

MODULUS AND POINCARÉ INEQUALITIES ON NON-SELF-SIMILAR SIERPIŃSKI CARPETS

JOHN M. MACKAY, JEREMY T. TYSON, AND KEVIN WILDRICK

ABSTRACT. A carpet is a metric space homeomorphic to the Sierpiński carpet. We characterize, within a certain class of examples, non-self-similar carpets supporting curve families of nontrivial modulus and supporting Poincaré inequalities. Our results yield new examples of compact doubling metric measure spaces supporting Poincaré inequalities: these examples have no manifold points, yet embed isometrically as subsets of Euclidean space.

1. INTRODUCTION

Metric spaces equipped with doubling measures that support Poincaré inequalities (also known as *PI spaces*) are ideal environments for first-order analysis and differential geometry [20], [10], [21], [24], [26]. Extending the scope of this theory by verifying Poincaré inequalities on new classes of spaces is a problem of high interest and relevance. Previously, several classes of spaces have been shown to support Poincaré inequalities:

- compact Riemannian manifolds or noncompact Riemannian manifolds satisfying suitable curvature bounds [9],
- Carnot groups and more general sub-Riemannian manifolds equipped with Carnot-Carathéodory (CC) metric [20], [19], [22],
- boundaries of certain hyperbolic Fuchsian buildings, see Bourdon and Pajot [6],
- Laakso's spaces [30],
- linearly locally contractible manifolds with good volume growth [34].

These examples fall into two (overlapping) classes: examples for which the underlying topological space is a manifold, and abstract metric examples which admit no bi-Lipschitz embedding into any finite-dimensional Euclidean space. Such bi-Lipschitz nonembeddability follows from Cheeger's celebrated Rademacher-style differentiation theorem in PI spaces, as explained in [10, §14]. Euclidean bi-Lipschitz nonembeddability is known, for instance, for all nonabelian Carnot groups and other regular sub-Riemannian manifolds, as well as for the examples of Bourdon and Pajot [6] and Laakso [30].

The preceding dichotomy should not be taken too seriously. Nonabelian Carnot groups equipped with the CC metric, for instance, have underlying space which is a topological manifold, yet do not admit any Euclidean bi-Lipschitz embedding. On the other hand, it is certainly possible to construct Euclidean subsets with some nonmanifold points which are PI spaces. This can be done, for instance, by appealing to various gluing theorems for PI

Date: January 9, 2012.

2010 Mathematics Subject Classification. 30L99; 31E05; 28A80.

Key words and phrases. Sierpiński carpet, doubling measure, modulus, Poincaré inequality, Gromov-Hausdorff tangent cone.

JMM and JTT supported by US National Science Foundation Grant DMS-0901620. KW supported by Academy of Finland Grants 120972 and 128144.

spaces, see [20, Theorem 6.15] (reproduced as Theorem 2.2 in this paper) for a general result along these lines. However, the following question appears to have been unaddressed in the literature until now.

Question 1.1. Do there exist sets $X \subset \mathbb{R}^N$ (for some N) with **no** manifold points which are PI spaces when equipped with the Euclidean metric and some suitable measure?

In connection with Question 1.1 we recall the examples constructed by Heinonen and Hanson [17]. For each $n \geq 2$, these authors construct a compact, geodesic, Ahlfors n -regular PI space of topological dimension n with no manifold points. They suggest [17, p. 3380], but do not check, that their nonmanifold example admits a bi-Lipschitz embedding into some Euclidean space. Note that the question about embeddability of the Heinonen–Hansen example is not resolved by Cheeger’s work, since such example admits almost everywhere unique tangent cones coinciding with \mathbb{R}^n .

The examples of PI spaces due to Bourdon and Pajot [6] comprise a class of compact metric spaces arising as the Gromov boundaries of certain hyperbolic groups acting geometrically on Fuchsian buildings. Topologically, all of the Bourdon–Pajot examples are homeomorphic to the Menger sponge. It is well-known that ‘typical’ Gromov hyperbolic groups have Menger sponge boundaries. While examples of Gromov hyperbolic groups with Sierpiński carpet boundary do exist, it is not presently known whether any such boundary can verify a Poincaré inequality in the sense of Heinonen and Koskela.

Question 1.2. Do there exist PI spaces whose underlying space is homeomorphic to the Sierpiński carpet?

In this paper we answer Questions 1.1 and 1.2 affirmatively. We identify a new class of doubling metric measure spaces supporting Poincaré inequalities. Our main results are Theorem 1.5 and 1.6. Our spaces have no manifold points, indeed, they are all homeomorphic to the Sierpiński carpet. On the other hand, all of our examples arise as explicit subsets of the plane equipped with the Euclidean metric and the Lebesgue measure. These are the first examples of compact subsets of Euclidean space without interior that support Poincaré inequalities for the usual Lebesgue measure.

To fix notation and terminology we recall the notion of Poincaré inequality on a metric measure space as introduced by Heinonen and Koskela [20]. Let (X, d, μ) be a metric measure space, i.e., (X, d) is a metric space and μ is a Borel measure which assigns positive and finite measure to all open balls in X . A Borel function $\rho : X \rightarrow [0, \infty]$ is an *upper gradient* of a function $u : X \rightarrow \mathbb{R}$ if $|u(x) - u(y)| \leq \int_\gamma \rho ds$ whenever γ is a rectifiable curve joining x to y .

Definition 1.3 (Heinonen–Koskela). Fix $p \geq 1$. The space (X, d, μ) is said to support a *p -Poincaré inequality* if there exist constants $C, \lambda \geq 1$ so that for any continuous function $u : X \rightarrow \mathbb{R}$ with upper gradient $\rho : X \rightarrow [0, \infty]$, the inequality

$$(1.1) \quad \int_B \left| u - \int_B u d\mu \right| d\mu \leq C \operatorname{diam}(B) \left(\int_{\lambda B} \rho^p d\mu \right)^{1/p}$$

holds for every ball $B = B(x, r) \subset X$. Here we denote, for a subset $E \subset X$ of positive measure, the mean value of a function $u : E \rightarrow \mathbb{R}$ by

$$\int_E u d\mu = \frac{1}{\mu(E)} \int_E u d\mu.$$

To each sequence $\mathbf{a} = (a_1, a_2, \dots)$ consisting of reciprocals of odd integers strictly greater than one we associate a modified Sierpiński carpet $S_{\mathbf{a}}$ by the following procedure. Let $T_0 = [0, 1]^2$ be the unit square and let $S_{\mathbf{a},0} = T_0$. Consider the standard tiling of T_0 by essentially disjoint closed congruent subsquares of side length a_1 . Let \mathcal{T}_1 denote the family of such subsquares obtained by deleting the central (concentric) subsquare, and let $S_{\mathbf{a},1} = \cup\{T : T \in \mathcal{T}_1\}$. Again, let \mathcal{T}_2 denote the family of essentially disjoint closed congruent subsquares of each of the elements of \mathcal{T}_1 with side length $a_1 a_2$ obtained by deleting the central (concentric) subsquare from each square in \mathcal{T}_1 , and let $S_{\mathbf{a},2} = \cup\{T : T \in \mathcal{T}_2\}$. Continuing this process, we construct a decreasing sequence of compact sets $\{S_{\mathbf{a},m}\}_{m \geq 0}$ and an associated carpet

$$S_{\mathbf{a}} := \bigcap_{m \geq 0} S_{\mathbf{a},m}.$$

For example, when $\mathbf{a} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \dots)$, the set $S_{\mathbf{a}}$ is the classical Sierpiński carpet (Figure 1). For any \mathbf{a} , $S_{\mathbf{a}}$ is a compact, connected, locally connected subset of the plane without interior and with no local cut points. By a standard fact from topology, $S_{\mathbf{a}}$ is homeomorphic to the Sierpiński carpet.

For each $k \in \mathbb{N}$, we will denote by $S_{1/(2k+1)}$ the self-similar carpet $S_{\mathbf{a}}$ associated to the constant sequence $\mathbf{a} = (\frac{1}{2k+1}, \frac{1}{2k+1}, \frac{1}{2k+1}, \dots)$. For each k , the carpet $S_{1/(2k+1)}$ has Hausdorff dimension equal to

$$(1.2) \quad Q_k = \frac{\log((2k+1)^2 - 1)}{\log(2k+1)} = \frac{\log(4k^2 + 4k)}{\log(2k+1)} < 2$$

and is Ahlfors regular in that dimension.

The starting point for our investigations was the following well-known fact.

Proposition 1.4. *For each k , the carpet $S_{1/(2k+1)}$, equipped with Euclidean metric and Hausdorff measure in its dimension Q_k , does **not** support any Poincaré inequality.*

Several proofs for Proposition 1.4 can be found in the literature. Bourdon and Pajot [7] provide an elegant argument involving the mutual singularity of one-dimensional Lebesgue measure and the push forward of the Q_k -dimensional Hausdorff measure on $S_{1/(2k+1)}$ under projection to a coordinate axis. A different argument involving modulus computations can be found in the monograph by the first two authors [31].

In this paper, we study non-self-similar carpets $S_{\mathbf{a}}$ for which \mathbf{a} is not a constant sequence. We are primarily interested in the case when $S_{\mathbf{a}}$ has Hausdorff dimension two. It is easy

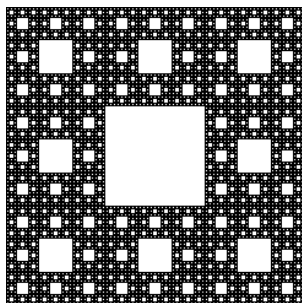


FIGURE 1. $S_{1/3}$

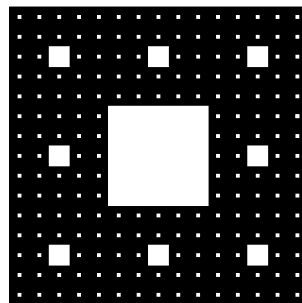


FIGURE 2. $S_{(1/3, 1/5, 1/7, \dots)}$

to see that this holds, for instance, if the sequence (a_m) of scaling ratios tends to zero, i.e., $\mathbf{a} \in c_0$. Figure 2 illustrates the set $S_{(1/3, 1/5, 1/7, \dots)}$.

Note that the left and right hand edges of $S_{\mathbf{a}}$ are separated by the generalized Cantor set

$$C_{\mathbf{a}} := S_{\mathbf{a}} \cap \left(\left\{ \frac{1}{2} \right\} \times [0, 1] \right).$$

This Cantor set will have positive length if and only if the length at each stage, $\prod_{j=1}^m (1 - a_j)$, remains bounded away from zero. After taking logarithms, this is seen to be equivalent to $\mathbf{a} \in \ell^1$.

In a similar fashion, we see that $\text{Area}(S_{\mathbf{a}}) = \mathcal{H}^2(S_{\mathbf{a}})$ is positive if and only if $\text{Area}(S_{\mathbf{a},m}) = \prod_{j=1}^m (1 - a_j^2)$ is bounded away from zero, i.e., $\mathbf{a} \in \ell^2$.

We equip $S_{\mathbf{a}}$ with the Euclidean metric d and the canonically defined measure μ arising as the weak limit of normalized Lebesgue measures on the precarpets $S_{\mathbf{a},m}$. For all \mathbf{a} , the measure μ is doubling. Under the assumption $\mathbf{a} \in \ell^2$, μ is Ahlfors 2-regular and is comparable (with constant depending only on $\|\mathbf{a}\|_2$) to the restriction of Lebesgue measure to $S_{\mathbf{a}}$. For these and other facts, see Proposition 3.1.

We now state our main theorems.

Theorem 1.5. *The carpet $(S_{\mathbf{a}}, d, \mu)$ supports a 1-Poincaré inequality if and only if $\mathbf{a} \in \ell^1$.*

Under the assumption of Theorem 1.5, the 1-modulus of all horizontal paths in $S_{\mathbf{a}}$ is easily seen to be positive. This fact follows from the usual Fubini argument, since the cut set $C_{\mathbf{a}}$ has positive length. The difficult part of the proof of Theorem 1.5 is the verification of the 1-Poincaré inequality. This is done using a theorem of Keith (Theorem 2.1) and a combinatorial procedure involving concatenation of curve families of positive 1-modulus.

Theorem 1.6. *The following are equivalent:*

- (a) $(S_{\mathbf{a}}, d, \mu)$ supports a p -Poincaré inequality for each $p > 1$,
- (b) $(S_{\mathbf{a}}, d, \mu)$ supports a p -Poincaré inequality for some $p > 1$,
- (c) $\mathbf{a} \in \ell^2$.

