \mathbf{F}, d)$:

(i) $|du(x)| \le \operatorname{Lip}(u)$ for all $u \in \mathcal{L}ip_{\mathrm{loc}}(X)$ and a.e. $x \in X$;

(ii) for each $c \in \mathbb{R}$, $du(x) = 0$ a.e. in the set $\{x \in X : u(x) = c\}$.

We call a weak $D$-structure $(X, \mathbf{F}, d)$ a *$D$-structure* if, in addition, $d$ is a derivation:

(iii) for all $u, v \in \mathcal{L}ip_{\mathrm{loc}}(X)$, $d(uv) = udv + vdu$.

**Examples 1.2.2.** (a) Every metric space admits the trivial $D$-structure $d \equiv 0$, where $\mathbf{F}$ is an arbitrary collection of Banach spaces.

(b) In $\mathbb{R}^n$, we may take $F_x = T_x\mathbb{R}^n = \mathbb{R}^n$ at each point $x \in \mathbb{R}^n$ and take $du = \nabla u$. This determines a $D$-structure for any Borel measure $\mu$ which is absolutely continuous with respect to the Lebesgue measure $m_n$.

(c) On every Riemannian manifold $M$ we have a canonical $D$-structure, given by the Riemannian distance, volume, gradient, and tangent spaces $T_xM$, $x \in M$. More generally, we may consider *Finsler manifolds*, where the cotangent spaces are equipped with only a norm; in each of these cases $du$ is the standard differential.

(d) Let $X$ be an $m$-rectifiable set in $\mathbb{R}^N$, $1 \leq m \leq N$, with locally finite Hausdorff measure $\mathcal{H}_m$. We may define a $D$-structure on $X$ as follows: take $F_x = \mathrm{Hom}(\mathrm{Tan}_m(\mathcal{H}_m \llcorner X, x), \mathbb{R})$ and $du(x) = \mathrm{ap}\, Du(x)$. Here $\mathrm{Tan}_m(\mathcal{H}_m \llcorner X, x)$ and $\mathrm{ap}\, Du(x)$ are the *approximate tangent space* and *approximate differential*; $F_x$ is an $m$-dimensional subspace of $\mathbb{R}^N$ so $F_x$ carries a natural inner product.

(e) **Carnot-Carathéodory spaces.** Let $X_1, \ldots, X_k$ be vector fields in $\mathbb{R}^N$, $N \geq 2$, with locally Lipschitz coefficients. For $a, b \in \mathbb{R}^N$, define $d_X(a, b)$ to be the infimum of those values $T > 0$ for which there exists a curve $\gamma : [0, T] \to \mathbb{R}^N$ such that $\gamma(0) = a$, $\gamma(T) = b$, and

$$\dot{\gamma} = \sum_{i=1}^{k} c_i X_i$$

with $\sum_{i=1}^{k} c_i^2 \leq 1$.
If $d_X(a, b)$ is finite for each pair of points $a, b \in \mathbb{R}^N$, we call the metric measure space

$$(\mathbb{R}^N, d_X, m_N)$$

a *Carnot-Carathéodory space*. If this space, in addition, is homeomorphic to the standard Euclidean space $\mathbb{R}^N$ via the identity map, we have a $D$-structure on $(\mathbb{R}^N, d_X, m_N)$, where $F_x = \mathbb{R}^k$ for each $x \in \mathbb{R}^N$ and $du = (X_1 u, \ldots, X_k u)$, $u \in \mathcal{L}ip_{\mathrm{loc}}(\mathbb{R}^N, d_X)$.

The requirement that $d_X(a, b)$ be finite for all $a, b$ is implied by *Hörmander's condition*: the vector fields $X_1, \ldots, X_k$ are $C^\infty$ and, for some fixed integer $p$, the commutators of $X_1, \ldots, X_k$ of length no more than $p$ span the tangent space $T_x\mathbb{R}^N$ at each $x$.

(f) A particular example of (e) is the *first Heisenberg group* $H_1$, where $N = 3$ (we denote by $(x, y, t)$ points in $\mathbb{R}^3$) and

$$X_1 = \partial_x + 2y\partial_t,$$
$$X_2 = \partial_y - 2x\partial_t.$$

The space $H_1 = (\mathbb{R}^3, d_X)$ has Hausdorff dimension 4, in fact, the Lebesgue measure of a ball of radius $R$ is equal to $c \cdot R^4$, where $c > 0$ is a fixed constant.

The trivial $D$-structure defined in (a) exists on every metric measure space. We would like to add some restrictions to our notion of $D$-structure to rule out this case.

**Definition 1.2.3.** Let $(X, \mathbf{F}, d)$ be a $D$-structure on a space $X$. We say that $(X, \mathbf{F}, d)$ is *nontrivial* if, for each $u \in \mathcal{L}ip_{\mathrm{loc}}(X)$, there is an upper gradient $\rho$ for $u$ which equals $|du|$ almost everywhere.

For example, if $\operatorname{lip} u(x) \le |du(x)|$ for a.e. $x \in X$ and every $u \in \mathcal{L}ip_{\mathrm{loc}}(X)$, then the $D$-structure is nontrivial (since $\operatorname{lip} u$ is an upper gradient of $u$, see Lemma 1.1.3).

## 1.3 Poincaré inequalities

Recall that the classical Poincaré inequality in $\mathbb{R}^n$ implies the following estimate for $C^\infty$ functions $u$ in a ball $B \subset \mathbb{R}^n$ of radius $r$:

$$\inf_{a\in\mathbb{R}} \int_B |u-a|\, dx \le Cr \int_B |\nabla u|\, d\mu. \tag{1.3.1}$$

We formulate an abstract version of this condition.

**Definition 1.3.2.** Let $X = (X, d, \mu)$ be a metric measure space. We say that $X$ satisfies the *weak $p$-Poincaré inequality for measurable (resp. continuous, resp. locally Lipschitz) functions* if there exist constants $C \ge 1$ and $\tau \ge 1$ so that

$$\fint_B |u - u_B|\, d\mu \le Cr \left( \fint_{\tau B} \rho^p\, d\mu \right)^{1/p} \tag{1.3.3}$$

for all bounded measurable (resp. continuous, resp. locally Lipschitz) functions $u$ on $\tau B$ with upper gradient $\rho$ and all balls $B$ in $X$ of radius $r$. If (1.3.3) holds with $\tau = 1$, we say that $X$ satisfies the *(strong) $p$-Poincaré inequality.* Here and henceforth, we employ the standard notation

$$f_E = \fint_E f\, d\mu := \frac{1}{\mu(E)} \int_E f\, d\mu$$

for measurable functions $f : X \to \mathbb{R}$, where $E \subset X$ has $\mu(E) > 0$.

Note that this condition becomes weaker as $p$ increases, by Hölder's inequality.

**Remarks 1.3.4.** (1) By a result of Heinonen and Koskela [9], if the space $X$ is *proper* (all closed balls of finite radius are compact) then $X$ satisfies the weak $p$-Poincaré inequality for Lipschitz functions if and only if it satisfies the weak $p$-Poincaré inequality for all measurable functions.

(2) By a result of Hajłasz and Koskela [6, Theorem 5.1], if the measure $\mu$ is doubling with doubling constant $C_d$ and if $X$ satisfies the weak $p$-Poincaré inequality for measurable functions, then it also satisfies the (stronger) Poincaré inequality

$$\left( \fint_B |u - u_B|^{p'}\, d\mu \right)^{1/p'} \le C' r \left( \fint_{5\tau B} \rho^p\, d\mu \right)^{1/p} \tag{1.3.5}$$

for all bounded measurable functions $u$ on $5\tau B$ with upper gradient $\rho$ and all balls $B$ in $X$ of radius $r$. Here the constants $p' > p$ and $C'$ depend only on $C_d$ and the constant $C$ of (1.3.3).

**Examples 1.3.6.** (1) Inequality (1.3.1) implies that the Euclidean space $\mathbb{R}^n$ (with the Euclidean metric and Lebesgue norm) satisfies the 1-Poincaré inequality. More generally, any complete Riemannian manifold $M$ with nonnegative Ricci curvature also satisfies the 1-Poincaré inequality; this can be derived, for example, from Buser's inequality [2]. Note that the canonical measure on $M$ is a doubling measure by the classical volume comparison theorem of Bishop-Gromov.

(2) Any connected, finite simplicial complex $X$ of pure dimension $n \geq 2$ for which the link of each vertex is connected satisfies the $n$-Poincaré inequality [8]. Here the relevant measure is the Lebesgue $n$-measure on each simplex in $X$. For the metric, we may use either the barycentric metric or the metric which $X$ inherits as a subset of some Euclidean space.

(3) Let $X = (\mathbb{R}^N, d_X, m_N)$ be a Carnot-Carathéodory space (see Example 1.2.2(e)) for which the vector fields $X_1, \ldots, X_k$ satisfy the Hörmander condition. Then $X$ is a doubling space satisfying the 1-Poincaré inequality [11]. The special case of *Carnot groups* (such as the first Heisenberg group) is of particular importance. Note that the nonabelian Carnot groups have the property that their Hausdorff dimension is strictly greater than their topological dimension.

(4) Other examples of doubling metric measure spaces which satisfy the 1-Poincaré inequality and have Hausdorff dimension strictly greater than their topological dimension have been given by Bourdon and Pajot [1] and Laakso [13]. The examples of Bourdon and Pajot arise in the context of geometric group theory; more specifically, they are the boundaries at infinity of certain two-dimensional hyperbolic buildings and have topological dimension one but Hausdorff dimension strictly greater than one. The allowable set of Hausdorff dimensions for the Bourdon-Pajot examples is a countable set of values which is dense in the interval $(1, \infty)$. Laakso has given for each $Q > 1$ a metric measure space of topological dimension one and Hausdorff dimension $Q$ which is doubling and also satisfies the 1-Poincaré inequality. His examples are finite-to-one quotients of certain compact subsets of Euclidean space.

## 1.4 Sobolev spaces defined in terms of a $D$-structure

Let $(X, \mathbf{F}, d)$ be a weak $D$-structure on a space $X$. For an open set $\Omega \subset X$ and $1 \leq p < \infty$, we can define the Sobolev space $H^{1,p}(\Omega)$ to be the closure (in the norm $|\cdot|_{H^{1,p}(\Omega)}$) of the collection of functions $u \in \mathcal{L}ip_{\text{loc}}(\Omega)$ for which the norm

$$|u|_{H^{1,p}(\Omega)} := \left(\int_\Omega |u|^p \, d\mu\right)^{1/p} + \left(\int_\Omega |du|^p \, d\mu\right)^{1/p} \tag{1.4.1}$$

is finite. Note that elements of $H^{1,p}(\Omega)$ are $L^p$-equivalence classes of functions.

Note also that $d$ is *a priori* only defined for functions in $\mathcal{L}ip_{\text{loc}}(X)$. However, any function $u \in \mathcal{L}ip_{\text{loc}}(\Omega)$ can be extended to a Lipschitz map on all of $X$. Combining this with Definition 1.2.1(ii), we see that $d$ has a unique extension from $\mathcal{L}ip_{\text{loc}}(X)$ to $\mathcal{L}ip_{\text{loc}}(E)$ for any measurable set $E \subset X$.