For $\mathbf{a} \in \ell^2 \setminus \ell^1$, the p -modulus of all horizontal paths in $S_{\mathbf{a}}$ is equal to zero for any p . However, the p -modulus ($p > 1$) of all rectifiable paths is positive. In section 5 we exhibit explicit path families with positive modulus. This provides a first step towards our eventual verification of the Poincaré inequality. Such verification in this context relies on the same theorem of Keith and a similar concatenation argument, starting from curve families of positive p -modulus as constructed above. The existence of path families with positive p -modulus follows easily from Ziemer's duality principle [37], [38]; see subsection 5.1. However, for our purposes such an abstract existence theorem is insufficient; we require an explicitly constructed and naturally parameterized curve family in order to implement our concatenation argument to verify Keith's criterion for the validity of Poincaré inequalities.

By Theorems 1.5 and 1.6, if $\mathbf{a} \in \ell^2 \setminus \ell^1$, then $(S_{\mathbf{a}}, d, \mu)$ supports a p -Poincaré inequality for each $p > 1$, but does not support a 1-Poincaré inequality. A significant recent result of Keith and Zhong [26] asserts that the set of values of p for which a given complete PI space supports a p -Poincaré inequality, is necessarily an open subset of $[1, +\infty)$.

Theorems 1.5 and 1.6 have a number of interesting consequences which we now enunciate.

Corollary 1.7. *There exist compact planar sets of topological dimension one that are Ahlfors 2-regular and 2-Loewner when equipped with the Euclidean metric and the Lebesgue measure.*

For each $\mathbf{a} \in \ell^2$, the carpet $S_{\mathbf{a}}$ verifies the conditions in Corollary 1.7. This follows from Theorem 1.6 and the equivalence of the Q -Loewner condition with the Q -Poincaré inequality in quasiconvex Ahlfors Q -regular spaces [20]. We remark that the examples of Bourdon–Pajot [6] and Laakso [30] are Q -regular Q -Loewner metric spaces of topological dimension one, however, these examples admit no bi-Lipschitz embedding into any finite-dimensional Euclidean space.

Corollary 1.8. *There exists a compact set $S \subset \mathbb{R}^2$, equipped with the Euclidean metric and a doubling measure, with the following properties: S supports no p -Poincaré inequality for any finite p , yet every strict weak tangent of S supports a 1-Poincaré inequality with universal constants. Moreover, S can be chosen to be quasiconvex and uniformly locally Gromov–Hausdorff close to planar domains.*

It is a general principle of analysis in metric spaces that quantitative geometric or analytic conditions often persist under Gromov–Hausdorff convergence. In particular, quantitative and scale-invariant conditions pass to weak tangent spaces. For instance, every weak tangent of a given doubling metric measure space satisfying a p -Poincaré inequality is again doubling and satisfies the same p -Poincaré inequality (see Theorem 2.5 for a version of this result used in this paper). Corollary 1.8 shows that weak tangent spaces can be significantly better behaved than the spaces from which they are derived, even in the presence of other good geometric properties.

The indicated example can be obtained by choosing $S = S_{\mathbf{a}}$ for any $\mathbf{a} \in c_0 \setminus \ell^2$. This follows from Theorem 1.6 and Proposition 4.8 discussed in section 4.1, where further details of the proof of Corollary 1.8 can be found.

A *carpet* is a metric measure space homeomorphic to $S_{1/3}$. There has been considerable interest of late in the problem of quasisymmetric uniformization of carpets by either round carpets or slit carpets [3], [4], [5], [32], [33]. The following two theorems are additional consequences of Theorem 1.6.

Theorem 1.9. *There exist round carpets in \mathbb{R}^2 which are Ahlfors 2-regular and support a p -Poincaré inequality for some $p < 2$.*

Theorem 1.10. *There exist parallel slit carpets which are Ahlfors 2-regular and support a p -Poincaré inequality for some $p < 2$.*

Recall that a planar carpet is said to be a *round carpet* if all of its peripheral circles are round geometric circles. A *slit carpet* is a carpet which is a Gromov–Hausdorff limit of a sequence of planar slit domains equipped with the internal metric. Recall that a domain $D \subset \mathbb{C}$ is a *slit domain* if $D = D' \setminus \bigcup_{i \in I} \gamma_i$, where D' is a simply connected domain and $\{\gamma_i\}_{i \in I}$ is a collection (of arbitrary cardinality) of disjoint closed arcs contained in D' . We admit the possibility that some of these arcs are degenerate, i.e., reduce to a point. A slit domain, resp. a slit carpet, is *parallel* if the nondegenerate arcs are parallel line segments, resp. if it is a limit of parallel slit domains.

Theorems 1.9 and 1.10 are proved in section 7. Theorem 1.9 follows from Theorem 1.6 and results of Bonk and Koskela–MacManus on quasisymmetric uniformization of carpets and quasisymmetric invariance of Poincaré inequalities on Ahlfors regular spaces. Theorem 1.10 follows from Theorem 1.6, Koebe’s uniformization theorem and the same work of Koskela–MacManus. Indeed, every carpet $S_{\mathbf{a}}$ with $\mathbf{a} \in \ell^2$ is quasisymmetrically equivalent to both a round carpet and also to a slit carpet with the stated properties.

1.1. Outline of the paper. In section 2 we recall general facts about analysis in metric spaces, particularly, facts about Poincaré inequalities in the sense of Definition 1.3. In section 3 we prove basic metric and measure-theoretic properties of the carpets $S_{\mathbf{a}}$. In particular, we show that the canonical measure on $S_{\mathbf{a}}$ is always a doubling measure, and we indicate in which situations it verifies upper or lower mass bounds.

Section 4 is devoted to the necessity of the ℓ^2 summability condition for the validity of Poincaré inequalities on the carpets $S_{\mathbf{a}}$. We first give an elementary combinatorial argument which employs a nonsharp summability hypothesis. More precisely, we show that $S_{\mathbf{a}}$ supports no Poincaré inequality whenever $\mathbf{a} \notin \ell^3$. Although this result is not sharp, we decided to include it to indicate the limitations of the classical folklore argument used to prove Proposition 1.4. The main result of section 4, Proposition 4.6, shows that ℓ^2 summability of \mathbf{a} is best possible for the validity of Poincaré inequalities on $S_{\mathbf{a}}$. In the final subsection of section 4 we describe in more detail the weak tangents of the carpets $S_{\mathbf{a}}$ and substantiate Corollary 1.8.

Our proof of the sufficiency of the summability criteria in Theorems 1.5 and 1.6 is contained in sections 5 and 6. In the former, we prove that the carpet $S_{\mathbf{a}}$ contains curve families of positive p -modulus for each $p > 1$ when $\mathbf{a} \in \ell^2$, and in the latter, we establish the validity of the appropriate range of Poincaré inequalities under either of the summability criteria in Theorems 1.5 and 1.6.

As mentioned before, the existence of curve families of positive modulus is an easy consequence of Ziemer's duality principle. We require an explicitly constructed and naturally parameterized curve family of this type. Our construction of such a curve family in subsection 5.3 is highly technical and detailed. The reader is invited to skim or omit this subsection on a first reading of the paper.

The key step in the proof of Theorem 1.6 is to perform, in the special case of the carpets $S_{\mathbf{a}}$, the following abstract procedure: in a metric space (X, d) endowed with a wide supply of rectifiable curves (in our case, \mathbb{R}^2), deform a given curve family so as to avoid a prespecified obstacle, at a small quantitative multiplicative cost to the p -modulus. Iterating this procedure produces curve families of positive p -modulus that avoid a countable family of obstacles of prespecified geometric sizes. Our implementation, while not completely general, covers a wider class of residual sets than just carpets: see Theorem 5.1 for a precise statement.

Our proof of the validity of suitable Poincaré inequalities resides in section 6. It makes substantial use of the precise rectilinear geometric structure of the carpets $S_{\mathbf{a}}$. Hence, the validity of a Poincaré inequality on the more general class of residual sets indicated in the preceding paragraph is less clear.

In section 7 we discuss uniformization of the carpets $S_{\mathbf{a}}$ by either round carpets or slit carpets. In particular, we establish Theorems 1.9 and 1.10. Section 8 contains concluding remarks and questions.

1.2. Acknowledgements. We are grateful to Mario Bonk for numerous discussions and especially for suggestions concerning uniformization of metric carpets. We would also like to thank Jasun Gong for useful comments about metric derivations on planar sets. We are particularly indebted to Hrant Hakobyan for pointing us to Ziemer's duality principle and for helpful remarks concerning conformal dimension.

Research for this paper was performed during visits of the last author to the University of Illinois and visits of all three authors to the Mathematics Institute at the University of Bern. The hospitality of both institutions is gratefully appreciated.

2. PRELIMINARIES

2.1. Basic definitions and notation. If $B = B(x, r)$ denotes a ball in a metric space $X = (X, d)$, we write λB for the dilated ball $B(x, \lambda r)$.

A *metric measure space* is a metric space (X, d) equipped with a Borel measure μ that is finite and positive on balls. The measure μ is *doubling* if there exists a constant $C > 0$ so that $\mu(B(x, 2r)) \leq C\mu(B(x, r))$ for all metric balls $B(x, r)$ in X . It is *Ahlfors Q -regular* for some $Q > 0$ if there exists a constant $C > 0$ so that $r^Q/C \leq \mu(B(x, r)) \leq Cr^Q$ for all metric balls $B(x, r)$ in X with $0 < r < \text{diam } X$. We say that μ is *Ahlfors regular* if it is Ahlfors Q -regular for some $Q > 0$. It is well known that any Ahlfors Q -regular measure on a metric space is comparable to the Hausdorff Q -measure \mathcal{H}^Q , and hence that \mathcal{H}^Q is also Ahlfors Q -regular in that case. Ahlfors regular measures are always doubling. Let us remark that we always denote by \mathcal{H}^s the s -dimensional Hausdorff measure in any metric space; we normalize these measures so that \mathcal{H}^n coincides with Lebesgue measure in \mathbb{R}^n .

A metric space (X, d) is said to be *quasiconvex* if there exists a constant C so that any pair of points $x, y \in X$ can be joined by a rectifiable path γ whose length is no more than $Cd(x, y)$. A metric space is quasiconvex if and only if it is bi-Lipschitz equivalent to a geodesic metric space.

Every doubling metric measure space admitting a Poincaré inequality is quasiconvex, see for instance [16] or [10]. By making use of quasiconvexity, we may assume that $\lambda = 1$ in (1.1), at the cost of increasing the value of C [16, Corollary 9.8].

2.2. Poincaré inequalities and moduli of curve families. The following result of Keith [23, Theorem 2] will be of great importance in this paper.

Theorem 2.1 (Keith). *Fix $p \geq 1$. Let (X, d, μ) be a complete, doubling metric measure space. Then X admits a p -Poincaré inequality if and only if there exist constants $C_1 > 0$ and $C_2 \geq 1$ so that*

$$(2.1) \quad d(x, y)^{1-p} \leq C_1 \text{mod}_p(\Gamma_{xy}; \mu_{xy}^{C_2})$$

for every pair of distinct points $x, y \in X$.

Here $\text{mod}_p(\Gamma_{xy}; \mu_{xy}^C)$ denotes the p -modulus of the curve family Γ_{xy} joining x to y , where the measure μ_{xy}^C is the symmetric Riesz kernel

$$\mu_{xy}^C(A) = \int_{A \cap B_{xy}^C} \frac{d(x, z)}{\mu(B(x, d(x, z)))} + \frac{d(y, z)}{\mu(B(y, d(y, z)))} d\mu(z),$$

where $B_{xy}^C = B(x, Cd(x, y)) \cup B(y, Cd(x, y))$. We recall that

$$\text{mod}_p(\Gamma; \nu) := \inf \int \rho^p d\nu$$

for a Borel measure ν on (X, d) . Here the infimum is taken over all nonnegative Borel functions ρ which are *admissible* for Γ , i.e., for which $\int_\gamma \rho ds \geq 1$ for all locally rectifiable curves $\gamma \in \Gamma$. When (X, d) is endowed with a fixed ambient measure μ , we abbreviate $\text{mod}_p \Gamma = \text{mod}_p(\Gamma; \mu)$.

2.3. Poincaré inequalities and metric gluings. The Poincaré inequality (1.1) is maintained under metric gluings. The following is a special case of a more general theorem of Heinonen and Koskela [20, Theorem 6.15], see also [17, Theorem 3.3].

Theorem 2.2 (Heinonen–Koskela). *Let X and Y be locally compact Ahlfors Q -regular metric measure spaces, $Q > 1$, let $A \subset X$ be a closed subset, and let $\iota : A \rightarrow Y$ be an isometric embedding. Let $p > 1$. If both X and Y support a p -Poincaré inequality and the inequalities*

$$\mathcal{H}_\infty^{Q-1}(A \cap B_X(x, r)) \geq cr^{Q-1}$$

and

$$\mathcal{H}_\infty^{Q-1}(\iota(A) \cap B_Y(y, r)) \geq cr^{Q-1}$$

hold for all $x \in A$, $y \in \iota(A)$ and $0 < r < \min\{\text{diam } X, \text{diam } Y\}$, where $c > 0$ is independent of x , y and r , then the metric gluing $X \cup_A Y$ supports a p -Poincaré inequality. The data for the p -Poincaré inequality on $X \cup_A Y$ depends quantitatively on the Ahlfors regular and Poincaré inequality data of X and Y , on p , and on the above constant c .

We recall that the metric gluing $X \cup_A Y$ is the quotient space obtained by imposing on the disjoint union $X \coprod Y$ the equivalence relation which identifies each $a \in A$ with its image $\iota(a)$. We equip this space with a natural metric which extends the metrics on X and Y as follows: for points $x \in X$ and $y \in Y$, let $d(x, y) = \inf\{d(x, a) + d(\iota(a), y) : a \in A\}$. Observe that the Q -regular measures on X and Y , respectively, combine to give a measure on $X \cup_A Y$ which is also Q -regular.

2.4. Gromov–Hausdorff convergence and weak tangents. A metric space (X, d) is *proper* if closed and bounded sets are compact.

Definition 2.3. A sequence of pointed proper metric measure spaces

$$\{(X_n, x_n, d_n, \mu_n)\}$$

converges to a pointed metric measure space (X, x, d, μ) if there exists a pointed proper metric space (Z, z, ρ) and isometric embeddings $f_n : X_n \rightarrow Z$, $f : X \rightarrow Z$ so that $f_n(x_n) = f(x)$ for all n , $f_n(X_n) \rightarrow f(X)$ in the sense of pointed Gromov–Hausdorff convergence, and $f_{\#}\mu_n \rightarrow f_{\#}\mu$ weakly.

We emphasize that the spaces X_n , X are not assumed to be compact. For the notion of pointed Gromov–Hausdorff convergence, see [23, §2.2] or [8, Chapter 7].

Definition 2.4. Let (X, d, μ) be a proper metric measure space. A pointed proper metric measure space (Y, y, ρ, ν) is called a *weak tangent* of (X, d, μ) if there exists a sequence of points $\{x_n\} \subset X$ and constants $\delta_n > 0$, $\lambda_n > 0$, so that the pointed proper metric measure spaces $\{(X, x_n, \frac{1}{\delta_n}d, \frac{1}{\lambda_n}\mu_n)\}$ converge to (Y, y, ρ, ν) .

We do not require that $\delta_n \rightarrow 0$. In the event that this occurs, we call the limit space a *strict weak tangent* of (X, d, μ) . If $x_n = x \in X$ for all n , we call (Y, y, ρ, ν) a *tangent* to X at x . The notion of *strict tangent* is defined similarly.

Keith has shown that Poincaré inequalities persist under Gromov–Hausdorff convergence. We state here a version of this result due to Keith [23], in a form which is suitable for our setting. For similar results, see Koskela [27] and Cheeger [10, §9].

Theorem 2.5 (Keith). *Suppose $X_1 \supset X_2 \supset \dots$ are subsets of \mathbb{R}^2 , and for each $n \in \mathbb{N}$, μ_n is a doubling measure supported on X_n , with uniform doubling constant. Let $X = \bigcap_{n \in \mathbb{N}} X_n$, and suppose that the measures $\{\mu_n\}$ converge weakly to a measure μ supported on X .*

If each (X_n, d, μ_n) supports a p -Poincaré inequality with uniform constants, then (X, d, μ) also supports a p -Poincaré inequality, with constants depending only on the doubling and Poincaré constants for the spaces X_n .

3. DEFINITION AND BASIC PROPERTIES OF THE CARPETS $S_{\mathbf{a}}$

We review the construction of the carpets $S_{\mathbf{a}}$. Fix a sequence

$$\mathbf{a} = (a_1, a_2, \dots)$$

where each a_m is an element of the set $\{\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \dots\}$. Starting from the unit square $T_0 = [0, 1]^2$ we set the *level parameter* $m = 1$ and iteratively apply the following two steps:

- Divide each current square into a_m^2 essentially disjoint closed congruent subsquares, where m denotes the current level parameter, and remove the central (concentric) subsquare from each square,
- Increase the level parameter m by 1.

We let \mathcal{T}_m denote the collection of all level m squares. For each m , \mathcal{T}_m consists of

$$\prod_{j=1}^m (a_j^{-2} - 1)$$

essentially disjoint closed squares, each of side length

$$s_m := \prod_{j=1}^m a_j.$$

The union of all squares in \mathcal{T}_m is the *level m precarpet*, denoted $S_{\mathbf{a},m}$. A *peripheral square* is a connected component of the boundary of a precarpet. Finally,

$$S_{\mathbf{a}} := \bigcap_{m \geq 0} S_{\mathbf{a},m} = \bigcap_{m \geq 0} \bigcup_{T \in \mathcal{T}_m} T.$$

Each carpet $S_{\mathbf{a}}$ is quasiconvex; this can be demonstrated using curves built by countable concatenations of horizontal and vertical segments. It is well-known that the usual Sierpiński carpet $S_{1/3}$ contains other nontrivial line segments, neither horizontal or vertical. Indeed, $S_{1/3}$ contains nontrivial line segments of each of the following slopes: 0, 1/2, 1, 2 and ∞ . An explicit description of the set of slopes of nontrivial line segments in all carpets $S_{\mathbf{a}}$ in terms of Farey fractions can be found in [13].

3.1. The natural measure on $S_{\mathbf{a}}$. There is a natural probability measure on $S_{\mathbf{a}}$. Since each precarpet $S_{\mathbf{a},m}$ has positive area, we define a measure μ_m on $[0, 1]^2$ which is Lebesgue measure restricted to the set $S_{\mathbf{a},m}$, renormalized to have total measure one. The sequence of measures (μ_m) converges weakly to a probability measure μ with support $S_{\mathbf{a}}$. To see this, note that on each square T of scale s_m that is not discarded, we have $\mu_n(T) = \mu_m(T)$ for all $n \geq m$, since later renormalizations merely redistribute mass within T . Therefore,

$$\mu(T) = \mu_m(T) = \prod_{j=1}^m (a_j^{-2} - 1)^{-1} =: v_m.$$

Moreover, for fixed $Q > 0$,

$$\frac{\mu(T)}{s_m^Q} = \prod_{j=1}^m a_j^{2-Q} (1 - a_j^2)^{-1}.$$

Note that if all $a_m = 1/(2k+1)$, then $\mu(T) = s_m^{Qk}$ for all $T \in \mathcal{T}_m$ and $\dim S_{1/(2k+1)} = Qk$. Here Q_k denotes the value in (1.2).

The following proposition describes the basic properties of μ . We write $a \lesssim b$ to mean that there exists a constant $C > 0$ so that $a \leq Cb$, where C depends only on the relevant data. Also, the notation $a \asymp b$ signifies that $a \lesssim b$ and $b \lesssim a$.

Proposition 3.1. *The metric measure space $(S_{\mathbf{a}}, d, \mu)$ has the following properties:*

- (i) *For any \mathbf{a} , μ is a doubling measure.*
- (ii) *For any \mathbf{a} , we have the lower mass bound $\mu(B(x, r)) \gtrsim r^2$ for all x and $r \leq 1$.*
- (iii) *If $\mathbf{a} \in c_0$, then for any $Q < 2$ we have $\mu(B(x, r)) \lesssim r^Q$ for all x and $r > 0$, hence $\dim S_{\mathbf{a}} = 2$.*
- (iv) *If $\mathbf{a} \in \ell^2$, then μ is comparable to Lebesgue measure with constant depending only on $\|\mathbf{a}\|_2$. Moreover, in this case, μ is an Ahlfors 2-regular measure on $S_{\mathbf{a}}$.*
- (v) *If $\mathbf{a} = (a_m)$ is eventually constant (and equal to $\frac{1}{2k+1}$), then μ is comparable to the Hausdorff measure \mathcal{H}^{Q_k} and is an Ahlfors Q_k -regular measure on $S_{\mathbf{a}}$.*

For $x \in S_{\mathbf{a}}$ and $r > 0$ define two integers $m(x, r)$ and $m(r)$ as follows:

- (1) $m(x, r)$ is the smallest integer m so that there exists $T \in \mathcal{T}_m$ with $x \in T \subset B(x, r)$,
- (2) $m(r)$ is the smallest integer m so that $s_m \leq r$.

First, an easy lemma:

Lemma 3.2. *For any x and r , $m(\sqrt{2}r) \leq m(x, r) \leq m(\frac{r}{\sqrt{2}}) + 1$.*

Proof. If $T \in \mathcal{T}_{m(x, r)}$ satisfies $x \in T \subset B(x, r)$, then $\sqrt{2}s_{m(x, r)} = \text{diam } T \leq \text{diam } B(x, r) \leq 2r$ which implies that $s_{m(x, r)} \leq \sqrt{2}r$ and $m(\sqrt{2}r) \leq m(x, r)$. Since $x \in T$ for some $T \in \mathcal{T}_{m(r/\sqrt{2})+1}$, and $\text{diam } T \leq \frac{r}{3}$, we have $m(x, r) \leq m(\frac{r}{\sqrt{2}}) + 1$. \square

We will derive the various parts of Proposition 3.1 from the following

Proposition 3.3. *For each $x \in S_{\mathbf{a}}$ and $0 < r \leq 1$,*

$$\mu(B(x, r)) \asymp h(r) := r^2 \prod_{j=1}^{m(r)} \left(\frac{1}{1 - a_j^2} \right).$$

Proof of Proposition 3.1. Note that $m(r)$ is a decreasing function of r . Part (i) follows easily:

$$\mu(B(x, 2r)) \lesssim (2r)^2 \prod_{j=1}^{m(2r)} \left(\frac{1}{1 - a_j^2} \right) \leq 4r^2 \prod_{j=1}^{m(r)} \left(\frac{1}{1 - a_j^2} \right) \lesssim \mu(B(x, r)).$$

Part (ii) is also clear, since the finite product term in the definition of $h(r)$ is always greater than or equal to one.

Next, we assert that $m(r) \leq m(2r) + 1$ for all $r > 0$. If not, we have $m(r) \geq m(2r) + 2$, so $m(r) - 1 \geq m(2r) + 1$, thus

$$r < s_{m(r)-1} \leq s_{m(2r)+1} \leq \frac{1}{3} s_{m(2r)} \leq \frac{1}{3} \cdot 2r,$$

a contradiction.

We now turn to part (iii). Assume that $\mathbf{a} \in c_0$, i.e., $a_m \rightarrow 0$. We will show that $\limsup_{r \rightarrow 0} \frac{\mu(B(x,r))}{r^Q}$ is finite for each $Q < 2$, uniformly in x . It suffices to show that

$$\limsup_{Q \rightarrow 0} \frac{h(r)}{r^Q} < \infty.$$

First we verify that $m(r) \leq -\log_2(r) + 1$. Suppose that n is the largest integer so that $2^n r \leq 1$. Since $m(1) = 0$,

$$m(r) \leq m(2r) + 1 \leq \dots \leq m(2^{n+1}r) + n + 1 \leq m(1) + n + 1 = n + 1 \leq -\log_2(r) + 1.$$

Now, since $a_j \rightarrow 0$, for any $\epsilon > 0$ there exists some $C = C(\epsilon)$ so that

$$\prod_{j=1}^{m(r)} \left(\frac{1}{1 - a_j^2} \right) \leq C(1 + \epsilon)^{m(r)} \leq C(1 + \epsilon)^{-\log_2 r + 1} \lesssim r^{-\log_2(1 + \epsilon)}.$$

If we choose ϵ so that $2 - Q > \log_2(1 + \epsilon)$, then we are done. From here part (iii) follows easily. Parts (iv) and (v) were discussed in the introduction. \square

Proof of Proposition 3.3. It is straightforward to bound $\mu(B(x,r))$ from above: cover $B(x,r)$ by squares from $\mathcal{T}_{m(r)}$. Then

$$\mu(B(x,r)) \leq \frac{(2r + 2s_{m(r)})^2}{s_{m(r)}^2} \cdot v_{m(r)} \leq \frac{16r^2}{s_{m(r)}^2} \cdot s_{m(r)}^2 \prod_{j=1}^{m(r)} \left(\frac{1}{1 - a_j^2} \right) \lesssim h(r).$$

To bound $\mu(B(x,r))$ from below, we split the proof into two cases.

Case 1. $r \leq 100s_{m(x,r)}$.

Since $B(x,r)$ contains a square of side $s_{m(x,r)}$, we use the obvious bound $\mu(B(x,r)) \geq v_{m(x,r)}$. Note that $m(r) - 1 \leq m(\frac{r}{\sqrt{2}}) - 1 \leq m(\sqrt{2}r) \leq m(x,r)$. Now,

$$v_{m(x,r)} = s_{m(x,r)}^2 \prod_{j=1}^{m(x,r)} \left(\frac{1}{1 - a_j^2} \right) \geq \left(\frac{1}{100} \right)^2 r^2 \cdot (1 - a_{m(r)}^2) \prod_{j=1}^{m(r)} \left(\frac{1}{1 - a_j^2} \right) \gtrsim h(r).$$

Case 2. $r > 100s_{m(x,r)}$.