Clearly, $d : \mathcal{L}ip_{\text{loc}}(\Omega) \cap H^{1,p}(\Omega) \to L^p(\Omega; \mathbf{F})$ is a linear operator (with the $L^p$-norm on each side). We say that $d$ is *p-closable* in $\Omega$ if whenever $(u_n)$ is a sequence of functions in $\mathcal{L}ip_{\text{loc}}(\Omega) \cap H^{1,p}(\Omega)$

tending to zero in $L^p(\Omega)$ so that $du_n \to s$ in $L^p(\Omega; \mathbf{F})$, then $s = 0$.

We call a weak $D$-structure on $X$ *$p$-closable* if it is $p$-closable for $\Omega = X$.

If $d$ is $p$-closable in $\Omega$, then $du$ may be unambiguously defined as an $L^p$-section of $\mathbf{F}$ over $\Omega$ for any $u \in H^{1,p}(\Omega)$.

**Theorem 1.4.2 (Semmes; Franchi-Hajłasz-Koskela).** *Let $\mu$ be a doubling measure on $X$ for which the metric measure space $(X, d, \mu)$ satisfies the weak $p$-Poincaré inequality for locally Lipschitz functions (for some $p \geq 1$). If $(X, \mathbf{F}, d)$ is a nontrivial weak $D$-structure, then $d$ is $p$-closable.*

See e.g. [5] or the first part of these notes by Nageswari Shanmugalingam.

Some natural questions arise at this point. For example, is the $D$-structure of Example 1.2.2(d) always $p$-closable? Are there weak $D$-structures which are $p$-closable for some values of $p$ and not for others?

**Remark 1.4.3.** If we assume in addition that $(X, \mathbf{F}, d)$ is a (strong) $D$-structure, i.e. that $d$ is a derivation, then $d$ is $p$-closable in each open set $\Omega$ if and only if it is $p$-closable (in $X$). This is easy to see by using appropriate Lipschitz cutoff functions to define a "discrete convolution".

# 2 Cheeger's theorem

## 2.1 Statement of the theorem

We now have the machinery in place to give (a slightly restated version of) the recent theorem of Cheeger (Theorem 4.38 of [4]) on differentiability of Lipschitz functions in metric measure spaces. In the ensuing theorem, the *data* of the metric measure space $(X, d, \mu)$ consists of the doubling constant $C_d$ for the measure $\mu$, the constants $C, \tau$ involved in the Poincaré inequality (1.3.3), and the exponent $p$. As in Section 1.2, we assume from now on that $\mu$ is a Borel measure satisfying $0 < \mu(B) < \infty$ for all balls $B$ in $X$.

**Theorem 2.1.1 (Cheeger).** *Let $X$ be a metric space supporting a doubling measure $\mu$ for which the space $X$ satisfies the weak $p$-Poincaré inequality for measurable functions for some $1 < p < \infty$.*

*Then there exists a nontrivial $D$-structure $(X, \mathbf{F}, d)$ on $X$. Moreover:*

*(i) There exists a constant $N$ depending only on the data of $X$ so that $\dim F_x \leq N$ for a.e. $x \in X$.*

*(ii) For each locally Lipschitz function $u$, the function $\rho_u(x) := |du(x)|$ is the minimal $p$-weak upper gradient of $u$. Note that $|du(x)|$ here denotes the norm of $du(x)$ as an element of the Banach space $F_x$ (recall Definition 1.2.1). For the definition of weak upper gradients, see the following section.*

*(iii) For each locally Lipschitz function $u$ and a.e. $x \in X$, we have*

$$|du(x)| = \operatorname{lip} u(x) = \operatorname{Lip} u(x).$$

*In particular, the limit*

$$\lim_{r \to 0} \frac{L(x, u, r)}{r}$$

*exists almost everywhere;*

*(iv) The Sobolev space $H^{1,p}(X)$ (defined in section 1.4) is reflexive.*

Note that $d$ is $p$-closable by Theorem 1.4.2. Nontriviality of the $D$-structure follows immediately from (iii), see the remarks at the end of section 1.2.

To be precise, Theorem 2.1.1 is only part of Cheeger's generalization of Rademacher's theorem. The full generalization is only obtained by combining this result with Theorem 10.2 of [4], where it is shown that Lipschitz functions on metric measure spaces $X$ as above satisfy a certain infinitesimal linearity property which, when specialized to the case of Euclidean space, coincides with the standard notion of differentiability (i.e. asymptotic linearity). See Sections 8 and 10 of [4].

Note that the exclusion of the case $p = 1$ from the above theorem is essentially immaterial since any space satisfying the weak 1-Poincaré inequality also satisfies the weak $p$-Poincaré inequality for any $p > 1$. However, this restriction is needed in (iv); the space $H^{1,1}(X)$ is not reflexive even in the case $X = \mathbb{R}^n$.

## 2.2 Newtonian spaces

**Definition 2.2.1.** Let $f$ be a function on a metric measure space $X = (X, \mu)$. We say that a nonnegative Borel function $\rho$ on $X$ is a *$p$-weak upper gradient* of $f$ if for $p$-a.e. rectifiable curve $\gamma : [a, b] \to X$,

$$|f(\gamma(a)) - f(\gamma(b))| \leq \int_\gamma \rho \, ds.$$

Recall that a collection $\Gamma$ of rectifiable curves in $X$ is called *$p$-null*, $1 < p < \infty$, if there exists a function $h \in L^p(X, \mu)$ so that $\int_\gamma h \, ds = \infty$ for each $\gamma \in \Gamma$. We say that a property $\mathcal{P}$ of curves holds for *$p$-a.e. curve* if the collection of curves for which $\mathcal{P}$ fails to hold is $p$-null.

The class of $p$-weak upper gradients of a function $f$ depends on the choice of the measure $\mu$ and on the value of $p$ (unlike the class of upper gradients of $f$). In general, it can be a strictly larger class, however, the upper gradients of $f$ are "dense" in the $p$-weak upper gradients in the following sense (Lemma 2.4 of [12]): if $\rho$ is any $p$-weak upper gradient of $f$, then there is a decreasing sequence $(\rho_n)$ of upper gradients of $f$ which tends to $\rho$ in $L^p(X, \mu)$. From this it is easy to see that the condition that $X$ satisfy the (weak) $p$-Poincaré inequality (Definition 1.3.2) is unchanged if we instead require (1.3.3) to hold for all functions $f$ and $p$-weak upper gradients $\rho$ of $f$.

**Definition 2.2.2.** Let $(X, \mu)$ be a metric measure space and let $U$ be an open subset of $X$. Following Shanmugalingam [16], we define $\tilde{N}^{1,p}(U)$ to be the collection of those functions $u \in L^p(U)$ which admit a $p$-weak upper gradient $\rho \in L^p(U)$. We introduce a pseudonorm on $\tilde{N}^{1,p}(U)$ by defining

$$|u|_{N^{1,p}} := \|u\|_p + \inf_\rho \|\rho\|_p,$$

where $\|\cdot\|_p$ denotes the norm in $L^p(U)$ and the infimum is taken over all $p$-weak upper gradients $\rho$ of $u$. The *Newtonian space* $N^{1,p}(U)$ is defined to be the collection of equivalence classes of elements of $\tilde{N}^{1,p}(U)$ under the equivalence relation

$$u \sim v \iff |u - v|_{N^{1,p}} = 0.$$

With the usual abuse of terminology, the elements of $N^{1,p}(U)$ are called functions.

We note some facts about the Newtonian space. For proofs, see [16].

(i) The normed space $(N^{1,p}(U), |\cdot|_{N^{1,p}(X)})$ is a Banach space.

(ii) Each function $u \in N^{1,p}(X)$ is absolutely continuous along $p$-a.e. curve.

(iii) When $p > 1$, there corresponds to each function $u \in N^{1,p}(X)$ a $p$-weak upper gradient $\rho_u \in L^p(X)$ which is minimal (in the sense that any other $p$-weak upper gradient $\rho$ of $u$ satisfies $\rho_u \le \rho$ a.e.). *A priori* the function $\rho_u$ depends also on $p$, but we will suppress this dependence in the notation. One of the consequences of Theorem 2.1.1 is that under the hypothesis of that theorem, the minimal $p$-weak upper gradient $\rho_u$ of any locally Lipschitz function $u$ is in fact independent of $p$; this follows directly from parts (ii) and (iii) of that theorem (recall that the definitions of the pointwise Lipschitz constants $\operatorname{Lip} u(x)$ and $\operatorname{lip} u(x)$ did not depend on $p$).

(iv) For any $c \in \mathbb{R}$, $\rho_u = 0$ a.e. on the set $\{x \in X : u(x) = c\}$. It follows from this that the space $N^{1,p}(X)$ is closed under the operations of maximum and minimum: if $u, v \in N^{1,p}(X)$ then $w_1 = \max\{u, v\}$ and $w_2 = \min\{u, v\}$ are also in $N^{1,p}(X)$ and $\rho_{w_i} \le \max\{\rho_u, \rho_v\}$ a.e. $i = 1, 2$.

(v) If $(u_n)$ is a sequence of functions converging to $u$ in $L^p(X)$ and $(\rho_n)$ is another sequence converging weakly to a function $\rho$ in $L^p(X)$ so that $\rho_n$ is a $p$-weak upper gradient of $u_n$ for each $n$, then $\rho$ is a $p$-weak upper gradient of $u$. (This is another reason why the introduction of the larger class of weak upper gradients is necessary. The corresponding result for upper gradients need not hold in general.)

If $U$ is an open subset of $X$, we define the space of *Newtonian functions with zero boundary values* $N_0^{1,p}(U)$ to be the closure (in the norm $|\cdot|_{N^{1,p}}$) of the set of functions $h \in N^{1,p}(U)$ whose support is compactly contained in $U$.

**Remark 2.2.3.** In section 2 of [4], Cheeger employs a different definition of an abstract Sobolev space. Let $(X, \mu)$ be a metric measure space and let $U$ be an open subset of $X$. Following Definition 2.2 of [4], we define $H^{1,p}(U)$ to be the collection of functions $f \in L^p(U)$ for which the quantity

$$|f|_{H^{1,p}} := \|f\|_p + \inf_{(\rho_i)} \liminf_{i\to\infty} \|\rho_i\|_p \tag{2.2.4}$$

is finite, where $\|\cdot\|_p$ denotes the norm in $L^p(U)$ and the infimum is taken over all sequences $(\rho_i)$ of nonnegative Borel functions on $U$ for which there exists a sequence $(f_i)$ converging to $f$ in $L^p(U)$ so that $\rho_i$ is an upper gradient for $f_i$ for each $i$.

A principal advantage of this definition is the fact that Rellich's theorem becomes a virtual tautology when expressed in this language, i.e., if $(f_k)$ is a sequence of functions in $H^{1,p}(U)$ which converge in $L^p(U)$ to a function $f$, then $f \in H^{1,p}(U)$ and

$$|f|_{H^{1,p}} \le \liminf_{k\to\infty} |f_k|_{H^{1,p}}. \tag{2.2.5}$$

This fact plays an important role in Cheeger's proof of a preliminary generalization of the Rademacher theorem to metric measure spaces[1]; see section 2.3.