Choose $T \in \mathcal{T}_{m(x,r)-1}$ so that $x \in T$. Since $T \not\subseteq B(x,r)$, the side length of T is at least $\frac{r}{\sqrt{2}}$. Since T is a square, $T \cap B(x,r)$ contains a (Euclidean) square V' of side $\frac{r}{4}$. Finally, since $s_{m(x,r)} \leq \frac{r}{100}$ and at most one square of generation $m(x,r)$ is deleted in T , V' contains a square V of side $s_v \in [\frac{r}{32}, \frac{r}{16}]$ consisting entirely of squares from $\mathcal{T}_{m(x,r)}$.

From the preceding facts we conclude that

$$\begin{aligned} \mu(B(x,r)) &\geq \mu(V) \geq \left(\frac{s_v}{s_{m(x,r)}} \right)^2 v_{m(x,r)} = s_v^2 \cdot \prod_{j=1}^{m(x,r)} \left(\frac{1}{1 - a_j^2} \right) \\ &\geq \frac{r^2}{32^2} \cdot (1 - a_{m(r)}^2) \prod_{j=1}^{m(r)} \left(\frac{1}{1 - a_j^2} \right) \gtrsim h(r). \end{aligned}$$

The proof is finished. \square

We make a final observation regarding the conformal dimension of $S_{\mathbf{a}}$. Recall that a metric space (X, d) is *minimal for conformal dimension* if its Hausdorff dimension is not lowered under any quasisymmetric map to any metric space. The self-similar carpets $S_{1/(2k+1)}$ are not minimal for conformal dimension. This result is a consequence of a theorem of Keith and Laakso [25], see also [31] for a brief recapitulation of the proof.

Corollary 3.4. *If $\mathbf{a} \in c_0$, then $S_{\mathbf{a}}$ is minimal for conformal dimension.*

Proof. By Proposition 3.1(iii), $S_{\mathbf{a}}$ has Hausdorff dimension 2. In a similar way, one shows that the Cantor set $C_{\mathbf{a}}$ has Hausdorff dimension 1. Since $S_{\mathbf{a}}$ contains the product of $C_{\mathbf{a}}$ and an interval, which has Hausdorff dimension 2, by [2, Section 5, Remark 1] the space $S_{\mathbf{a}}$ is minimal for conformal dimension. \square

The conformal dimensions of the carpets $S_{\mathbf{a}}$ when $\mathbf{a} \notin c_0$ remain unknown. Determining the conformal dimension of $S_{1/3}$ is a longstanding open problem.

4. FAILURE OF THE POINCARÉ INEQUALITY

In this section we provide conditions under which the p -Poincaré inequality fails to be satisfied on $S_{\mathbf{a}}$ for various choices of p and \mathbf{a} . In doing so we verify the necessity of the summability criteria in Theorems 1.5 and 1.6.

In order to show that the Poincaré inequality fails to be satisfied, it suffices to show that the corresponding modulus is trivial. On the carpets $S_{\mathbf{a}}$, it suffices merely to show that the corresponding modulus of the curves joining two opposite edges of the carpet is trivial. This is the content of our first two propositions.

Proposition 4.1. *Let (X, d, μ) be a complete, doubling metric measure space. If (X, d, μ) supports a p -Poincaré inequality, then $\text{mod}_p \Gamma > 0$ for some curve family Γ .*

Proposition 4.2. *Let $(S_{\mathbf{a}}, d, \mu)$ be a carpet as described in subsection 3.1. If $(S_{\mathbf{a}}, d, \mu)$ supports a p -Poincaré inequality, then $\text{mod}_p \Gamma > 0$ for the family Γ of rectifiable curves joining the left and right hand edges of $S_{\mathbf{a}}$.*

Proposition 4.1 is well known; we provide a short proof for the convenience of the reader.

Proof of Proposition 4.1. By Theorem 2.1, the weighted modulus $\text{mod}_p(\Gamma_{xy}; \mu_{xy}^C)$ is positive for all distinct points $x, y \in X$. Fix two such points x and y with $r = d(x, y)$, fix $\epsilon > 0$ with $\text{mod}_p(\Gamma_{xy}; \mu_{xy}^C) \geq \epsilon$, and consider the set $A := B(x, \frac{2}{3}r) \setminus B(x, \frac{1}{3}r)$. Let Γ_A be the family of rectifiable curves joining $B(x, \frac{1}{3}r)$ to $X \setminus B(x, \frac{2}{3}r)$. If ρ is admissible for Γ_A , then $\rho \cdot \chi_A$ is admissible for the family of curves joining x to y . Consequently,

$$\epsilon \leq \int_A \rho^p d\mu_{xy}^C.$$

Observing that

$$d\mu_{xy}^C \llcorner A \leq \frac{\frac{2}{3}r}{\mu(B(x, \frac{1}{3}r))} d\mu \llcorner A$$

we conclude that

$$c \leq \int_A \rho^p d\mu$$

for some $c = c(\epsilon, r, \mu(B(x, r/3))) > 0$. Hence $\text{mod}_p \Gamma_A \geq c > 0$. \square

Remark 4.3. We omit the proof of Proposition 4.2, but note that in Lemma 4.3.4 and Proposition 4.3.3 of [31] the following results are proved: (i) for each $p \geq 1$, the p -modulus of the curves joining the left and right hand edges of $S_{1/3}$ is equal to zero, and (ii) for each $p \geq 1$, the p -modulus of any curve family Γ contained in $S_{1/3}$ is equal to zero. We refer the reader to the proof of Proposition 4.3.3 of [31] for a discussion of how these two results are related. A similar argument derives Proposition 4.2 from Proposition 4.1.

It is also possible to prove Proposition 4.2 directly. Later in this section, we will show that the p -modulus of the family of curves joining the left and right edges of the carpet is equal to zero. To this end, we will construct admissible functions ρ_n with $\|\rho_n\|_p \rightarrow 0$. Integrating these functions yields functions u_n on $S_{\mathbf{a}}$ with the property that $u_n = 0$ on the left edge of $S_{\mathbf{a}}$, $u_n = 1$ on the right edge of $S_{\mathbf{a}}$, and u_n has upper gradient ρ_n . Such family of functions cannot satisfy (1.1) with a uniform constant C , for any fixed $p \geq 1$.

According to Proposition 4.2, in order to disprove the validity of a Poincaré inequality on $S_{\mathbf{a}}$ it suffices to consider the modulus of the family of curves joining the left and right hand edges. We use this observation systematically in what follows.

Proposition 4.4. *If $\mathbf{a} \notin \ell^1$, then $S_{\mathbf{a}}$ does not support a 1-Poincaré inequality.*

Proof. It suffices to show that $\text{mod}_1 \Gamma = 0$, where Γ denotes the family of curves joining the left and right hand edges of $S_{\mathbf{a}}$.

Define functions $\rho_m : S_{\mathbf{a}} \rightarrow [0, \infty]$, $m \geq 1$, as follows: let $\rho_m \equiv \frac{1}{s_m}$ on the vertical middle strip of width s_m , and $\rho_m \equiv 0$ elsewhere. Observe that ρ_m is admissible for Γ . We compute

$$\int \rho_m d\mu = \prod_{i=1}^m \left(a_i^{-1} \cdot \frac{a_i - a_i^2}{1 - a_i^2} \right) = \prod_1^m \frac{1}{1 + a_i}.$$

Since $\mathbf{a} \notin \ell^1$, the right hand side goes to zero as $m \rightarrow \infty$. Hence $\text{mod}_1 \Gamma = 0$ as desired. \square

We now consider the case $p > 1$. The following proposition is superseded by the stronger Proposition 4.6. However, the method of proof of Proposition 4.5 is somewhat more flexible. For example, it applies to other carpets where at each stage we remove squares that are not necessarily centrally symmetric, since the argument only depends on the number of removed squares in each column of the carpet.

Proposition 4.5. *If $\mathbf{a} \notin \ell^3$, then $S_{\mathbf{a}}$ does not support a p -Poincaré inequality for any $p \geq 1$.*

Proof. It suffices to consider the case $p > 1$. As discussed above, it is enough for us to prove that $\text{mod}_p \Gamma = 0$, where Γ denotes the family of rectifiable curves joining the left and right hand edges of $S_{\mathbf{a}}$.

For each m we will build an upper gradient ρ_m . To describe the construction, consider the following prototypical case. Let $\rho_- \equiv 1$ on the unit square S_- . Given $a \in \{\frac{1}{3}, \frac{1}{5}, \dots\}$, remove the central square of side a , and call the resulting closed set S . To renormalize the measure on S , we multiply by $(1 - a^2)^{-1}$.

Choose parameters $\alpha \geq 0$, $\beta \geq 0$, and define ρ on S to be β on the central strip of width a , and α everywhere else. In order to keep $\int_{\gamma} \rho ds$ equal to one for curves joining the left of S to the right, we assume that

$$\alpha(1 - a) + \beta a = 1,$$

which implies that

$$\beta = \frac{1}{a}(1 - (1 - a)\alpha).$$

Before, we had

$$\int_{S_-} \rho_-^p d\mu = 1;$$

now we have

$$(4.1) \quad \int_S \rho^p d\mu = \frac{(1-a)\alpha^p + a(1-a)\beta^p}{1-a^2} = \frac{\alpha^p + a^{1-p}(1-(1-a)\alpha)^p}{1+a}.$$

Taking derivatives of the right hand side with respect to α for fixed p , we observe that the right hand side is minimized when

$$\alpha = \frac{(1-a)^{1/(p-1)}}{a + (1-a)^{p/(p-1)}}.$$

Substituting into (4.1) yields

$$(4.2) \quad \int_S \rho^p d\mu = 1 - \frac{pa^3}{2(p-1)} + \frac{(p-2)pa^4}{6(p-1)^2} + O[a]^5.$$

Note that the a^3 coefficient is strictly less than zero.

We now construct our function ρ_m on $S_{\mathbf{a}}$. This is done by iterating the above procedure. At the first stage, we perform the construction above for $a = a_1$ to create ρ_1 with $\alpha = \alpha_1$.

At the second stage, on each square of side s_1 , ρ_1 is constant. Given $a = a_2$ we find $\alpha = \alpha_2$ as above. We define ρ_2 on each square of side s_1 by redistributing the value of ρ_1 across the square using the proportions α, β specified by $a = a_2$. We iterate this procedure m times, obtaining ρ_m .

By construction, ρ_m is admissible for Γ . Moreover, the inductive definition of ρ_m yields

$$(4.3) \quad \int_{S_{\mathbf{a}}} \rho_m^p d\mu = \prod_{i=1}^m [\alpha_i^p + a_i^{1-p}(1-(1-a_i)\alpha_i)^p] \frac{1}{1+a_i}.$$

By (4.1) and (4.2), if $\mathbf{a} \notin \ell^3$, then the right hand side of (4.3) tends to zero as m goes to infinity. Hence $\text{mod}_p \Gamma = 0$ as desired. \square

The carpets considered in this paper have a very specific geometry that can be used to improve the hypotheses of the previous proposition. The following result is sharp.

Proposition 4.6. *If $\mathbf{a} \notin \ell^2$, then $S_{\mathbf{a}}$ does not support a p -Poincaré inequality for any $p \geq 1$.*

Proof. We will build an admissible function ρ for the curve family Γ joining the left and right hand edges of $S_{\mathbf{a}}$. This function ρ will have essential supremum zero, which shows that $\text{mod}_p \Gamma = 0$ for all p . By Proposition 4.2, $S_{\mathbf{a}}$ does not support a p -Poincaré inequality for any $p \geq 1$.

Let γ be a rectifiable curve joining the left and right hand edges of $S_{\mathbf{a}}$. By passing to a subcurve if necessary we may assume that γ is an arc, i.e., that it is injective.

Step 0. Let $A_0 = S_{\mathbf{a}}$, and $\Gamma_0 = \{\gamma\}$.

We divide $S_{\mathbf{a}}$ into $m_1 = s_1^{-1}$ vertical strips of width a_1 . These strips are bounded by vertical cut sets V_0, V_1, \dots, V_{m_1} , where $V_0 = \{0\} \times [0, 1]$, $V_1 = \{a_1^{-1}\} \times [0, 1]$, and so on, with the exception that for the two vertical cut sets that meet the deleted square of side a_1 , we delete the two open intervals of length a_1 that are on the side of that square.

We now split $\Gamma_0 = \{\gamma\}$ into a disjoint family of curves. We parametrize γ by arc length, with $\gamma(0) \in V_0$, and $\gamma(1) \in V_1$. Let $t_0^+ \geq 0$ be the last time γ meets V_0 . Let $t_1^- \geq t_0^+$ be the next time after that γ meets V_1 . Let γ_1 be the subpath of γ given by $[t_0^+, t_1^-]$.

Continue inductively, letting $t_{i-1}^+ \geq t_{i-1}^-$ be the last time γ meets V_{i-1} , and $t_i^- \geq t_{i-1}^+$ be the next time γ meets V_i . Let γ_i be the subpath of γ given by $[t_{i-1}^+, t_i^-]$.

By construction, $\Gamma_1 = \{\gamma_1, \dots, \gamma_{m_1}\}$ is a family of m_1 curves, where each γ_i joins V_{i-1} to V_i , and is contained between them. (See Figure 3, where the deleted subpaths are indicated by dotted lines.)

Note that, of course, the length of Γ_1 (i.e., the sum of the lengths of $\gamma_1, \dots, \gamma_{m_1}$), is at most the length of Γ_0 , that is $\text{length}(\gamma)$.

Step i (fold in). We are given as input a collection $\Gamma_i = \{\gamma_j\}$ of $m_i = s_i^{-1}$ curves and vertical slices V_0, \dots, V_{m_i} , where γ_j joins V_{j-1} to V_j and is contained between them, for each $j = 1, \dots, m_i$.

Choose the largest $l_i \in \mathbb{N}$ so that $l_i a_{i+1} \leq \frac{1}{3}$. Note that since $a_{i+1} \in \{\frac{1}{3}, \frac{1}{5}, \dots\}$, we have $l_i a_{i+1} > \frac{1}{3} - \frac{1}{5}$.

Let D_i be the collection of open rectangles of width $l_i a_{i+1} s_i$ and height $(1 - 2l_i a_{i+1}) s_i$ centered on either the left or right sides of the deleted squares of side s_i . Notice that this leaves two squares of side $l_i a_{i+1} s_i$ on the side of each deleted square of side s_i . Divide each of these squares into two triangles along the diagonal that meets the corner of the square of side s_i . Let R_i be the collection of those triangles that share a side with a deleted square of side s_i . See Figure 4 for part of an example where these regions are labeled.

We now define a folding map $F_i : S_{\mathbf{a}} \rightarrow S_{\mathbf{a}}$ by declaring F_i to be the identity except on R_i , where the map folds each triangle across the diagonal, so that the image of the horizontal edge of D_i is now vertical.

Notice that in Figure 4, the region D_i will not overlap with D_{i+1} . However, when $a_{i+1} = \frac{1}{3}$, D_i may contain an entire square Q of side s_{i+1} that is to the left or right of a particular deleted square of side s_{i+1} , but it cannot contain both such squares. When this happens, we leave the square Q untouched at step i , and do not include it in D_i .

We apply F_i to the collection of curves Γ_i , and then build Γ_{i+1} using the same inductive construction as we used in Step 0 to build Γ_1 from Γ_0 . Here Γ_{i+1} will be a collection of $m_{i+1} := s_{i+1}^{-1}$ curves inside the m_{i+1} vertical strips of width s_{i+1} , joining V_0 to V_1 , and so on. Moreover, by construction Γ_{i+1} will lie inside $A_i = A_{i-1} \setminus (D_i \cup R_i)$, and the length of Γ_{i+1} is not greater than the length of Γ_i .

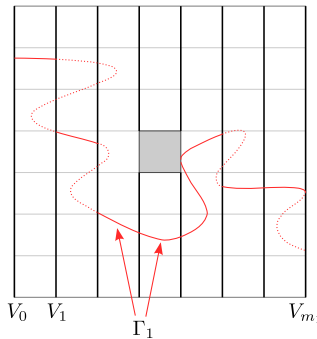


FIGURE 3. Curve splitting

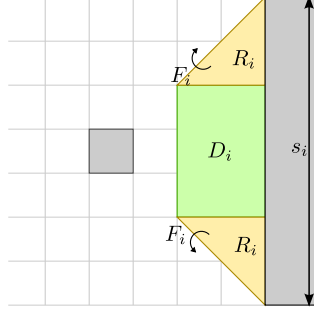


FIGURE 4. Unfolding

Conclusion. We continue this construction until step n , when we have a collection of curves Γ_{n+1} that lies in A_n , and have deleted the rectangles in the collections D_1, \dots, D_n from the sides of the deleted squares. We let $X_n^{(n)} = A_n$, and now proceed to unfold our measure and curves back out into the regions R_1, \dots, R_n .

Define inductively

$$X_N^{(i)} = F_i^{-1}(X_n^{(i+1)}),$$

for $i = n - 1, n - 2, \dots, 0$.

It is clear that $S_{\mathbf{a}} = X_0^{(0)} \supset X_1^{(0)} \supset X_2^{(0)} \supset \dots$. Let $X = \bigcap_{i=0}^{\infty} X_i^{(0)}$. For each n , and for any γ in Γ , our construction implies that

$$\begin{aligned} \text{length}(\gamma \cap X_n^{(0)}) &= \text{length}(\Gamma_0 \cap X_n^{(0)}) \geq \text{length}(\Gamma_1 \cap X_n^{(0)}) \\ &\geq \text{length}(\Gamma_2 \cap X_n^{(1)}) \geq \dots \geq \text{length}(\Gamma_{n+1} \cap X_n^{(n)}) \\ &= \text{length}(\Gamma_{n+1} \cap A_n) \geq 1, \end{aligned}$$

since Γ_{n+1} is a chain of paths crossing each vertical strip of width s_{n+1} from the left to the right, and Γ_{n+1} lives in A_n .

Therefore, $\text{length}(\gamma \cap X) \geq 1$, since $\text{length} = \mathcal{H}^1$ is a measure on arcs.

Let ρ be the characteristic function of X . We have shown that ρ is admissible for Γ .

It remains to prove that $\mu(X) = 0$. Consider each deleted square of side s_i . Out of the neighboring $(a_i^{-2} - 1)$ boxes of side s_i , from at least one (two if $a_i \neq \frac{1}{3}$) of these we will delete a rectangle in D_i whose μ -measure, as a proportion of a square of side s_i , is at least $(1 - 2l_i a_{i+1})l_i a_{i+1} \geq \frac{2}{45}$. Since all the D_i are pairwise disjoint, we have

$$\mu(X) \leq \prod_{i=1}^{\infty} \left(1 - \frac{1}{a_i^{-2} - 1} \frac{2}{45}\right) = \prod_{i=1}^{\infty} \left(1 - \frac{2}{45} a_i^2 + \dots\right),$$

which converges to zero since $\mathbf{a} \notin \ell^2$. □

Remark 4.7. The argument also shows that $S_{\mathbf{a}}$ does not support an ∞ -Poincaré inequality. See [12] for the definition. Note that the ∞ -Poincaré inequality is weaker than the p -Poincaré inequality for any finite p .

4.1. Weak tangents of Sierpiński carpets. Weak tangents of metric spaces describe infinitesimal behavior at a point or along a sequence of points. In this section we characterize the strict weak tangents of non-self-similar carpets. More precisely, we prove the following proposition.

Proposition 4.8. *Let $\mathbf{a} \in c_0$. Then every strict weak tangent of $S_{\mathbf{a}}$ is of the form $(\mathbb{R}^2 \setminus T, d, \nu)$ where T is a generalized square and ν is proportional to Lebesgue measure restricted to $\mathbb{R}^2 \setminus T$.*

By a *generalized square* we mean a set $T \subsetneq \mathbb{R}^2$ of the type

$$T = (a, b) \times (c, d)$$

where $-\infty \leq a \leq b \leq \infty$, $-\infty \leq c \leq d \leq \infty$, and $b - a = d - c$ if one (hence both) of these values is finite. (We interpret the degenerate interval (a, b) , $a = b$, as the empty set.) Thus T is either the empty set, an open square, a quadrant or a half-space.

Proof of Proposition 4.8. Suppose W is a strict weak tangent arising as the limit of the sequence of metric spaces $\{X_n = (S_{\mathbf{a}}, x_n, \frac{1}{\delta_n}d)\}$, where $x_n \in S_{\mathbf{a}}$, $\delta_n \in (0, \infty)$, and $\delta_n \rightarrow \infty$.

The following lemma indicates why W can omit at most one large square.

Lemma 4.9. *There exist $R_n \rightarrow \infty$ and $r_n \rightarrow 0$ so that in the ball $B(x_n, R_n) \subset X_n$ there is at most one square of side greater than 1 removed, and all other squares removed have size at most r_n .*

Proof. Fix n , and let $m = m(\delta_n)$, i.e., $s_m \leq \delta_n < s_{m-1}$.

Either $\delta_n \in [s_m, s_{m-1}\sqrt{a_m})$, or $\delta_n \in [s_{m-1}\sqrt{a_m}, s_{m-1})$. In the first case, squares of size at least $\frac{s_m}{\delta_n}$ are $\frac{s_{m-1}}{\delta_n} \geq \frac{1}{\sqrt{a_m}}$ separated, while all others have size at most $\frac{s_{m+1}}{\delta_n} \leq a_{m+1}$. In the second case, squares of size at least $\frac{s_{m-1}}{\delta_n} \geq 1$ are $\frac{s_{m-2}}{\delta_n} \geq \frac{1}{a_{m-1}}$ separated, and all others have size at most $\frac{s_m}{\delta_n} \leq \sqrt{a_m}$.

Setting $R_n = \min(\frac{1}{a_{m-1}}, \frac{1}{\sqrt{a_m}})$ and $r_n = \max(\sqrt{a_m}, a_{m+1})$, we have proved the lemma. \square

Using the lemma, we can reduce the proof of Proposition 4.8 to consideration of limits of $\mathbb{R}^2 \setminus T_n$ where T_n is either a square of side at least one, or the empty set. It is easy to see that W will be isometric to $\mathbb{R}^2 \setminus T$, where T is either a square (of side at least one), a quarter-plane, half-plane or the empty set.

Since the measure μ on $S_{\mathbf{a}}$ agrees with the weak limit of renormalized Lebesgue measure on the domains $S_{\mathbf{a},m}$, by the lemma, if we look at measures of balls in X_n of size much larger than r_n , they will agree with a constant multiple of Lebesgue measure up to small error. Consequently, the possible measures on W will arise as limits of rescaled Lebesgue measure on $\mathbb{R}^2 \setminus T_n$. The only possible non-trivial Radon measure of this type is a constant multiple of Lebesgue measure restricted to $\mathbb{R}^2 \setminus T$. This finishes the proof of Proposition 4.8. \square

Note that all of the weak tangent spaces identified in the conclusion of Proposition 4.8 support a 1-Poincaré inequality, with uniform constants (i.e., independent of the choice of such a weak tangent space). This is because we have only a finite number of similarity types of spaces (full space, half space, quarter space, or the complement of a square), and the Poincaré inequality data is invariant under similarities. The quasiconvexity of the original carpets $S_{\mathbf{a}}$ is a standard fact. Indeed, arbitrary pairs of points can be joined by quasiconvex curves which are comprised of countable unions of horizontal and vertical line segments.

Definition 4.10. A metric space (X, d) is *locally Gromov–Hausdorff close to planar domains* if for each $x \in X$ and each $\epsilon > 0$, there exists $r > 0$ and a domain $\Omega \subset \mathbb{R}^2$ so that the Gromov–Hausdorff distance between the metric ball $B(x, r) \subset X$ and Ω is at most ϵ . Furthermore, (X, d) is *uniformly locally Gromov–Hausdorff close to planar domains* if ϵ can be chosen independently of x .

The fact that $S_{\mathbf{a}}$ is uniformly locally Gromov–Hausdorff close to planar domains follows easily from the construction and the condition $\mathbf{a} \in c_0$.

The preceding discussion understood, the proof of Corollary 1.8 is complete.

5. POSITIVITY OF MODULUS

In this section, we show that the carpets $S_{\mathbf{a}}$ support curve families of positive p -modulus for $p > 1$, when $\mathbf{a} \in \ell^2$. In fact, we verify this conclusion for a significantly wider class of compact planar sets. The following theorem gives a general sufficient condition for the positivity of p -modulus.

Theorem 5.1. *Let $D \subset \mathbb{R}^2$ be the closure of a domain and let $\mathbf{a} = (a_1, a_2, \dots) \in \ell^2$ with $a_m \in (0, 1)$ for all m . For each m , let $s_m = \prod_{j=1}^m a_j$, and let \mathcal{U}_m be a family of disjoint open subsets of D . Assume that the following two conditions are satisfied:*

- for all $U \in \mathcal{U}_m$, $\text{diam } U \leq 2s_m$, and
- for all $U \in \mathcal{U}_m$ and all $V \in \mathcal{U}_1 \cup \mathcal{U}_2 \cup \dots \cup \mathcal{U}_m$, $V \neq U$, we have $\text{dist}(U, V) \geq \frac{2}{5}s_{m-1}$.

Let

$$S_M := D \setminus \bigcup_{U \in \mathcal{U}_1 \cup \dots \cup \mathcal{U}_M} U$$

and

$$S = \bigcap_{M \geq 0} S_M.$$

Then for all $p > 1$ and all open balls $B \subset S$, there exists a curve family contained in B with positive p -modulus.