For each $1 \leq p < \infty$, the space $H^{1,p}(U)$ is a Banach space (Theorem 2.7 of [4]). This can be easily verified as a consequence of (2.2.5). In [15, Theorem 2.3.2], Shanmugalingam proves that the spaces $H^{1,p}(X)$ and $N^{1,p}(X)$ are isometrically equivalent when $p > 1$. In fact, the $p$-weak upper gradient $\rho_u$ of a function $u \in N^{1,p}(X)$ coincides with the *minimal generalized upper gradient* of $u$ considered as an element of $H^{1,p}(X)$ (see Definition 2.9 and Theorem 2.10 of [4]). It follows that the norm (2.2.4) of a function $f \in H^{1,p}(X)$, $p > 1$, is given by

$$|f|_{H^{1,p}} = \|f\|_p + \|\rho_f\|_p, \tag{2.2.6}$$

and so (by part (ii) of Theorem 2.1.1)

$$|f|_{H^{1,p}} = \|f\|_p + \|df\|_p.$$

Thus we see (in light of Theorem 2.1.1) that the notation $H^{1,p}(X)$ used here agrees with the notation used in section 1.4.

## 2.3 Asymptotic linearity

**Definition 2.3.1 (Cheeger).** A function $u \in N^{1,p}(X)$ (equivalently $H^{1,p}(X)$) is said to be *asymptotically generalized linear (with respect to $\rho_u$)* at a point $x_0 \in X$ if the following two conditions hold:

(i) $x_0$ is a Lebesgue point of the function $\rho_u^p$,

(ii) $u$ asymptotically minimizes the $p$-energy over all Newtonian functions with equal boundary values, i.e.

$$\rho_u(x_0)^p = \lim_{r\to 0} \fint_{B(x_0,r)} \rho_u^p \, d\mu \leq \liminf_{r\to 0} \inf_{h_r} \fint_{B(x_0,r)} \rho_{u+h_r}^p \, d\mu,$$

where the infimum is taken over all $h_r \in N_0^{1,p}(B(x_0,r))$. (Here $B(x_0,r)$ denotes the **closed** ball of radius $r$ centered at $x_0$.)

If $u \in N^{1,p}(X)$ is asymptotically generalized linear at $x_0$, we write $x_0 \in \mathrm{AGL}_p(u)$.

**Theorem 2.3.2 (Theorem 3.7 of [4]).** *Let $(X, d, \mu)$ be a Vitali space (see p. 2) and let $1 < p < \infty$. If $f : X \to \mathbb{R}$ is Lipschitz, then $\mathrm{AGL}_p(f)$ is a set of full measure in $X$.*

*Proof.* Suppose that $f$ is a Lipschitz function on $X$ for which the conclusion fails to hold. Since $\mu$-a.e. point of $X$ is a Lebesgue point of $\rho_f^p$, it must be the case that the set

$$A = \left\{ x \in X : \lim_{r\to 0} \fint_{B(x,r)} \rho_f^p \, d\mu > \liminf_{r\to 0} \inf_{h_r} \fint_{B(x,r)} \rho_{f+h_r}^p \, d\mu \right\}$$

[1] According to Cheeger [3], the definition of a Sobolev space in terms of the norm in (2.2.4) was motivated in large part by the fact that it yielded this preliminary version of the Rademacher theorem.

has positive measure. Without loss of generality, we may assume that $A$ is bounded and that

$$\lim_{r\to 0} \fint_{B(x,r)} \rho_f^p \, d\mu \geq \liminf_{r\to 0} \inf_{h_r} \fint_{B(x,r)} \rho_{f+h_r}^p \, d\mu + \epsilon$$

for all $x \in A$ and some fixed $\epsilon > 0$. Thus to each point $a \in A$ we may associate a sequence of closed balls $B(a,r)$, $r \to 0$, and a sequence of functions $k_{a,r} \in N_0^{1,p}(B(a,r))$ so that

$$\fint_{B(a,r)} \rho_f^p \, d\mu \geq \fint_{B(a,r)} \rho_{f+k_{a,r}}^p \, d\mu + \epsilon. \tag{2.3.3}$$

Since $X$ is a Vitali space, for each $n = 1, 2, \ldots$ we may choose a countable disjoint collection $B_1^n, B_2^n, \ldots,\ B_i^n = B(a_i^n, r_i^n)$, so that $\operatorname{diam} B_i^n \leq 1/n$ for each $i$ and

$$\mu(A \setminus \cup_i B_i^n) = 0. \tag{2.3.4}$$

For simplicity, we write $k_i^n := k_{a_i^n, r_i^n}$.

For each $n$, define a function $f_n$ on $X$ by

$$f_n(x) = \begin{cases} f + k_i^n, & \text{on } B_i^n, \\ f & \text{otherwise.} \end{cases}$$

The functions $k_i^n$ can be chosen so that $f_n$ converges to $f$ in $L^\infty$. Indeed, if

$$(f + k_i^n)(y) > b := \sup\{f(x) : x \in B_i^n\}$$

or

$$(f + k_i^n)(y) < a := \inf\{f(x) : x \in B_i^n\}$$

at any point $y \in B_i^n$, we may truncate the functions $k_i^n$, noting that the average value of $\rho_{f+k_i^n}$ over the ball $B_i^n$ must decrease. In other words, let $\tilde{k}_i^n := (f + k_i^n)_a^b - f$, where, for $-\infty < a < b < \infty$ and $u : X \to \mathbb{R}$, we define $u_a^b := \max\{\min\{u, b\}, a\}$. (Note that $\rho_{f+\tilde{k}_i^n} \leq \rho_{f+k_i^n}$ a.e. in $B_i^n$.) Then

$$\|f - f_n\|_\infty = \sup_i \|\tilde{k}_i^n\|_\infty \leq b - a \leq L \operatorname{diam} B_i^n \leq L/n,$$

where $L$ is the Lipschitz constant of $f$. By (2.3.3),

$$\int_{B_i^n} \rho_{f+\tilde{k}_i^n}^p \, d\mu \leq \int_{B_i^n} \rho_f^p \, d\mu - \epsilon\mu(B_i^n).$$

Summing over $i$ and using (2.3.4), we get

$$\int_A \rho_{f_n}^p \, d\mu \leq \int_{\cup_i B_i^n} \rho_{f_n}^p \, d\mu \leq \int_{\cup_i B_i^n} \rho_f^p \, d\mu - \epsilon\mu(A) \leq \int_X \rho_f^p \, d\mu - \epsilon\mu(A) \tag{2.3.5}$$

for all $n$. Thus $(\rho_{f_n})$ is a bounded sequence in $L^p(A)$ and so there exists a subsequence $\rho_k := \rho_{f_{n_k}}$ converging weakly to some function $\rho_0$ in $L^p(A)$. By Mazur's lemma, a sequence of convex combinations of $(\rho_k)$ (which we continue to denote by $\rho_k$) converges to $\rho_0$ in $L^p(A)$. Since $\rho_0$ is a $p$-weak upper gradient of $f$ (see section 2.2) we have

$$\int_A \rho_k^p \, d\mu \to \int_A \rho_0^p \, d\mu \geq \int_A \rho_f^p \, d\mu.$$

But passing to the limit in (2.3.5) yields

$$\int_A \rho_0^p \, d\mu \leq \int_A \rho_f^p \, d\mu - \epsilon\mu(A).$$

This contradiction completes the proof. □

**Remarks 2.3.6.** (1) An analysis of the proof reveals that we need only assume that $f$ is continuous with $\rho_f \in L^p$, provided that $X$ is locally compact.

(2) Theorem 2.3.2 already represents a partial generalization of Rademacher's differentiation theorem for Lipschitz functions valid only under the Vitali condition on the measure $\mu$. It is well-known (see e.g. Chapter 2 of [14]) that the Vitali condition holds for any Radon measure in Euclidean space. It is not at all clear exactly what property of Lipschitz functions in Euclidean space is isolated by Theorem 2.3.2 for an arbitrary Radon measure $\mu$. Note that in the case when $\mu$ is the Lebesgue measure in $\mathbb{R}^n$, the minimal weak upper gradient $\rho_u$ of a Lipschitz function $u$ coincides with its classical gradient $|\nabla u|$ (defined a.e.), but this fact will not in general hold for an arbitrary measure $\mu$.

(3) The proof of Theorem 2.3.2 can be somewhat simplified by making use of the Sobolev space $H^{1,p}(X)$ defined in Remark 2.2.3 and in particular the version of Rellich's theorem given in (2.2.5). We pick up the story at (2.3.5). Letting $U$ be an arbitrary bounded open set containing $\cup_i B_i^n$ and noting that $\rho_{f_n} = \rho_f$ a.e. in $U \setminus \cup_i B_i^n$, we see that (2.3.5) implies that

$$\int_U \rho_{f_n}^p \, d\mu \leq \int_U \rho_f^p \, d\mu - \epsilon\mu(A) \tag{2.3.7}$$

for all $n$. Since $(f_n)$ converges to $f$ in $L^\infty$, it converges in $L^p(U)$ and so

$$\begin{aligned} |f|_{H^{1,p}(U)} - \|f\|_{L^p(U)} &\leq \liminf_{n\to\infty} |f_n|_{H^{1,p}(U)} - \lim_{n\to\infty} \|f_n\|_{L^p(U)} \\ &= \liminf_{n\to\infty} \left(|f_n|_{H^{1,p}(U)} - \|f_n\|_{L^p(U)}\right) = \liminf_{n\to\infty} \|\rho_{f_n}\|_{L^p(U)} \end{aligned}$$

by (2.2.5) and (2.2.6). Hence (2.3.7) implies that

$$(|f|_{H^{1,p}(U)} - \|f\|_{L^p(U)})^p \leq \|\rho_f\|_{L^p(U)}^p - \epsilon\mu(A) = (|f|_{H^{1,p}(U)} - \|f\|_{L^p(U)})^p - \epsilon\mu(A)$$

which contradicts the assumption that $A$ has positive measure.

## 2.4 Reverse Poincaré inequalities

For Lipschitz functions $u$ with compact support in a ball $B \subset \mathbb{R}^n$ of radius $r$, one version of the Poincaré inequality states that

$$\int_B |u|^p \, dx \le C r^p \int_B |\nabla u|^p \, dx \tag{2.4.1}$$

for each $p \ge 1$, where $C$ is a constant depending only on $p$ and the dimension $n$. In general, the inequality in (2.4.1) can not be reversed. However, we have the following

**Proposition 2.4.2.** *Let $u$ be a $p$-harmonic function in a domain $\Omega \subset \mathbb{R}^n$. Then the following* weak reverse Poincaré inequality *holds for all balls $B \subset \Omega$:*

$$r^p \int_{\frac{1}{2}B} |\nabla u|^p \, dx \le C \int_B |u|^p \, dx. \tag{2.4.3}$$

*The constant $C$ here depends only on $n$ and $p$.*

Recall that a function $u \in W^{1,p}(\Omega)$ is said to be *$p$-harmonic* if

$$\int_B |\nabla u|^p \, dx \le \int_B |\nabla v|^p \, dx$$

for all balls $B \subset \Omega$ and all functions $v \in u + W_0^{1,p}(B)$ (i.e., $v - u \in W_0^{1,p}(B)$).