The coefficients 2 and $\frac{2}{5}$ in Theorem 5.1 have been fixed for the sake of definiteness and can be varied without changing the result.

In the setting of the carpets $S_{\mathbf{a}}$, the elements of \mathcal{U}_m correspond to the omitted squares at level m . We obtain the following corollary.

Corollary 5.2. *For any $p > 1$ and $\mathbf{a} \in \ell^2$, there exists a positive constant $C = C(p, \mathbf{a})$ so that the p -modulus of the curve family joining the left hand edge to the right hand edge of $S_{\mathbf{a}}$ is at least C . If \mathbf{a} is monotone decreasing, then $C = C(p, \|\mathbf{a}\|_2)$.*

Whenever \mathbf{a} is in ℓ^2 , Proposition 3.1(iv) states that μ is comparable to Hausdorff 2-measure \mathcal{H}^2 restricted to $S_{\mathbf{a}}$. The argument given there extends with minimal modification to the setting of Theorem 5.1. For simplicity we will work with \mathcal{H}^2 in this section and the next.

We provide two proofs of Theorem 5.1 and Corollary 5.2. First we give a simple proof using the modulus of cut sets and a duality principle due to Ziemer. We are indebted to Hrant Hakobyan for indicating this method of proof. As this approach is not useful for our later purposes, we only write out the proof carefully in the setting of our carpets $S_{\mathbf{a}}$ (Corollary 5.2) and leave its extension to the more general context of Theorem 5.1 to the reader.

In order to show that $S_{\mathbf{a}}$ admits Poincaré inequalities we need a more explicit construction of curve families. The argument in section 5.1 merely establishes positivity of the modulus of the family of **all** curves joining two specified boundary continua. We therefore give a second, more technical and difficult, proof for Theorem 5.1. The technical aspects of this second proof are all related to the issue of building an explicit, nicely parameterized, curve family of positive modulus.

5.1. A duality principle. Our first proof of Corollary 5.2 uses a duality principle for p -moduli due to Ziemer. The formulation which we give can be found in Theorem 3.13 of [37] in the conformal case ($p = n$ in \mathbb{R}^n); for the extension to general $1 < p < \infty$ we refer to [38].

Theorem 5.3 (Ziemer). *Let $W \subset \mathbb{R}^2$ be a bounded domain, and let C_0 and C_1 be disjoint compact sets contained in the closure of W . Let Σ be the family of sets which separate C_0 and C_1 in W and let Γ denote the family of curves joining C_0 to C_1 in W .*

Then for any $p \in (1, \infty)$, with $q = p/(p - 1)$, we have

$$\text{mod}_p(\Gamma) = \text{mod}_q(\Sigma)^{1-p}.$$

We remark that the p -modulus of the family Σ is defined exactly as for curve families, even though the members of Σ need not be paths.

First proof of Corollary 5.2. By Keith [23, Theorem 1], it suffices to show that the p -modulus of the families of curves joining the left edge to the right edge in S_m is bounded away from zero independently of m . Since modulus is upper semicontinuous, this bound will continue to the limit.

By Theorem 5.3, it suffices to show that the q -modulus of the family Σ_m of curves separating the left edge from the right edge in S_m is bounded from above independently of m .

Let ρ_0 be the constant function on S_m of value 1, and observe that ρ_0 is admissible for Σ_0 , and therefore that

$$\text{mod}_q(\Sigma_0) = M_0 \in [0, \infty).$$

We now define, by induction, admissible functions ρ_m for Σ_m . Suppose ρ_{m-1} has been defined. If U is an open square of side s_m deleted at stage m with $s_m/2$ -neighborhood U' , set

$$\rho_m(x) = 3 \max\{\rho_{m-1}(y) : y \in U'\},$$

for $x \in U'$. (Note that these neighborhoods are disjoint.) Otherwise, set $\rho_{m-1} = \rho_m$. Clearly, ρ_m is admissible for Σ_m .

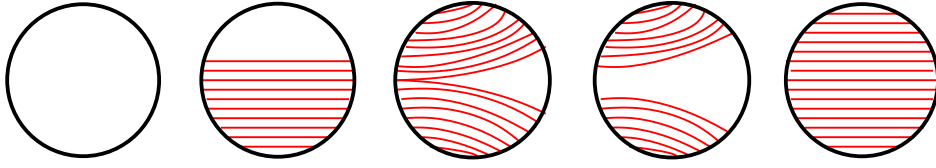
Since the maximum value of ρ_{m-1} on U' is attained on an area proportional to s_{m-1}^2 , and we are increasing the value by 3 on an area proportional to s_m^2 , a quick calculation shows that, when we do this for every such U' , the following integral estimate holds:

$$\int_{S_m} \rho_m^q d\mathcal{H}^2 \leq (1 + Ca_m^2) \int_{S_{m-1}} \rho_{m-1}^q d\mathcal{H}^2.$$

Since $\mathbf{a} \in \ell^2$, $\text{mod}_q(\Sigma_m)$ is bounded from above independently of m . □

Remark 5.4. The above proof should extend to cover the more general class of spaces in Theorem 5.1. The only potential difficulty in adapting the argument to this setting may lie in the choice of the domain V . We leave this point to the reader and turn our attention at this time to an alternate, more constructive, proof for Theorem 5.1.

5.2. Bending curve families. The basic idea of the construction in this section is as follows. We present an algorithm which accepts as input a family of curves in the plane and which yields as output a new family of curves which avoids a prespecified obstacle at a small quantitative multiplicative cost to the p -modulus. We apply this algorithm recursively to avoid all of the omitted sets. The algorithm in question works by splitting the family of input curves in two pieces which are deformed to pass on either side of the obstacle. (Similar ideas appear in a paper of Chris Bishop [1] on A_1 deformations of the plane.)

FIGURE 5. Structure of $\overline{\text{spt } \Gamma} \cap B(z, r)$

The curve families that we consider are axiomatized in the following definition.

Definition 5.5. An *open measured family of C^2 curves* is a collection Γ of disjoint C^2 curves in a set $X \subset \mathbb{R}^2$, together with a measure σ on Γ , such that the union of all the curves in Γ , denoted $\text{spt } \Gamma$, is an open subset of X . We will denote such a pair by (Γ, σ) , or just by Γ if the measure σ is understood.

There is a natural measure $\nu_\Gamma = \nu_{(\Gamma, \sigma)}$ defined on $\text{spt } \Gamma$ by

$$(5.1) \quad \nu_\Gamma(V) = \int_\Gamma \mathcal{H}^1(V \cap \gamma) d\sigma(\gamma).$$

At this point, the integral in (5.1) should be interpreted as an upper integral with value in $[0, \infty]$. However, under the conditions of Definition 5.6, ν_Γ will be a finite Borel measure.

We assume that each curve in Γ is parameterized with nonzero speed. Since each curve in Γ is C^2 , there is a vector field $\dot{\Gamma}$ defined on $\text{spt } \Gamma$, where $\dot{\Gamma}(x)$ is the (unoriented) unit tangent vector to the unique curve $\gamma \in \Gamma$ that passes through x . In fact,

$$\dot{\Gamma}(\gamma(t)) = \gamma'(t)/|\gamma'(t)|.$$

Definition 5.6. Fix $\delta_0 \geq 0$ and $r_0 > 0$. We say an open measured family of C^2 curves (Γ, σ) is a δ_0 -good family of curves on scales less than r_0 if the Radon-Nikodym derivative $w_\Gamma = d\nu_\Gamma/d\mathcal{H}^2$ exists and is a nonnegative locally Hölder continuous function on $\cup \Gamma$. Here ν_Γ denotes the natural measure defined in (5.1). We also require that for any ball $B(z, r)$, $z \in X$, $0 < r \leq r_0$ we have the following properties:

- (A) The complement of the closure of $\text{spt } \Gamma$ in $B(z, r)$ is a connected open set.
- (B) We can choose orientations on $\dot{\Gamma}$ consistently in $B(z, r)$ so that

$$\angle(\dot{\Gamma}(x), \dot{\Gamma}(y)) \leq \delta_0 \left(\frac{d(x, y)}{2r_0} \right)^{2/3},$$

for any $x, y \in B(z, r)$, where $\angle(\mathbf{v}, \mathbf{w})$ denotes the angle between vectors \mathbf{v} and \mathbf{w} .

- (C) There is a constant $A_{z,r} \in (0, \infty)$ so that $w_\Gamma|_{B(z,r)}$ is $\frac{2}{3}$ -Hölder continuous with constant $(2r_0)^{-2/3} A_{z,r} \delta_0$ and

$$w_\Gamma(B(z, r)) \subset [(1 + \delta_0)^{-1} A_{z,r}, (1 + \delta_0) A_{z,r}].$$

The first condition ensures that $\overline{\text{spt } \Gamma} \cap B(z, r)$ is either the empty set, one half of the ball, all of the ball except for one open gap, or all of the ball. See Figure 5 for an illustration.

The last two conditions guarantee that the vector field $\dot{\Gamma}$ is $\frac{2}{3}$ -Hölder continuous (with suitable constant) and that the Radon-Nikodym derivative $w_\Gamma = \frac{d\nu_\Gamma}{d\mathcal{H}^2}$ exists and is locally close to constant on $\text{spt } \Gamma$.

Why are the vector fields only Hölder continuous? If they were Lipschitz continuous, then by the uniqueness of solutions to ODE with Lipschitz coefficients, the curves could not

split to bend round an obstacle. The choice of $\frac{2}{3}$ is fixed in view of the cubic spline which we construct in Lemma 5.8(4). This choice is merely a convenience; any Hölder exponent strictly less than one would serve our purposes equally well.

Second proof of Theorem 5.1 and Corollary 5.2. By Keith [23, Theorem 1], it suffices to construct curve families Γ'_m contained in S_m for each m with p -modulus uniformly bounded from below. Since modulus is upper semicontinuous, this bound will continue to the limit.

Fix a ball $B \subset D$ so that $B \cap S$ has positive measure.

Consider $V = B \cap S_m$. If $V \cap S$ has nonempty interior we are done, since open sets in \mathbb{R}^2 certainly contain curve families with positive p -modulus. Otherwise V contains omitted sets U from collections \mathcal{U}_m with arbitrarily large m . Choose one such $U \in \mathcal{U}_M$, where M will be chosen below, whose distance to ∂V is at least $5 \operatorname{diam} U$, choose a square $W \subset V$ of side $\operatorname{diam} U$ whose distance to U is between $\operatorname{diam} U$ and $3 \operatorname{diam} U$, and consider the family of horizontal curves in V . Call this family Γ_0 and equip it with the uniform measure associated to Lebesgue measure on the vertical axis. Then Γ_0 is 0-good on scales below r_0 for some suitably chosen r_0 , away from the edges of W .

Suppose there exists a measured curve family (Γ_m, σ_m) which is δ_m -good on scales below r_m , for some δ_m and r_m . Let $\nu_m = \nu_{\Gamma_m}$ be the natural measure on $\operatorname{spt} \Gamma_m$ and let $w_m = d\nu_m/d\mathcal{H}^2$ be the corresponding weight. If $\rho : S_m \rightarrow [0, \infty]$ is admissible for $\operatorname{mod}_p \Gamma_m$, that is, $1 \leq \int_\gamma \rho d\mathcal{H}^1$ for each $\gamma \in \Gamma_m$, then by averaging over Γ_m with respect to σ_m , we see that

$$(5.2) \quad 1 \leq \int_{\Gamma_m} \int_\gamma \rho d\mathcal{H}^1 d\sigma_m(\gamma) = \int_{\operatorname{spt} \Gamma_m} \rho d\nu_m = \int_{S_m} \rho w_m d\mathcal{H}^2 \leq \|\rho\|_p \|w_m\|_q,$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Therefore, to bound $\operatorname{mod}_p \Gamma_m \leq \operatorname{mod}_p \Gamma'_m$ from below, it suffices to bound $\|w_m\|_q$ from above. If this bound can be made independent of m , then the proof will be complete.

The following proposition provides the key inductive step in building Γ_m and establishing this bound. We postpone its proof.

Proposition 5.7. *There exist positive constants δ_0, a_* and D with the following property.*

Suppose (Γ_i, σ_i) is a δ_0 -good family of curves on scales smaller than s_i in \mathbb{R}^2 . Let ν_i be the natural measure and let $w_i = d\nu_i/d\mathcal{H}^2$ be the corresponding weight. If $a_{i+1} \leq a_$ and we assume that $B(z, 2s_{i+1})$ meets $\cup \Gamma_i$ for some $z \in S_n$, with no curves in Γ_i stopping inside $B(z, s_i)$, then we can deform (Γ_i, σ_i) inside $B(z, s_i/3)$ into a new measured curve family $(\Gamma_{i+1}, \sigma_{i+1})$ that is δ_0 -good on scales smaller than s_{i+1} , so that $\cup \Gamma_{i+1}$ does not meet $B(z, 2s_{i+1})$ and*

$$(5.3) \quad \int_{B(z, s_i/3)} |w_{i+1}|^q d\mathcal{H}^2 \leq (1 + Da_{i+1}^2) \int_{B(z, s_i/3)} |w_i|^q d\mathcal{H}^2.$$

Intuitively, Proposition 5.7 asserts that we can deform Γ_i inside a ball B on a given scale s_i so as to avoid a prespecified obstacle of size s_{i+1} (in this case, a ball of radius $2s_{i+1}$ concentric with B) and so that the ℓ^q norm of the associated weight increases multiplicatively by at most a factor of $1 + Ca_{i+1}^2$, where C is independent of \mathbf{a} . The point is that we can repeatedly apply the proposition (on smaller and smaller scales) without losing control of δ_0 .

We now complete the proof of Theorem 5.1.

Choose a_* as in Proposition 5.7 and choose M so that $a_i < a_*$ when $i \geq M$. Note that if \mathbf{a} is monotone decreasing, then M depends only on $\|\mathbf{a}\|_2$.

The construction is inductive. Assume that we have constructed a measured curve family (Γ_m, σ_m) in $S_{\mathbf{a}, M}$ that is δ_0 -good on scales below s_m . The squares which we delete at stage $m+1$ are all at least s_m apart, so we can apply Proposition 5.7 at each location independently. Applying (5.3) near each location, we have

$$\|w_{m+1}\|_q \leq (1 + Da_{m+1}^2)^{1/q} \|w_m\|_q.$$

Iterating this bound, we see that for all m ,

$$\|w_m\|_q \leq \left(\prod_{i=1}^{\infty} (1 + Da_i^2) \right)^{1/q} \|w_0\|_q \leq \exp\left(\frac{D}{q} \|\mathbf{a}\|_2^2\right) \|w_0\|_q;$$

since $\mathbf{a} \in \ell^2$ we are done. \square

Proof of Corollary 5.2. Again, choose a_* as in Proposition 5.7 and choose M so that $a_i < a_*$ when $i \geq M$. The carpet $S_{\mathbf{a}}$ splits into finitely many smaller carpets constructed from the sequence (a_M, a_{M+1}, \dots) and glued along their edges. If the modulus of the family of curves joining the left and right edges of each of the small carpets is positive, so too will be the modulus of the family of curves joining the left and right of the entire carpet. This follows from basic properties of the modulus.

It thus suffices to consider the case when $a_i < a_*$ for all i . In this case we repeat the argument in the previous proof starting with the family of curves which join the left and right edges of the carpet. Performing the inductive construction and taking advantage of Proposition 5.7 completes the proof.

Finally, we note that in the case when \mathbf{a} is monotone decreasing, then M can be chosen only depending on $\|\mathbf{a}\|_2$ (and not on the actual sequence \mathbf{a}). The p -modulus of the eventual curve family depends only on p and M . This establishes the final claim of the corollary. \square

It remains to establish Proposition 5.7. This is the goal of the following section. The argument is rather technical although essentially elementary. The reader is invited to skip over the following subsection on a first reading of the paper.

5.3. Compressing curve families: the proof of Proposition 5.7. The following construction is standard. For the convenience of the reader we provide a short proof.

Lemma 5.8. *There is a C^2 function $\varphi : [-1, 1] \rightarrow [0, 1]$ which satisfies*

- (1) $\varphi|_{[-0.1, 0.1]} \equiv 1$,
- (2) *the support of φ lies in $[-0.9, 0.9]$,*
- (3) $|\varphi'| \leq 5/2$, $|\varphi''| \leq 25/2$, *and*
- (4) $|\varphi'| \leq 14|\varphi|^{2/3}$.

Proof. Choose $\varphi'' : [-1, 1] \rightarrow \mathbb{R}$ to be the simplest piecewise linear function whose graph passes through the points $(\pm 1, 0)$, $(\pm 0.9, 0)$, $(\pm 0.7, b)$, $(\pm 0.3, -b)$, $(\pm 0.1, 0)$, and $(0, 0)$, where $b > 0$ is a constant to be determined.

Assuming that $\varphi'(-1) = \varphi(-1) = 0$, we integrate to find φ . Note that $\varphi|_{[-0.1, 0.1]} \equiv 0.08b$, so we choose $b = 1/0.08 = 25/2$. With this choice, $|\varphi'| \leq 0.2b = 5/2$ and $|\varphi''| \leq b = 25/2$. Hence conditions (1), (2) and (3) are satisfied.

Finally, note that for $0 \leq h \leq 0.2$, $\varphi''(-0.9+h) = 5bh = \frac{125}{2}h$, $\varphi'(-0.9+h) = \frac{5}{2}bh^2 = \frac{125}{4}h^2$ and $\varphi(-0.9+h) = \frac{5}{6}bh^3 = \frac{125}{12}h^3$, so

$$|\varphi'(-0.9+h)| = 5 \left(\frac{3}{2}\right)^{2/3} |\varphi(-0.9+h)|^{2/3} \leq 10|\varphi(-0.9+h)|^{2/3}.$$

This bound also applies for $x \in [0.7, 0.9]$. On the other hand, for $x \in [-0.7, 0.7]$ we have $\varphi(x) \geq \varphi(-0.7) = \frac{1}{12}$ and $|\varphi'(x)| \leq \frac{5}{2}$, so $|\varphi'(x)| \leq \frac{5}{2} = \frac{5}{2} \cdot 12^{2/3} \left(\frac{1}{12}\right)^{2/3} \leq 14|\varphi(x)|^{2/3}$. \square

We now begin the proof of Proposition 5.7.

We fix a positive constant $\delta_0 \leq \frac{1}{200}$. We will choose a large positive integer $N = N(\delta_0) \geq 4$; the precise choice will be made later in the proof. Finally, we assume that $a_* \leq 10^{-2-N}$; we will only consider $a_i < a_*$.

Let (Γ_i, σ_i) be a δ_0 -good family of curves on scales smaller than s_i and suppose that $B(z, 2s_{i+1})$ meets $\text{spt } \Gamma_i$. We can apply an isometry of \mathbb{R}^2 to reduce to the case when $z = 0$,

$$\begin{aligned} B(z, 2s_{i+1}) &\subset P := [-2s_{i+1}, 2s_{i+1}]^2 \\ &\subset Q := [-10^N s_{i+1}, 10^N s_{i+1}]^2 \\ &\subset R := [-s_i/10, s_i/10]^2 \subset B(z, s_i/3), \end{aligned}$$

and

$$(5.4) \quad \text{there exists a point in } P \text{ such that } \dot{\Gamma}_i \text{ has horizontal slope.}$$

Note that (5.4), in conjunction with (B) from Definition 5.6, implies that all curves in R have slopes within $(1/5)\delta_0$ of zero. In particular, each curve $\gamma \in \Gamma_i$ is a graph over the x -axis inside R . Henceforth we will assume that each curve is given in graph form: $y = \gamma(x)$. Nevertheless, we continue to denote by $\gamma = \{(x, \gamma(x))\}$ the graph itself.

We choose a curve γ_0 which passes near P and either bounds an existing gap in Q , or is far from an existing gap in Q . To be precise, let U be the open set from condition (A), which is bounded by one or two C^2 curves whose slopes satisfy, along with Γ_i , condition (B). If U meets $L = \{0\} \times [-3s_{i+1}, 3s_{i+1}]$, choose γ_0 which bounds the edge of U meeting L . If U does not meet L , choose $\gamma_0 \in \Gamma_i$ which passes through $(0, -3s_{i+1})$ or $(0, 3s_{i+1})$, chosen so that

$$(5.5) \quad d(\gamma_0(0), U) \geq 5s_{i+1}$$

We will compress the curves inside Q into the complement of P , leaving everything unchanged in $R \setminus Q$. To build Γ_{i+1} we will delete γ_0 from Γ_i if necessary, and apply a diffeomorphism on $X \setminus \gamma_0$ to compress the remaining curves around P . The two options in the choice of γ_0 correspond to either enlarging an existing gap in Γ_i , or creating a new gap at least $4s_{i+1}$ from any previous gap.

We rescale the function φ from Lemma 5.8 to the scale of Q by defining

$$\tilde{\varphi}(x) = 6s_{i+1}\varphi\left(\frac{x}{10^N s_{i+1}}\right).$$

Note that $|\tilde{\varphi}'| \leq 10^{2-N}$, $|\tilde{\varphi}''| \leq 10^{2-2N}s_{i+1}^{-1}$, and

$$(5.6) \quad |\tilde{\varphi}'| \leq 10^{2-N} \left(\frac{\tilde{\varphi}}{2s_{i+1}}\right)^{2/3}.$$

We now define the *local compression map* $H : Q \setminus \{\gamma_0\} \rightarrow Q$. Let $g : Q \setminus \{\gamma_0\} \rightarrow [-1, 1]$ be given by

$$g(x, y) = \begin{cases} \varphi\left(\frac{y - \gamma_0(x)}{10^N s_{i+1} - \gamma_0(x)}\right) & \text{if } y \in (\gamma_0(x), 10^N s_{i+1}) \\ -\varphi\left(\frac{\gamma_0(x) - y}{10^N s_{i+1} + \gamma_0(x)}\right) & \text{if } y \in (-10^N s_{i+1}, \gamma_0(x)). \end{cases}$$

Since the functions γ_0 and φ are C^2 and $10^N s_{i+1} - \gamma_0$ and $10^N s_{i+1} + \gamma_0$ take values in $[0.99 \cdot 10^N s_{i+1}, 1.01 \cdot 10^N s_{i+1}]$, g is C^2 . The function g varies from 0 near the top of Q to 1 just above γ_0 , and from -1 just below γ_0 to 0 near the bottom of Q . Next let

$$h(x, y) = y + \tilde{\varphi}(x) \cdot g(x, y),$$

and define

$$H(x, y) = (x, h(x, y)).$$

Both h and H are C^2 , moreover, H is a diffeomorphism.

Extend g to be zero outside Q and extend H to be the identity outside Q . The new collection of curves is defined by pushing forward by the local compression map H :

$$\Gamma_{i+1} = \{H(\gamma) : \gamma \in \Gamma_i, \gamma \neq \gamma_0\}.$$

The measure σ_i on Γ_i pushes forward in the obvious way to a measure on Γ_{i+1} that we denote by σ_{i+1} . We define $\tilde{\Gamma}_{i+1}$ and ν_{i+1} as indicated in Definition 5.6. Since $H(Q \setminus \{\gamma_0\})$ and P are disjoint, so are $\text{spt } \Gamma_{i+1}$ and P .

Proposition 5.9. $(\Gamma_{i+1}, \sigma_{i+1})$ is a δ_0 -good family of curves on scales below $3s_{i+1}$.

Assuming for the moment the validity of Proposition 5.9 we quickly complete the proof of Proposition 5.7. We first show that $\nu_i(Q)$ is sufficiently small compared to $\nu_i(R)$.

Lemma 5.10. *We have $\nu_i(Q)/\nu_i(R) \leq 10^{3N} a_{i+1}^2$.*

Proof. By condition (A), we know that $\mathcal{H}^2(\text{spt } \Gamma_i \cap R) \geq \frac{1}{3} \mathcal{H}^2(R) = 75^{-1} s_i^2$. By condition (C), we have, for $A = A_{z, s_i}$,

$$\nu_i(R) \geq (1 + \delta_0)^{-1} A \mathcal{H}^2(\text{spt } \Gamma_i \cap R) \geq 75^{-1} (1 + \delta_0)^{-1} A s_i^2.$$

Similarly, we see that

$$\nu_i(Q) \leq (1 + \delta_0) A \mathcal{H}^2(Q) \leq (1 + \delta_0) A \cdot 4 \cdot 10^{2N} s_{i+1}^2.$$

Combining these two equations gives

$$\frac{\nu_i(Q)}{\nu_i(R)} \leq \frac{(1 + \delta_0) A \cdot 4 \cdot 10^{2N} s_{i+1}^2}{75^{-1} (1 + \delta_0)^{-1} A s_i^2} \leq 10^{3N} a_{i+1}^2.$$

This finishes the proof of Lemma 5.10. □

Note that on Q , we have $w_{i+1} \leq 2w_i$. Therefore

$$(5.7) \quad \int_R w_{i+1}^q d\mathcal{H}^2 \leq \int_{R \setminus Q} w_i^q d\mathcal{H}^2 + 2^q \int_Q w_i^q d\mathcal{H}^2.$$

However, with $A > 0$ as chosen above,

$$\begin{aligned}
 \int_Q w_i^q d\mathcal{H}^2 &\leq (1 + \delta_0)^{q-1} A^{q-1} \int_Q w_i d\mathcal{H}^2 \\
 &\leq (1 + \delta_0)^{q-1} A^{q-1} 10^{3N} a_{i+1}^2 \int_R w_i d\mathcal{H}^2 \\
 &\leq (1 + \delta_0)^{q-1} A^{q-1} 10^{3N} a_{i+1}^2 (1 + \delta_0)^{-(1-q)} A^{1-q} \int_R w_i^q d\mathcal{H}^2 \\
 &= (1 + \delta_0)^{2q-2} 10^{3N} a_{i+1}^2 \int_R w_i^q d\mathcal{H}^2.
 \end{aligned}$$

Applying this to (5.7), we see that there is some constant $D = D(q)$ so that

$$\int_R w_{i+1}^q d\mathcal{H}^2 \leq (1 + D a_{i+1}^2) \int_R w_i^q d\mathcal{H}^2.$$

This completes the proof of Proposition 5.7. \square

The proof of Proposition 5.9 is divided into three lemmas.