*Proof.* We use the Euler-Lagrange formulation of $p$-harmonicity: for all $C^\infty$ functions $\varphi$ with compact support in $\Omega$,

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \, dx = 0. \tag{2.4.4}$$

It follows by a standard completion argument that (2.4.4) holds more generally for all functions $\varphi \in W_0^{1,p}(\Omega)$.

Fix a ball $B \subset \Omega$. Choose a nonnegative $C^\infty$ function $\psi$ supported in $B$ for which $\psi|_{\frac{1}{2}B} \equiv 1$ and

$$\sup_{x \in B \setminus \frac{1}{2}B} |\nabla \psi(x)| \le C/r.$$

Then let $\varphi := -u\psi^p \in C_0^\infty(B) \subset C_0^\infty(\Omega)$. We calculate

$$0 = \int_B |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \, dx = \int_B |\nabla u|^{p-2} \nabla u \cdot (-\psi^p \nabla u - p u \psi^{p-1} \nabla \psi) \, dx$$

which implies that

$$\begin{aligned} \int_B |\nabla u|^p \psi^p \, dx &\le p \int_B \psi^{p-1} |\nabla \psi| |u| |\nabla u|^{p-1} \, dx \\ &\le p \left( \int_B \psi^p |\nabla u|^p \, dx \right)^{(p-1)/p} \left( \int_B |\nabla \psi|^p |u|^p \, dx \right)^{1/p}. \end{aligned}$$

Thus

$$\int_{\frac{1}{2}B} |\nabla u|^p \, dx \le p^p \int_B |\nabla \psi|^p |u|^p \, dx \le \frac{C}{r^p} \int_B |u|^p \, dx$$

which completes the proof of (2.4.3). □

**Remark 2.4.5.** Note that (2.4.4) makes sense as soon as $u \in W^{1,p-1}(\Omega)$. When $p = 2$ this is *Weyl's lemma*: if $\nabla u \in L^1(\Omega)$ and $\int_\Omega \nabla u \cdot \nabla \varphi \, dx = 0$ for all $\varphi \in C_0^\infty(\Omega)$, then $u$ is harmonic. For arbitrary $p$, it is an open question whether any function $u \in W^{1,p-1}(\Omega)$ which satisfies (2.4.4) is again $p$-harmonic (note that it would suffice to show that $u \in W^{1,p}(\Omega)$).

Proposition 2.4.6 and Corollary 2.4.10 provide abstract versions of the above discussion. In Cheeger's paper, these two results are Lemma 3.10 and Theorem 3.14, respectively.

**Proposition 2.4.6.** *Let $X = (X, d, \mu)$ be an arbitrary metric measure space and let $u \in N^{1,p}(B)$, $1 < p < \infty$, where $B \subset X$ is a ball of radius $r$. Assume that there exist constants $1 \le b < a < \infty$ so that*

$$a \left( \int_{B \setminus \frac{1}{2}B} \rho_u^p \, d\mu \right)^{1/p} \le \left( \int_B \rho_u^p \, d\mu \right)^{1/p} \tag{2.4.7}$$

*and*

$$\left( \int_B \rho_u^p \, d\mu \right)^{1/p} \le b \left( \int_B \rho_{u+k}^p \, d\mu \right)^{1/p} \tag{2.4.8}$$

*for all $k \in N_0^{1,p}(B)$. Then there exists a constant $C = C(a, b)$ so that*

$$r \left( \int_B \rho_u^p \, d\mu \right)^{1/p} \le C \left( \int_{\frac{3}{4}B \setminus \frac{1}{2}B} |u|^p \, d\mu \right)^{1/p}. \tag{2.4.9}$$

*Proof.* Let $\varphi : X \to [0, 1]$ be a Lipschitz function which equals 1 in $\frac{1}{2}B$ and equals zero in $X \setminus \frac{3}{4}B$, and for which $|\mathrm{Lip}\, \varphi| \le 4/r$. Write $v = (1 - \varphi)u$. Then $v - u \in N_0^{1,p}(B)$ and $v = 0$ in $\frac{1}{2}B$. It is easy to verify that

$$\rho_v \le \mathrm{Lip}(1 - \varphi) \cdot |u| + (1 - \varphi)\rho_u$$

almost everywhere (this is the Leibniz rule for differentiation of products for $p$-weak upper gradients). Using (2.4.8) and Minkowski's inequality together with the bound for $|\mathrm{Lip}\, \varphi|$, we see that

$$\begin{aligned} r \left( \int_B \rho_u^p \, d\mu \right)^{1/p} &\le br \left( \int_B \rho_v^p \, d\mu \right)^{1/p} \\ &\le br \left( \frac{4}{r} \left( \int_{\frac{3}{4}B \setminus \frac{1}{2}B} |u|^p \, d\mu \right)^{1/p} + \left( \int_{B \setminus \frac{1}{2}B} \rho_u^p \, d\mu \right)^{1/p} \right) \\ &\le 4b \left( \int_{\frac{3}{4}B \setminus \frac{1}{2}B} |u|^p \, d\mu \right)^{1/p} + \frac{b}{a} r \left( \int_B \rho_u^p \, d\mu \right)^{1/p}, \end{aligned}$$

where the last line follows from (2.4.7). Now (2.4.9) follows with $C = C(a, b) = 4ab/(a - b)$. □

**Corollary 2.4.10.** *Let $X = (X, d, \mu)$ be doubling and let $f \in N^{1,p}(X)$ be asymptotically generalized linear at a point $x_0$ with $\rho_f(x_0) > 0$. Then there exists $C$ depending only on the doubling constant $C_d$ so that*

$$r\left(\int_{B(x_0,r)} \rho_f^p \, d\mu\right)^{1/p} \leq C\left(\int_{B(x_0,\frac{3}{4}r)\setminus B(x_0,\frac{1}{2}r)} |f|^p \, d\mu\right)^{1/p}.$$

*for sufficiently small $r$.*

*Proof.* It suffices to verify that (2.4.7) and (2.4.8) hold for sufficiently small $r$ with $a$ and $b$ depending only on the doubling constant $C_d$. Note first that the asymptotic generalized linearity of $f$ at $x_0$ together with the fact that $\rho_f(x_0) > 0$ implies that for each $\epsilon > 0$ and for sufficiently small $r$, we have

$$(1-\epsilon)\rho_f(x_0) \leq \left(\fint_{B(x_0,r)} \rho_f^p \, d\mu\right)^{1/p} \leq (1+\epsilon)\rho_f(x_0) \tag{2.4.11}$$

and

$$\rho_f(x_0) \leq (1+\epsilon)\left(\fint_{B(x_0,r)} \rho_{f+h_r}^p \, d\mu\right)^{1/p}$$

for all $h_r \in N_0^{1,p}(B(x_0,r))$. Thus we see that (2.4.8) holds for each $b > 1$ for sufficiently small $r$. To see that (2.4.7) holds, we rewrite it in the form

$$\alpha \int_{B(x_0,r)} \rho_f^p \, d\mu \leq \int_{B(x_0,\frac{1}{2}r)} \rho_f^p \, d\mu \tag{2.4.12}$$

where $\alpha = 1 - a^{-p} < 1$. Upon applying (2.4.11) to the balls $B(x_0,r)$ and $B(x_0,\frac{1}{2}r)$, we conclude that (2.4.12) holds, provided that

$$\alpha \leq \frac{1}{C_d}\left(\frac{1-\epsilon}{1+\epsilon}\right)^p$$

for some $\epsilon > 0$, i.e. that $\alpha < 1/C_d$. We see that (2.4.7) and (2.4.8) hold for sufficiently small $r$ for every choice of $a$ and $b$ satisfying $1 < b < a < (1 - C_d^{-1})^{-1/p}$. □

## 2.5 Finite dimensionality of the space of functions satisfying the reverse Poincaré inequality

**Proposition 2.5.1 (Theorem 4.8 of [4]).** *Let $X = (X, d, \mu)$ be a doubling metric measure space with doubling constant $C_d$ which satisfies the weak $p$-Poincaré inequality (for measurable functions). Fix a ball $B \subset X$ of radius $r$ and fix $K \geq 1$. Let $E = E(B, K)$ denote the set of functions $f \in N^{1,p}(B)$ for which*

$$r^p \int_B \rho_f^p \, d\mu \leq K^p \int_{\frac{3}{4}B} |f|^p \, d\mu. \tag{2.5.2}$$

*Then there exists a constant $N$ (depending only on $K$, $p$ and the data of $X$) and there exists a bounded linear map $\Phi : N^{1,p}(B) \to \mathbb{R}^N$ which is injective on linear subspaces of $E|_{\frac{3}{4}B} := \{f|_{\frac{3}{4}B} : f \in E\}$. Hence the dimension of any linear subspace of $E|_{\frac{3}{4}B}$ is at most $N$.*

*Proof.* Fix a (small) constant $s$ depending on $K$ and the data of $X$ (the exact choice will be given later). Choose balls $B_i = B(x_i, sr)$, $i = 1, 2, \ldots, N$, so that

(i) $\tau B_i \subset B$ for each $i$ (where $\tau$ is the constant appearing in (1.3.3)),

(ii) $\frac{3}{4}B \subset \bigcup_i B_i$,

(iii) $\frac{1}{2}B_i \cap \frac{1}{2}B_j = \emptyset$ when $i \neq j$.

The doubling condition implies that we can make such a choice with $N = N(s, C_d)$. Moreover, condition (iii) implies that the collection of balls $\tau B_1, \ldots, \tau B_N$ has *bounded overlap*: no point in $X$ lies in more than $M$ of these balls, where $M = M(\tau, C_d)$.

Define a linear map $\Phi : N^{1,p}(B) \to \mathbb{R}^N$ by

$$\Phi(f) = \left(\mu(B_1)^{1/p} f_{B_1}, \ldots, \mu(B_N)^{1/p} f_{B_N}\right).$$

The result will follow easily once we prove that

$$\int_{\frac{3}{4}B} |f|^p \, d\mu \leq C\Phi(f) \tag{2.5.3}$$

for each $f \in E$ with $C$ depending only on the data of $X$.

To see (2.5.3), note that

$$\int_{\frac{3}{4}B} |f|^p \, d\mu \leq \sum_{i=1}^{N} \int_{B_i} |f|^p \, dx \leq 2^p \left( \sum_{i=1}^{N} \int_{B_i} |f - f_{B_i}|^p \, dx + \sum_{i=1}^{N} \mu(B_i) \left| ⨍_{B_i} f \right|^p \right).$$

Using Remark 1.3.4(2), we see that

$$\int_{\frac{3}{4}B} |f|^p \, d\mu \leq 2^p C' \sum_{i=1}^{N} (sr)^p \int_{\tau B_i} \rho_f^p \, d\mu + 2^p \|\Phi(f)\|_p^p,$$

where we use the norm $\|x\|_p := \left(\sum_{i=1}^N |x_i|^p\right)^{1/p}$ for $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$. Thus

$$\begin{aligned}
\int_{\frac{3}{4}B} |f|^p \, d\mu &\leq (2s)^p MC' \cdot r^p \int_B \rho_f^p \, d\mu + 2^p \|\Phi(f)\|_p^p \\
&\leq (2Ks)^p MC' \int_{\frac{3}{4}B} |f|^p \, d\mu + 2^p \|\Phi(f)\|_p^p.
\end{aligned}$$

by (2.5.2). This establishes (2.5.3) if $s$ is chosen so small that $(2Ks)^p MC' \leq \frac{1}{2}$. □

## 2.6 Nontriviality of the minimal generalized upper gradient

In this section, we discuss the issue of when the minimal generalized upper gradient $\rho_f$ of a Lipschitz function $f : X \to \mathbb{R}$ is (in some sense) nontrivial. Note that if $X$ does not contain any rectifiable curves, then $\rho_f$ is always identically zero.

**Proposition 2.6.1 (Proposition 4.26 of [4]).** *Let $X = (X, d, \mu)$ be a doubling space which satisfies the weak $p$-Poincaré inequality. Then there exists a constant $C \geq 1$ depending only on the data of $X$ so that for each Lipschitz function $f : X \to \mathbb{R}$, we have*

$$\operatorname{Lip} f(x_0) \leq C\rho_f(x_0) \tag{2.6.2}$$

*for a.e. $x_0 \in X$.*

Recall that

$$\rho_f(x_0) \leq \operatorname{lip} f(x_0) \leq \operatorname{Lip} f(x_0)$$

for a.e. $x_0 \in X$ by Lemma 1.1.3.

Note that the exceptional set (the collection of points $x \in X$ for which (2.6.2) fails to hold) will in general depend on $f$ and $p$. In fact, the proof reveals that (2.6.2) holds for every *$p$-strong Lebesgue point* of $\rho_f$, that is, every point $x_0 \in X$ for which

$$\lim_{r \to 0} \fint_{B(x_0,r)} |\rho_f - \rho_f(x_0)|^p \, d\mu = 0. \tag{2.6.3}$$

*Proof.* Let $x_0$ be a point in $X$ for which (2.6.3) holds. For $p \geq 1$ and $R > 0$, define the *restricted $p$-maximal operator* $M_{p,R}$ on functions $h \in L^1_{\text{loc}}(X, \mu)$ by

$$M_{p,R}h(x) := \sup_{0<\rho<R} \Big( \fint_{B(x,\rho)} |h|^p \, d\mu \Big)^{1/p},$$

We note two properties of this operator:

- if $h \in L^1_{\text{loc}}(X, \mu)$, $R > 0$ and $c \in \mathbb{R}$, then $M_{p,R}(h + c) \leq M_{p,R}h + c$,
- $M_{p,R} : L^1_{\text{loc}}(X, \mu) \to \text{weak} - L^1_{\text{loc}}(X, \mu)$ in the following sense: there exist constants $C$ and $\kappa$ so that

$$\frac{\mu\{x \in B(x_0, r) : M_{p,R}h(x) > \lambda\}}{\mu(B(x_0, r))} \leq \frac{C}{\lambda^p}(R/r)^\kappa \fint_{B(x_0,2R)} |h|^p \, d\mu \tag{2.6.4}$$

  for $x_0 \in X$, $h \in L^1_{\text{loc}}(X, \mu)$ and $R \geq r > 0$, $\lambda > 0$.

Applying (2.6.4) with $h(x) = \rho_f(x) - \rho_f(x_0)$ and using (2.6.3), we see that

$$\lim_{r \to 0} \frac{\mu(B(x_0, r) \cap F_\lambda^{\sigma r})}{\mu(B(x_0, r))} = 0$$

for each $\lambda > 0$ and $\sigma > 1$, where $F_\lambda^R := \{x \in X : M_{p,R}(\rho_f - \rho_f(x_0)) > \lambda\}$.

For $0 < \epsilon < 1$, choose $\lambda = \rho_f(x_0) + \epsilon$. Pick $0 < r < r_0$, where $r_0 = r_0(x_0)$ will be determined later, and let $x$ satisfy $d(x, x_0) = r$. Since $(X, \mu)$ is a doubling space, there exists a constant $c(\epsilon) > 0$ so that

$$\frac{\min\{\mu(B(x, \epsilon r)), \mu(B(x_0, \epsilon r))\}}{\mu(B(x_0, r))} \geq c(\epsilon)$$

(see p. 2). If $r_0$ is chosen so small that

$$\mu(B(x_0, r) \cap F_\lambda^{\sigma r}) < c(\epsilon)\mu(B(x_0, r))$$

for all $r < r_0$, then $B(x_0, \epsilon r) \setminus F_\lambda^{\sigma r}$ and $B(x, \epsilon r) \setminus F_\lambda^{\sigma r}$ are nonempty. Choose $y_0 \in B(x_0, \epsilon r) \setminus F_\lambda^{\sigma r}$ and $y \in B(x, \epsilon r) \setminus F_\lambda^{\sigma r}$. Using the triangle inequality, we estimate

$$\frac{|f(x_0) - f(x)|}{r} \leq 2L\epsilon + \frac{|f(y_0) - f(y)|}{r},$$

where $L$ is the Lipschitz constant of $f$.

The satisfaction of the Poincaré inequality (1.3.3) for a pair of functions $f \in L^1_{\mathrm{loc}}(X)$ and $\rho \geq 0$ implies the following pointwise estimate (*Hajłasz's estimate*) for a.e. $a, b \in X$:

$$|f(a) - f(b)| \leq Cd(a, b)(M_{p,R}\rho(a) + M_{p,R}\rho(b)), \tag{2.6.5}$$

where $C$ is a constant depending only on the data of $X$ and $R = 2\tau d(a, b)$. See, for example Theorem 3.2 of [6]. Applying this with $a = y_0$, $b = y$ and $\rho = \rho_f$, we see that

$$|f(y_0) - f(y)| \leq Cd(y_0, y)(M_{p,6\tau r}\rho_f(y_0) + M_{p,6\tau r}\rho_f(y)).$$

Thus

$$\begin{aligned}\frac{|f(x_0) - f(x)|}{r} &\leq 2L\epsilon + C(M_{p,6\tau r}\rho_f(y_0) + M_{p,6\tau r}\rho_f(y)) \\ &\leq 2L\epsilon + C\lambda \leq C\rho_f(x_0) + (2L + C)\epsilon\end{aligned}$$

which implies

$$\frac{L(x_0, f, r)}{r} \leq C\rho_f(x_0) + (2L + C)\epsilon.$$

The result follows in the limit as $r$ and then $\epsilon$ tend to zero. □

## 2.7 Vectors of Lipschitz functions

We now want to extend the results of the previous section to cover the case of *vectors* of functions, i.e. Lipschitz maps $F = (f_1, \ldots, f_k) : X \to \mathbb{R}^k$, $k \geq 1$. We will do this by postcomposing with a linear functional $\lambda : \mathbb{R}^k \to \mathbb{R}$ and applying what we already know to the real-valued function $\lambda \circ F$. The crucial estimate (2.7.2) indicates (roughly speaking) that the construction is continuous in $\lambda$.

Let $k \geq 1$. We identify $\mathrm{Hom}(\mathbb{R}^k, \mathbb{R})$ with $\mathbb{R}^k$ via the canonical identification $\lambda \leftrightarrow (\lambda_1, \dots, \lambda_k)$, where $\lambda_i = \langle \lambda, e_i \rangle$, $i = 1, \dots, k$. We give $\mathrm{Hom}(\mathbb{R}^k, \mathbb{R})$ the $L^2$ norm $\|\lambda\| := (\sum_{i=1}^k |\lambda_i|^2)^{1/2}$.

An element $\lambda$ will be called *rational* if $\lambda_i \in \mathbb{Q}$ for each $i$.

**Proposition 2.7.1 (Lemma 4.32 in [4]).** *Let $X = (X, d, \mu)$ be a Vitali space. Let $F : X \to \mathbb{R}^k$ be Lipschitz with Lipschitz constant $L$. Fix $1 < p < \infty$. Then $\cap_\lambda \mathrm{AGL}_p(\langle \lambda, F \rangle)$ is a set of full measure in $X$. Moreover,*

$$\|\rho_{\langle \lambda, F \rangle} - \rho_{\langle \lambda', F \rangle}\|_\infty \leq L\|\lambda - \lambda'\|. \tag{2.7.2}$$

Recall that $\mathrm{AGL}_p(f)$ denotes the set of points in $X$ at which $f : X \to \mathbb{R}$ is asymptotically generalized linear; see section 2.3.

*Proof.* By Theorem 2.3.2,

$$\bigcap_{\substack{\lambda \\ \lambda \text{ is rational}}} \mathrm{AGL}_p(\langle \lambda, F \rangle)$$

is a set of full measure in $X$. Since asymptotic generalized linearity is preserved under (global) uniform convergence, the first claim in Proposition 2.7.1 will follow from (2.7.2).

To prove (2.7.2), note that for any $x \in X$,

$$|\rho_{\langle \lambda, F \rangle}(x) - \rho_{\langle \lambda', F \rangle}(x)| \leq |\rho_{\langle \lambda, F \rangle - \langle \lambda', F \rangle}(x)| = |\rho_{\langle \lambda - \lambda', F \rangle}(x)| \leq \mathrm{Lip}\, \langle \lambda - \lambda', F \rangle(x)$$

which is clearly bounded by $L\|\lambda - \lambda'\|$. ☐

In a similar way, assuming that $(X, d, \mu)$ is doubling and satisfies the weak $p$-Poincaré inequality, we can show that there exists a set of full measure in $X$ on which

$$\rho_{\langle \lambda, F \rangle}(x) \leq \mathrm{lip}\, \langle \lambda, F \rangle(x) \leq \mathrm{Lip}\, \langle \lambda, F \rangle(x) \leq C \rho_{\langle \lambda, F \rangle}(x)$$

for all $\lambda \in \mathrm{Hom}(\mathbb{R}^k, \mathbb{R})$. See Lemma 4.35 of [4].

A crucial role in the proof of Cheeger's theorem is played by the following result, which states that the target dimension $k$ is bounded from above if the maps $\langle \lambda, F \rangle$, $\lambda \neq 0$, satisfy a certain nontriviality condition.

**Proposition 2.7.3 (Lemma 4.37 of [4]).** *Let $X = (X, d, \mu)$ be a doubling space satisfying the weak $p$-Poincaré inequality. Let $F : X \to \mathbb{R}^k$ be Lipschitz, $k \geq 1$. If there exists a point $x_0 \in X$ for which $\langle \lambda, F \rangle$ is asymptotically generalized linear at $x_0$ for each $\lambda \in \mathrm{Hom}(\mathbb{R}^k, \mathbb{R})$ and $\rho_{\langle \lambda, F \rangle}(x_0) > 0$ for each $\lambda \neq 0$, then $k \leq N$ where $N$ is a constant depending only on the data of $X$.*

*Proof.* Without loss of generality we may assume $F(x_0) = 0$. We will show that there exists a ball $B = B(x_0, r)$ so that (2.5.2) holds for all functions $f = \langle \lambda, F \rangle$, $\lambda \in \mathrm{Hom}(\mathbb{R}^k, \mathbb{R})$. The conclusion will then follow from Proposition 2.5.1.

Let $S^{k-1}$ denote the unit sphere in $\mathrm{Hom}(\mathbb{R}^k, \mathbb{R})$ and let

$$\epsilon = \delta \cdot L^{-1} \cdot \inf\{\rho_{\langle \lambda, F\rangle}(x_0) : \lambda \in S^{k-1}\}$$

where $L$ is a Lipschitz constant for $F$ and $\delta > 0$ is a (small) constant to be chosen later. Note that $\epsilon > 0$ since the mapping $\lambda \mapsto \rho_{\langle \lambda, F\rangle}(x_0)$ is positive and continuous (in fact Lipschitz) as a real-valued function on the compact set $S^{k-1}$ by (2.7.2).

Let $\lambda^1, \ldots, \lambda^m$ be an $\epsilon$-dense collection in $S^{k-1}$. Since this collection is finite, we may choose $r_0 > 0$ so that

$$\frac{1}{2}\rho_{\langle \lambda^j, F\rangle}(x_0) \le \left(\fint_{B(x_0,r)} \rho^p_{\langle \lambda^j, F\rangle}\, d\mu\right)^{1/p} \le \frac{C}{r}\left(\fint_{B(x_0,\frac{3}{4}r)} |\langle \lambda^j, F\rangle|^p\, d\mu\right)^{1/p} \tag{2.7.4}$$

for all $r < r_0$ and $j = 1, \ldots, m$. Here the first inequality follows since $x_0$ is a Lebesgue point of the function $\rho^p_{\langle \lambda^j, F\rangle}$ while the second inequality follows from Corollary 2.4.10.

Fix $0 < r < r_0$ and let $B = B(x_0, r)$. For ease of notation, we define

$$W(\lambda) := \left(\fint_B \rho^p_{\langle \lambda, F\rangle}\, d\mu\right)^{1/p} \quad \text{and} \quad Y(\lambda) := \left(\fint_{\frac{3}{4}B} |\langle \lambda, F\rangle|^p\, d\mu\right)^{1/p}.$$

Thus the second inequality in (2.7.4) states that $rW(\lambda^j) \le C\,Y(\lambda^j)$ for all $j$. Our goal is to show that for some (slightly larger) constant $C'$, we have

$$rW(\lambda) \le C'\,Y(\lambda) \tag{2.7.5}$$

for all $\lambda \in S^{k-1}$ (and hence all $\lambda \ne 0$, since (2.7.5) is scale-invariant in $\lambda$).

Let $\lambda \in S^{k-1}$ and choose $j$ so that $\|\lambda - \lambda^j\| \le \epsilon$. First, we use the Cauchy-Schwartz inequality to estimate

$$|Y(\lambda - \lambda^j)| \le \|\lambda - \lambda^j\| \left(\fint_{\frac{3}{4}B} \|F\|^p\, d\mu\right)^{1/p} \le \epsilon \left(\fint_{\frac{3}{4}B} \|F - F(x_0)\|^p\, d\mu\right)^{1/p} \le L\epsilon r,$$

where $\|F(x)\|$ denotes the $L^2$-norm of $F(x) \in \mathbb{R}^k$. By our choice of $\epsilon$ and $r_0$,

$$L\epsilon \le \delta\, \rho_{\langle \lambda^j, F\rangle}(x_0) \le 2\delta\, W(\lambda^j)$$

and so Minkowski's inequality implies that

$$r\,W(\lambda^j) \le C\,Y(\lambda^j) \le C\,Y(\lambda) + CY(\lambda - \lambda^j) \le C\,Y(\lambda) + 2C\delta \cdot r\,W(\lambda^j),$$

which in turn implies that

$$r\,W(\lambda^j) \le 2C\,Y(\lambda)$$

if $\delta = 1/(4C)$. To complete the proof of (2.7.5), we note that $|W(\lambda) - W(\lambda^j)| \le L\epsilon$ by (2.7.2) and so

$$r\,W(\lambda) \le r(W(\lambda^j) + L\epsilon) \le r(W(\lambda^j) + 2\delta W(\lambda^j)) \le C\,Y(\lambda).$$

□

## 2.8 Existence of coordinate functions and differentiability almost everywhere of Lipschitz functions

We come now to the main result of section 4 of [4], which gives a version of the Rademacher differentiation theorem which holds on any doubling metric measure space satisfying the weak $p$-Poincaré inequality for some $p > 1$.

**Theorem 2.8.1 (Theorem 4.38 of [4]).** *Let $X = (X, d, \mu)$ be a doubling space satisfying the weak p-Poincaré inequality, $p > 1$. Then*

$$X = \bigcup_{\alpha} U_\alpha \cup Z,$$

*where $\mu(U_\alpha) > 0$ for all $\alpha$ and $\mu(Z) = 0$, and to each $\alpha$, there correspond real-valued Lipschitz functions $x_1^\alpha, \ldots, x_k^\alpha$ on $U_\alpha$. Here*

$$k = k(\alpha) \leq N,$$

*where $N$ is a fixed constant which depends only on the data of $X$. The collection of "coordinate charts" $(U_\alpha, x_1^\alpha, \ldots, x_k^\alpha)$ satisfies the following properties:*

*(i) $x_1^\alpha, \ldots, x_k^\alpha$ are linearly independent as functions on $U_\alpha$, i.e. if $\mathbf{X}^\alpha = (x_1^\alpha, \ldots, x_k^\alpha) : U_\alpha \to \mathbb{R}^k$ then $\langle \lambda, \mathbf{X}^\alpha \rangle \equiv 0$ on $U_\alpha$ if and only if $\lambda = 0$;*

*(ii) for each $\alpha$ and each $\lambda \in \mathrm{Hom}(\mathbb{R}^k, \mathbb{R})$, we have $U_\alpha \subset \mathrm{AGL}_p(\langle \lambda, \mathbf{X}^\alpha \rangle)$; moreover, for each $x_0 \in U_\alpha$ and $\lambda \neq 0$, we have $\rho_{\langle \lambda, \mathbf{X}^\alpha \rangle}(x_0) > 0$;*

*(iii) $k$ is the "maximal" integer for which (i) and (ii) hold.*

Note that condition (i) follows from condition (ii).

The statement in (iii) requires some clarification. If $V$ is a subset of $X$ of positive $\mu$-measure, define $k(V)$ to be the supremum of the values $k$ for which there exist real-valued Lipschitz functions $\varphi_1, \ldots, \varphi_k$ on $V$ satisfying (i) and (ii). Clearly, if $W$ is a subset of $V$ which also has positive $\mu$-measure, then $k(W) \geq k(V)$ (just restrict the functions $\varphi_1, \ldots, \varphi_k$ to $W$). Let us say that $V$ is *saturated* if $k(W) = k(V)$ for all such sets $W$. Then (iii) can be restated as saying that each of the sets $U_\alpha$ is saturated. Note that this is easy to guarantee in the situation of Theorem 2.8.1. Indeed, suppose we have found a decomposition $X = \cup_\alpha U_\alpha \cup Z$ satisfying (i) and (ii). If any of the sets $U_\alpha$ is not saturated, we may decompose it further into subsets on which the value of $k(\alpha)$ is increased by (at least) one. Note that this process must terminate since we have an *a priori* finite upper bound for the values $k(\alpha)$ via Proposition 2.7.3.

The proof of the following easy measure-theoretic fact is left to the reader.

**Lemma 2.8.2.** *Suppose that $\mathcal{P}$ is a property of measurable sets in a $\sigma$-finite measure space $(X, \mu)$. Assume that every measurable set $A \subset X$ with $\mu(A) > 0$ has a subset $B \subset A$ with $\mu(B) > 0$ so that $B$ has property $\mathcal{P}$. Then*

$$X = \bigcup_{k=1}^{\infty} U_k \cup Z$$

*where $U_k$ has property $\mathcal{P}$ for each $k$ and $\mu(Z) = 0$.*

We begin the proof of Theorem 2.8.1 by showing that any set of positive measure contains a subset of positive measure on which a single function satisfying 2.8.1(ii) can be found. See [4, Lemma 4.31].

**Lemma 2.8.3.** *Let $X$ be as in Theorem 2.8.1 and let $A \subset X$ with $\mu(A) > 0$. Then there exists a Lipschitz function $u : A \to \mathbb{R}$ so that $\rho_u(x) > 0$ for all $x \in C$, where $C \subset \mathrm{AGL}_p(u)$ with $\mu(C) > 0$.*

*Proof.* Let $x_0 \in A$ be a point of density of $A$ and set $u(x) = d(x, x_0)$. For a radius $r > 0$, we denote by $B_r$ the ball $B(x_0, r)$.

If the conclusion is false, then it must be the case that $\rho_u(x) = 0$ a.e. in $A$. We use the Poincaré inequality to deduce that

$$\begin{aligned} ⨍_{B_r} |u - u_{B_r}|\, d\mu &\le Cr \left( ⨍_{B_r} \rho_u^p \, d\mu \right)^{1/p} \\ &= Cr \left( \frac{1}{\mu(B_r)} \int_{B_r \setminus A} \rho_u^p \, d\mu \right)^{1/p} \le Cr \left( \frac{\mu(B_r \setminus A)}{\mu(B_r)} \right)^{1/p} \end{aligned}$$

since $\rho_u \le 1$. Since $x_0$ is a point of density of $A$, we conclude that $r^{-1} ⨍_{B_r} |u - u_{B_r}|\, d\mu \to 0$ as $r \to 0$. However, one easily verifies that

$$⨍_{B_r} |u - u_{B_r}|\, d\mu \ge cr \tag{2.8.4}$$

for all small $r > 0$, where $c > 0$ depends only on the doubling constant of $X$. □

**Remark 2.8.5.** Although the constant $c$ in (2.8.4) depends only on the doubling constant, it is necessary to use the Poincaré inequality in the proof of (2.8.4). In fact, the Poincaré inequality implies that *small spheres are nonempty*: for all $x_0 \in X$ and sufficiently small $r$, the set of points $x \in X$ for which $d(x, x_0) = r$ is nonempty. From this we deduce that the following *decay property*: there exist positive constants $\delta$ and $\epsilon$ so that for all $x_0 \in X$ and sufficiently small $r$, we have $\mu(B_{\delta r}) \le (1-\epsilon)\mu(B_r)$. Thus

$$u_{B_r} = ⨍_{B_r} u \, d\mu \ge \delta r \, \frac{\mu(B_r \setminus B_{\delta r})}{\mu(B_r)} \ge \delta \epsilon r$$

and so

$$⨍_{B_r} |u - u_{B_r}|\, d\mu \ge \frac{\mu(B_{\frac{1}{2}\delta\epsilon r})}{\mu(B_r)} \left( u_{B_r} - \frac{1}{2}\delta\epsilon r \right) \ge c(\delta\epsilon/2)^{\kappa+1} r.$$

Here we have used the volume decay property of the doubling measure $\mu$ (see p. 2).

*Proof of Theorem 2.8.1.* Let $\mathcal{P}_1$ be the following property for a set $U \subset X$: the $\mu$-measure of $U$ is positive and there exists a Lipschitz function $u : U \to \mathbb{R}$ so that almost every point of $U$ is a point of asymptotic generalized linearity of $u$ and $\rho_u > 0$ a.e. in $U$. Using Lemmas 2.8.3 and 2.8.2, we deduce that $X$ can be written as a union of sets $U'_\alpha$, together with a $\mu$-null set, so that each of the sets $U'_\alpha$ has property $\mathcal{P}_1$.

The proof is completed by taking $k$ maximal for the two properties (i) and (ii) (see the discussion following the statement of the theorem). □

As an immediate consequence of Theorem 2.8.1, we deduce the following version of Rademacher's theorem (Lipschitz functions are almost everywhere differentiable) on metric measure spaces which are doubling and satisfy the Poincaré inequality.

**Theorem 2.8.6 (Theorem 4.38(iii) and Corollary 4.41 of [4]).** *Let the space $X$ be as above and let $f : X \to \mathbb{R}$ be a Lipschitz function. For each $\alpha$, we have the following property: to almost every point $x_0 \in U_\alpha$, there corresponds a (unique) linear functional $\lambda = \lambda(x_0, f, \alpha) \in \mathrm{Hom}(\mathbb{R}^{k(\alpha)}, \mathbb{R})$ so that*

$$\rho_{-f+\langle\lambda,\mathbf{X}^\alpha\rangle}(x_0) = \lim_{\substack{x\to x_0\\ x\in U_\alpha}} \frac{|f(x) - f(x_0) - \langle\lambda, \mathbf{X}^\alpha(x) - \mathbf{X}^\alpha(x_0)\rangle|}{d(x, x_0)} = 0. \tag{2.8.7}$$

*We write $d^\alpha f(x_0) := \lambda(x_0, f, \alpha)$. For fixed $\alpha$, the map $x \mapsto d^\alpha f(x)$ is in $L^\infty(U_\alpha, \mu)$. At a.e. point $x_0 \in U_\alpha$, we have*

$$\rho_f(x_0) = \rho_{\langle d^\alpha f(x_0),\mathbf{X}^\alpha\rangle}(x_0). \tag{2.8.8}$$

Roughly speaking, (2.8.8) says that the "derivative" of the map $f$ at $x_0$ agrees with the "derivative" of its first-order Taylor approximation $\langle d^\alpha f(x_0), \mathbf{X}^\alpha\rangle$.

Note that we have restricted in (2.8.7) to $x \in U_\alpha$ since the function $\mathbf{X}^\alpha$ is *a priori* only defined on $U_\alpha$. However, it is not hard to see that if we extend $\mathbf{X}^\alpha$ to a Lipschitz map from $X$ to $\mathbb{R}$, then the limit in (2.8.7) is still zero if we allow $x$ to approach $x_0$ throughout $X$. Moreover, this fact is independent of what extension of $\mathbf{X}^\alpha$ we use.

*Proof.* Write $f_\alpha := f|_{U_\alpha}$. For any set $A \subset U_\alpha$ of positive measure, the functions $x_1^\alpha, \ldots, x_k^\alpha, -f_\alpha$ form a collection of $k+1$ Lipschitz functions on $A$. Since $k$ is maximal, we conclude that there exists a subset $C \subset A$ of positive measure so that for each $x_0 \in C$, there exists $\tilde\lambda(x_0) \in \mathrm{Hom}(\mathbb{R}^{k+1}, \mathbb{R})$ with

$$\rho_{\langle\tilde\lambda(x_0),(-f_\alpha,\mathbf{X}^\alpha)\rangle}(x_0) = 0. \tag{2.8.9}$$

Write $\tilde\lambda(x_0) = (\tilde\lambda_0, \tilde\lambda_1, \ldots, \tilde\lambda_k)$. Note that $\tilde\lambda_0 \neq 0$ by property (ii). Define

$$\lambda(x_0, f, \alpha) := (\tilde\lambda_1/\tilde\lambda_0, \ldots, \tilde\lambda_k/\tilde\lambda_0).$$

Then (2.8.9) and Proposition 2.6.1 together imply that

$$\mathrm{Lip}(-\tilde\lambda_0 f_\alpha + \sum_{i=1}^k \tilde\lambda_i x_i^\alpha)(x_0) = 0$$

from which (2.8.7) follows. The theorem now follows by again using Lemma 2.8.2, letting $\mathcal{P}$ be the following property of a set $A \subset U_\alpha$: to each point $x_0 \in A$, there corresponds a linear functional $\lambda$ so that (2.8.7) holds.

To show uniqueness of the quantity $\lambda(x_0, f, \alpha)$, we note that if (2.8.7) holds for two functionals $\lambda$ and $\lambda'$, then

$$\liminf_{x\to x_0} \frac{|\langle\lambda - \lambda', \mathbf{X}^\alpha(x) - \mathbf{X}^\alpha(x_0)\rangle|}{d(x, x_0)} = 0$$

which implies that $\rho_{\langle\lambda-\lambda',\mathbf{X}^\alpha\rangle}(x_0) = 0$ and so $\lambda = \lambda'$ by property (ii).

Finally, to show (2.8.8), we note that for any $\lambda \in \mathrm{Hom}(\mathbb{R}^k, \mathbb{R})$, we have

$$|\rho_f(x_0) - \rho_{\langle\lambda,\mathbf{X}^\alpha\rangle}(x_0)| \le \rho_{f-\langle\lambda,\mathbf{X}^\alpha\rangle}(x_0)$$

for a.e. $x_0 \in U_\alpha$. Equation (2.8.8) follows upon choosing $\lambda = d^\alpha f(x_0)$ and using (2.8.7).

Boundedness of the map $x \mapsto d^\alpha f(x)$ is clear as $f$ is Lipschitz. In order to show that this map is $L^\infty$, it thus suffices to prove that it is measurable. □

**Lemma 2.8.10.** *For each $\alpha$, the map $x \mapsto d^\alpha f(x)$ is a measurable function.*

*Proof.* It suffices to show that the set

$$E_\Omega := \{x \in U_\alpha : d^\alpha f(x) \in \Omega\}$$

is relatively Borel in $U_\alpha$ for each open set $\Omega \subset \mathbb{R}^{k(\alpha)}$. Note that $E_\Omega$ consists of those points $x \in U_\alpha$ for which $\mathrm{Lip}(-f + \langle\lambda, \mathbf{X}^\alpha\rangle)(x_0) = 0$ for some $\lambda \in \Omega$.

For a general Lipschitz function $h : U_\alpha \to \mathbb{R}$, define

$$h_\delta(x) := \delta^{-1} L(x, h, \delta) = \delta^{-1} \sup_{\substack{y \\ d(x,y)\le\delta}} |h(x) - h(y)|.$$

Then $h_\delta \to \mathrm{Lip}\, h$ as $\delta \to 0$. Moreover, if $x, x' \in X$ satisfy $d(x, x') \le \eta$, then

$$\delta\, h_\delta(x) \le (\delta + \eta) h_{\delta+\eta}(x'). \tag{2.8.11}$$

For $\delta, \epsilon > 0$, let $G_\delta(\epsilon)$ denote the collection of points $x \in X$ for which $h_\beta(x) < \epsilon$ for some $\beta < \delta$. We claim that $U_\delta(\epsilon)$ is a relatively open set in $U_\alpha$. Indeed, let $x \in G_\delta(\epsilon)$ and choose $\beta < \delta$ so that $h_\beta(x) < \epsilon$. Choose $\eta < \beta$ so that

$$\frac{\beta}{\beta - \eta} h_\beta(x) < \epsilon.$$

If $x' \in U_\alpha \cap B(x, \eta)$, then $(\beta - \eta) h_{\beta-\eta}(x') \le \beta h_\beta(x)$ by (2.8.11) and so $h_{\beta-\eta}(x') < \epsilon$ and $x' \in B_\delta(\epsilon)$.

Returning to the proof of the lemma, let $G^\lambda_\delta(\epsilon) \subset U_\alpha$ be the set constructed as above for the map $h := -f + \langle\lambda, \mathbf{X}^\alpha\rangle$. By the above remarks, $G^\lambda_\delta(\epsilon)$ is relatively Borel in $U_\alpha$. One easily verifies that

$$E_\Omega = \bigcup_{\lambda\in\Omega} \bigcap_{m=1}^{\infty} \limsup_{n\to\infty} G^\lambda_{1/n}(1/m)$$

where $\limsup_{n\to\infty} A_n = \cap_k \cup_{n\ge k} A_n$ denotes the set of points which lie in infinitely many of the sets $A_n$. Thus $E_\Omega$ is relatively Borel in $U_\alpha$ and the proof is complete. □

We next study how the quantity $d^\alpha f(x_0)$ varies as a function of $\alpha$. Note that for a fixed $\alpha$, the $k$-tuple $d^\alpha x_1^\alpha, \ldots, d^\alpha x_k^\alpha$ forms a basis of $\mathrm{Hom}(\mathbb{R}^k, \mathbb{R})$; in fact, using the canonical identification, we have $d^\alpha x_i^\alpha = e_i$, the standard $k$th basis vector. If we now consider a pair of indices $\alpha, \beta$, we see that the collection $(d^\beta x_i^\alpha)_{i=1}^k$ is also a basis at a.e. point of the intersection $U_\alpha \cap U_\beta$. In particular, we see that if $\mu(U_\alpha \cap U_\beta) > 0$, then $k(\alpha) = k(\beta) = k$ by conditions (ii) and (iii) and we can write

$$d^\beta x_i^\alpha = M^{\alpha\beta} \cdot d^\alpha x_i^\alpha$$

where $M^{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{R}^{k \times k}$. The collection of such matrices satisfies the usual cocycle conditions: $(M^{\alpha\beta})^{-1} = M^{\beta\alpha}$ and

$$M^{\alpha\gamma} = M^{\alpha\beta} M^{\beta\gamma}$$

a.e. on the intersection $U_\alpha \cap U_\beta \cap U_\gamma$.

At each point $x_0 \in U_\alpha$, we introduce a norm $|\cdot|_{\alpha, x_0}$ on $\mathrm{Hom}(\mathbb{R}^{k(\alpha)}, \mathbb{R})$ as follows:

$$|\lambda|_{\alpha, x_0} = \rho_{\langle \lambda, \mathbf{X}^\alpha \rangle}(x_0).$$

The fact that this is a norm follows from Theorem 2.8.1(ii). By (2.8.8), we have

$$\rho_f(x_0) = |d^\alpha f(x_0)|_{\alpha, x_0}. \tag{2.8.12}$$

We have now essentially completed the proof of parts (i) and (ii) of Theorem 2.1.1. At each point $x \in U_\alpha$, we have the Banach space

$$F_x = (\mathrm{Hom}(\mathbb{R}^{k(\alpha)}, \mathbb{R}), |\cdot|_{\alpha, x})$$

and the dimensions of these Banach spaces are uniformly bounded by some constant $N$ depending only on the data of $X$. These Banach spaces combine to give a finite dimensional $L^\infty$ vector bundle $\mathbf{F}$ over $X$, which we call the *generalized cotangent bundle* and denote by $T^*X$. One then defines a derivation operator

$$d : \mathcal{L}ip_{\mathrm{loc}}(X) \to \Gamma(T^*X) \tag{2.8.13}$$

by $df(x) = d^\alpha f(x)$ whenever $x \in U_\alpha$; we call $df$ the *Cheeger differential* of $f$. Theorem 2.1.1(ii) follows from (2.8.12) and shows that the resulting $D$-structure is nontrivial. We leave as an exercise to the reader to show that $d$ is a derivation (Definition 1.2.1(iii)).

We now have two different definitions for the Sobolev class of functions on $X$: the space $H^{1,p}(X)$ with norm $|\cdot|_{H^{1,p}(X)}$ defined in section 1.4 (equivalently in 2.2.3) and the Newtonian space $N^{1,p}(X)$ with norm $|\cdot|_{N^{1,p}(X)}$. As mentioned in 2.2.3, these two spaces are isometrically isomorphic when $p > 1$. The fact that $H^{1,p}(X)$ embeds into $N^{1,p}(X)$ is clear from Theorem 2.1.1(ii). For the converse, see [15]. Henceforth, for simplicity, we denote the norm in this space by $|\cdot|_{1,p}$.

The reflexivity of this space when $p > 1$ now follows via the discussion on p. 460 of [4], which we briefly review. First, we recall that any norm $|\cdot|$ on a $k$-dimensional vector space is comparable (with constants depending on $k$) to an inner product (and hence uniformly convex) norm $\|\cdot\|$. It thus follows that the space $H^{1,p}(X)$ has a canonical uniformly convex norm $\|\cdot\|_{1,p}$ which is comparable to $|\cdot|_{1,p}$ and is obtained by integrating with respect to the measure $\mu$ the pointwise norm obtained by replacing each of the norms $|\cdot|_{\alpha,x}$ on the space $\mathrm{Hom}(\mathbb{R}^{k(\alpha)}, \mathbb{R})$ by the associated inner product norm. The reflexivity of the space $H^{1,p}(X)$ follows immediately.

We will not discuss the proof of Theorem 2.1.1(iii). See sections 5 and 6 of [4]. Instead, we now briefly illustrate the theory by returning to several of the examples discussed in section 1.3.

**Examples 2.8.14.** (1) The first Heisenberg group $H_1 = (\mathbb{R}^3, d_X)$ (see Example 1.2.2(f)) has topological dimension three and Hausdorff dimension four. The *horizontal distribution* is generated by the two-dimensional Lie subalgebra of $T_0 H_1$ spanned by the vector fields $X_1$ and $X_2$. The generalized cotangent bundle constructed above coincides with the dual of this horizontal distribution and hence also has dimension two (that is, the values $k = k(\alpha)$ described above are all equal to 2).

(2) The examples of Bourdon-Pajot and Laakso have topological dimension one and Hausdorff dimension strictly greater than one. For these spaces, one can show that the generalized cotangent bundle is again one-dimensional.

One can use these facts to show that none of these spaces – the first (or any nonabelian) Heisenberg group, the Bourdon-Pajot examples, and the Laakso examples – can be embedded into a Euclidean space via a bi-Lipschitz mapping; see sections 13 and 14 of Cheeger's paper. Near the end of section 4 of his paper, Cheeger makes the following conjecture:

**Conjecture 2.8.15.** *For any space $X$ satisfying the assumptions of Theorem 2.1.1 we have*

$$\mathcal{H}_{k(\alpha)}(\mathbf{X}^\alpha(U_\alpha)) > 0$$

*for each $\alpha$, where $\mathcal{H}_s$ denotes $s$-dimensional Hausdorff measure.*

If true, this result would imply that $X$ cannot be embedded into Euclidean space via a bi-Lipschitz mapping whenever the dimension of the generalized cotangent bundle $T^*X$ is strictly smaller than the Hausdorff dimension of $X$.

## 2.9 Mappings between metric spaces

This section is taken from Remarks 4.44 and 4.45 of [4] and also contains announcements of some additional results to be found in [10].

Let $X = (X, d, \mu)$ be a doubling space satisfying the weak $p$-Poincaré inequality for some $p > 1$. Let $Y = (Y, d')$ be another metric space (for now, we make no further assumptions on $Y$). If $F : X \to Y$ and $f : Y \to \mathbb{R}$ are Lipschitz functions, then $f \circ F : X \to \mathbb{R}$ is also Lipschitz and so by Theorem 2.8.6 the differentials $d^\alpha(f \circ F)$ are defined on sets $U_\alpha$ which cover $\mu$-almost all of $X$. If now $Y$ is given a measure $\nu$ so that $(Y, d', \nu)$ is also doubling and satisfies the weak $p$-Poincaré inequality, then the differentials $d^\beta f$ are also defined on sets $V_\beta$ covering $\nu$-almost all of $Y$. At a.e. point $x_0 \in F^{-1}(V_\beta) \cap U_\alpha$, there exists a unique linear map

$$D_F^{\alpha\beta}(x_0)^T : \mathrm{Hom}(\mathbb{R}^{k(\beta)}, \mathbb{R}) \to \mathrm{Hom}(\mathbb{R}^{k(\alpha)}, \mathbb{R}),$$

called the (transposed) *Jacobian matrix* of $F$ at $x_0$, which satisfies the relation

$$D_F^{\alpha\beta}(x_0)^T \cdot d^\beta f(y_0) = d^\alpha (f \circ F)(x_0), \tag{2.9.1}$$

where $y_0 = F(x_0)$. The existence and uniqueness of this mapping are obvious. Indeed, taking $f = x_j^\beta$, $j = 1, \ldots, k(\beta)$, in (2.9.1) and using the canonical identification of $\mathrm{Hom}(\mathbb{R}^k, \mathbb{R})$ with $\mathbb{R}^k$ we see that the $j$th column of the $(k(\alpha) \times k(\beta))$ matrix representing $D_F^{\alpha\beta}(x_0)^T$ is given by the vector in $\mathbb{R}^{k(\alpha)}$ corresponding to $d^\alpha(x_j^\beta \circ F)(x_0)$.

Note, however, that the transposed Jacobian matrix is only defined on the set

$$\bigcup_{\alpha,\beta} F^{-1}(V_\beta) \cap U_\alpha$$

in $X$. We would like to guarantee that this is a set of full measure in $X$ but in general this is not true. In order to ensure that this is the case, we must further assume that $\nu(A) = 0$ implies $\mu(F^{-1}A) = 0$, that is, that the *push-forward* measure $F_*\mu$ is absolutely continuous with respect to $\nu$. When this is the case, we have a natural induced map $F^* : T^*Y \to T^*X$ satisfying

$$F^*(df) = d(f \circ F), \tag{2.9.2}$$

where $d$ is the Cheeger differential. Expressed in coordinate charts $U_\alpha \subset X$ and $V_\beta \subset Y$, $F^*$ is just the transposed Jacobian matrix $D_F^{\alpha\beta}(x_0)^T$ and (2.9.2) becomes (2.9.1).

Now assume that $X = (X, d, \mu)$ and $Y = (Y, d', \nu)$ are locally compact metric measure spaces which are *Ahlfors $Q$-regular* for some $Q > 1$. Recall that Ahlfors $Q$-regularity for a measure $\mu$ on $X$ means that there exist constants $0 < c \le C < \infty$ so that

$$cr^Q \le \mu(B(x, r)) \le Cr^Q$$

for all points $x \in X$ and $r < \operatorname{diam} X$. Then $(X, \mu)$ and $(Y, \nu)$ are both doubling spaces in the sense of Definition 1.1.4. Let $F : X \to Y$ be a *quasisymmetric* homeomorphism, that is, assume that there exists an increasing homeomorphism $\eta : [0, \infty) \to [0, \infty)$ so that

$$\frac{d'(F(x), F(y))}{d'(F(x), F(z))} \le \eta\left(\frac{d(x, y)}{d(x, z)}\right)$$

for all $x, y, z \in X$. Assume also that $X$ satisfies the weak $p$-Poincaré inequality for some $1 < p \le Q$. We cite without proof the following results:

(1) If $p < Q$, then $Y$ satisfies the weak $p'$-Poincaré inequality for some $p' < Q$ which depends only on $p$, $\eta$, and the regularity data of $X$ and $Y$, see [12]. On the other hand, if $p = Q$ and $X$ is in addition *proper* (closed balls are compact) and pathwise connected, then $Y$ also satisfies the weak $Q$-Poincaré inequality. This follows by combining Corollary 1.6 of [17] with Corollary 5.8 and Theorem 5.12 of [8].

(2) Under the same assumptions as in (1), the push-forward measure $F_*\mu$ is absolutely continuous with respect to $\nu$ (when $p < Q$, this is Corollary 7.13 in [8] while when $p = Q$, it is Theorem 7.12 of [10]).

(3) For any $x_0 \in X$, the mapping $f_{x_0} : X \to \mathbb{R}$ given by

$$f_{x_0}(x) = d'(F(x), F(x_0))$$

is an element of the local Newtonian space $N^{1,Q}_{\text{loc}}(X)$, that is, $f_{x_0} \in N^{1,Q}(B)$ for all balls $B$ in $X$. Moreover, the minimal $Q$-weak upper gradient $\rho_{f_{x_0}} = \rho_F$ is independent of $x_0$. When $p < Q$, we in fact have $f_{x_0} \in \bigcup_{q>Q} N^{1,q}_{\text{loc}}(X)$ by Theorem 9.3 of [8]; the case $p = Q$ is Theorem 7.8 of [10].

By using this last fact in conjuction with Hajłasz's estimate (2.6.5), we deduce that there is a constant $C$ depending only on the data of $X$ and $Y$ and $\eta$ so that for all balls $B$ of radius $R$ in $X$ and all $x, y \in B$ we have

$$d'(F(x), F(y)) \le C d(x, y)(M_{Q,4\tau R}\rho_F(x) + M_{Q,4\tau R}\rho_F(y)).$$

For $j = 1, 2, \ldots$ define

$$E_j(B) = \{x \in B : M_{Q,4\tau R}\rho_F(x) \le j\}.$$

Then $F$ is Lipschitz with constant $C j$ when restricted to the set $E_j(B)$. A standard covering argument shows that

$$\mu(B \setminus E_j(B)) \le \frac{C}{j^Q} \int_{100\tau R} \rho_F^Q \, d\mu,$$

which tends to zero as $j \to \infty$. Thus the collection of sets $E_{jk} := E_j(B(x_0, k))$, where $x_0$ is a fixed basepoint in $X$, covers $\mu$-almost all of $X$. If $f : Y \to \mathbb{R}$ is any Lipschitz function, then $f \circ F$ is Lipschitz when restricted to any of the sets $E_{jk}$ and the discussion in the first part of this section is valid. We deduce from this the following result, which is contained in Remark 4.45 of [4] in the case $p < Q$ and Theorem 9.8 of [10] in the case $p = Q$.

**Proposition 2.9.3.** *Let $X = (X, d, \mu)$ and $Y = (Y, d', \nu)$ be Ahlfors $Q$-regular spaces, $Q > 1$, and let $F : X \to Y$ be a quasisymmetric homeomorphism. Assume either that (a) $X$ is locally compact and satisfies the weak $p$-Poincaré inequality for some $p = p(X) \in (1, Q)$, or (b) $X$ is proper and pathwise connected and satisfies the weak $Q$-Poincaré inequality. Then $Y$ also satisfies the same conditions. There is a natural induced map $F^* : T^*Y \to T^*X$ which satisfies*

$$F^*(df) = d(f \circ F),$$

*where $d$ is the operator given in (2.8.13).*

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