Lemma 5.11. *Condition (A) of Definition 5.6 is satisfied.*

Proof. If γ_0 was chosen to bound the open set U in the complement of $\text{spt } \Gamma_i$, then the deformation H has only enlarged this set, and it is easy to see that Γ_{i+1} will satisfy condition (A).

On the other hand, if $\gamma_0 \in \Gamma_i$ was chosen so that $d(\gamma_0(0), U) \geq 5s_{i+1}$, then we need to check that the new open set opened up along γ_0 will not result in two open gaps in $\text{spt } \Gamma_{i+1}$ in the same s_{i+1} -ball.

Denote the curve which bounds the edge of U closest to γ_0 by γ_1 . Without loss of generality, we may assume that $\gamma_0(x) \leq \gamma_1(x)$ for all $x \in I := [-10^N s_{i+1}, 10^N s_{i+1}]$. To complete the proof of this lemma, it suffices to show that $\gamma_1(x) - \gamma_0(x) \geq 4s_{i+1}$ for all $x \in I$, since then in the image they will remain sufficiently far apart.

Let σ_0 be the σ_i measure of those curves of Γ_i which lie between γ_0 and γ_1 . For $x_1 \leq x_2$, let

$$T[x_1, x_2] = \{(x, y) \in Q \cap \text{spt } \Gamma_i : x_1 \leq x \leq x_2, \gamma_0(x) < y < \gamma_1(x)\}.$$

By (5.1), for any $x \in I$, $h > 0$ we have $\sigma_0 h \leq \nu_i(T[x, x+h])$. On the other hand, by condition (C) we have $\nu_i(T[x, x+h]) \leq (\gamma_1(x) - \gamma_0(x) + h/100)h(1 + \delta_0)A$, for the appropriate value of the constant $A = A_{z,r}$. Combining these and letting h go to zero, we see that

$$(5.8) \quad \sigma_0 \leq (1 + \delta_0)A(\gamma_1(x) - \gamma_0(x)).$$

Likewise, by considering $T[0, h]$, we see that

$$(\gamma_1(0) - \gamma_0(0) - h/100)h(1 + \delta_0)^{-1}A \leq \nu_i(T[0, h]) \leq 1.01\sigma_0 h,$$

and therefore

$$(5.9) \quad (\gamma_1(0) - \gamma_0(0))(1 + \delta_0)^{-1}A \leq 1.01\sigma_0.$$

Combining (5.5), (5.8) and (5.9), we see that

$$5s_{i+1} \leq \gamma_1(0) - \gamma_0(0) \leq (1 + \delta_0)A^{-1}1.01\sigma_0 \leq (1 + \delta_0)^2 1.01(\gamma_1(x) - \gamma_0(x)).$$

Therefore γ_0 and γ_1 are always at least $4s_{i+1}$ apart in Q . \square

Lemma 5.12. *Condition (B) of Definition 5.6 is satisfied.*

Proof. Let us consider $u_1, u_2 \in R \cap (\text{spt } \Gamma_{i+1})$ so that $d(u_1, u_2) \leq s_{i+1}$. Note that $u_k = H(v_k)$ for some $v_k \in \text{spt } \Gamma_i \cap R$, $k \in \{1, 2\}$, with $d(v_1, v_2) \leq 1.01d(u_1, u_2)$. Write $v_k = (x_k, \gamma_k(x_k))$ for some $\gamma_k \in \Gamma_i$.

We calculate the differential of the function $h \circ (\text{id} \otimes \gamma_k)$ as follows:

$$(5.10) \quad (h \circ (\text{id} \otimes \gamma_k))'(x_k) = C_{1k} + C_{2k} + C_{3k},$$

where $C_{1k} = \gamma_k'(x_k)$, $C_{2k} = \tilde{\varphi}'(x_k)g(v_k)$, and

$$C_{3k} = \tilde{\varphi}(x_k)(g \circ (\text{id} \otimes \gamma_k))'(x_k).$$

Eventually, we want to estimate the difference between $(h \circ (\text{id} \otimes \gamma_1))'(x_1)$ and $(h \circ (\text{id} \otimes \gamma_2))'(x_2)$. In view of (5.10), we write ΔC_1 , ΔC_2 , and ΔC_3 for the differences of the summands.

To estimate ΔC_1 we use the following elementary fact: for any $v, w \in [-1/4, 1/4]$, the vectors $\mathbf{v} = (1, v)$ and $\mathbf{w} = (1, w)$ satisfy

$$(5.11) \quad \frac{1}{2}|v - w| < \left| |\mathbf{v}|^{-1}\mathbf{v} - |\mathbf{w}|^{-1}\mathbf{w} \right| < 2|v - w|.$$

Recall that $\dot{\Gamma}_i(v_k) = \frac{(1, \gamma_k'(x_k))}{\sqrt{1 + \gamma_k'(x_k)^2}}$ for $k \in \{1, 2\}$. Since $\dot{\Gamma}_i(v_k)$ is a unit vector,

$$(5.12) \quad |\dot{\Gamma}_i(v_1) - \dot{\Gamma}_i(v_2)| = 2 \sin\left(\frac{1}{2}\angle(\dot{\Gamma}_i(v_1), \dot{\Gamma}_i(v_2))\right) \leq \angle(\dot{\Gamma}_i(v_1), \dot{\Gamma}_i(v_2)).$$

An application of (5.11) gives

$$(5.13) \quad \begin{aligned} |\Delta C_1| &= |\gamma_1'(x_1) - \gamma_2'(x_2)| \leq 2|\dot{\Gamma}_i(v_1) - \dot{\Gamma}_i(v_2)| \leq 2\angle(\dot{\Gamma}_i(v_1), \dot{\Gamma}_i(v_2)) \\ &\leq 2\delta_0 \left(\frac{d(v_1, v_2)}{2s_i}\right)^{2/3} \leq 3\delta_0 a_{i+1}^{2/3} \left(\frac{d(u_1, u_2)}{2s_{i+1}}\right)^{2/3}. \end{aligned}$$

If one of the points v_1 and v_2 lies outside Q , then both are close to the edge of Q , where H is the identity, thus $|\Delta C_2| = |\Delta C_3| = 0$. We therefore assume that both v_1 and v_2 are in Q .

Suppose v_1 and v_2 lie on opposite sides of γ_0 . Since $d(v_1, v_2) \leq 2s_{i+1}$, we can show that $|\tilde{\varphi}(x_1)| + |\tilde{\varphi}(x_2)| \leq 2d(u_1, u_2)$. Therefore, using (5.6) we see that

$$(5.14) \quad \begin{aligned} |\Delta C_2| &= |\tilde{\varphi}'(x_1)g(v_1) - \tilde{\varphi}'(x_2)g(v_2)| \leq |\tilde{\varphi}'(x_1)| + |\tilde{\varphi}'(x_2)| \\ &\leq 10^{2-N} \left(\frac{\tilde{\varphi}(x_1)}{2s_{i+1}}\right)^{2/3} + 10^{2-N} \left(\frac{\tilde{\varphi}(x_2)}{2s_{i+1}}\right)^{2/3} \leq 4 \cdot 10^{2-N} \left(\frac{d(u_1, u_2)}{2s_{i+1}}\right)^{2/3}. \end{aligned}$$

Since v_1 and v_2 are both close to γ_0 , they are both in the region where $|g| = 1$, whence

$$(5.15) \quad \Delta C_3 = 0.$$

It remains to bound ΔC_2 and ΔC_3 when v_1 and v_2 lie on the same side of γ_0 . Without loss of generality, we may assume that both are above γ_0 . Then $g(v_k) = \varphi(A_k/B_k)$ where $A_k = \gamma_k(x_k) - \gamma_0(x_k)$ and $B_k = 10^N s_{i+1} - \gamma_0(x_k)$. Note that

$$\begin{aligned} |A_k| &\leq 1.01 \cdot 10^N s_{i+1}, \quad |B_k| \in [0.99 \cdot 10^N s_{i+1}, 1.01 \cdot 10^N s_{i+1}], \\ |A_1 - A_2| &\leq 2d(v_1, v_2), \quad |B_1 - B_2| \leq \frac{1}{100}d(v_1, v_2). \end{aligned}$$

To estimate $|g(v_1) - g(v_2)|$ we use the simple estimate

$$(5.16) \quad \left| \frac{A_1}{B_1} - \frac{A_2}{B_2} \right| \leq \frac{|A_1 - A_2| \cdot |B_2| + |A_2| \cdot |B_1 - B_2|}{|B_1 B_2|} \\ \leq 3 \cdot 10^{-N} s_{i+1}^{-1} d(v_1, v_2) \leq 10^{1-N} \left(\frac{d(u_1, u_2)}{2s_{i+1}} \right)^{2/3}.$$

Thus

$$(5.17) \quad |g(v_1) - g(v_2)| = \left| \varphi \left(\frac{A_1}{B_1} \right) - \varphi \left(\frac{A_2}{B_2} \right) \right| \leq \|\varphi'\|_\infty \left| \frac{A_1}{B_1} - \frac{A_2}{B_2} \right| \leq 10^{2-N} \left(\frac{d(u_1, u_2)}{2s_{i+1}} \right)^{2/3}.$$

Also

$$(5.18) \quad |\tilde{\varphi}'(x_1) - \tilde{\varphi}'(x_2)| \leq \|\tilde{\varphi}''\|_\infty |x_1 - x_2| \leq 10^{2-2N} s_{i+1}^{-1} |x_1 - x_2| \leq 10^{3-2N} \left(\frac{d(u_1, u_2)}{2s_{i+1}} \right)^{2/3}.$$

We combine (5.17) and (5.18) to get

$$(5.19) \quad |\Delta C_2| \leq |\tilde{\varphi}'(x_1)| |g(v_1) - g(v_2)| + |\tilde{\varphi}'(x_1) - \tilde{\varphi}'(x_2)| |g(v_2)| \\ \leq 10^{2-N} \cdot 10^{2-N} \left(\frac{d(u_1, u_2)}{2s_{i+1}} \right)^{2/3} + 10^{3-2N} \left(\frac{d(u_1, u_2)}{2s_{i+1}} \right)^{2/3} \\ \leq 10^{5-2N} \left(\frac{d(u_1, u_2)}{2s_{i+1}} \right)^{2/3}.$$

Finally, we must bound $|\Delta C_3|$. As v_1, v_2 both lie above γ_0 , we have

$$(g \circ (\text{id} \otimes \gamma_k))'(x_k) = E_k \varphi'(A_k/B_k),$$

where A_k and B_k are as defined above and

$$E_k = \frac{A'_k B_k - A_k B'_k}{B_k^2}, \quad A'_k = \gamma'_k(x_k) - \gamma'_0(x_k), \quad B'_k = \gamma'_0(x_k).$$

Now $\max\{|A'_k|, |B'_k|\} \leq \frac{1}{100}$, so $|E_k| \leq 3 \cdot 10^{-2-N} s_{i+1}^{-1}$ and $(g \circ (\text{id} \otimes \gamma_k))'(x_k) \leq 10^{-1-N} s_{i+1}^{-1}$.

Since v_1 and v_2 lie on the same side of γ_0 , we have, denoting $\alpha = \left(\frac{d(u_1, u_2)}{2s_{i+1}} \right)^{2/3}$,

$$|A_1 - A_2|, |B_1 - B_2| \leq 10s_{i+1}\alpha, \quad |A'_1 - A'_2|, |B'_1 - B'_2| \leq \alpha,$$

so

$$|E_1 - E_2| = \left| \frac{A'_1 B_1 - A_1 B'_1}{B_1^2} - \frac{A'_2 B_2 - A_2 B'_2}{B_2^2} \right| \\ \leq 10^{1-4N} s_{i+1}^{-4} \left| (A'_1 B_1 - A_1 B'_1) B_2^2 - (A'_2 B_2 - A_2 B'_2) B_1^2 \right| \\ = 10^{1-4N} s_{i+1}^{-4} \left| B_1 B_2^2 (A'_1 - A'_2) + A'_2 B_1 B_2 (B_2 - B_1) + A_2 B_1^2 (B'_2 - B'_1) \right. \\ \left. + B'_1 B_1^2 (A_2 - A_1) + A_1 B'_1 (B_1 + B_2) (B_1 - B_2) \right| \\ \leq 10^{1-4N} s_{i+1}^{-4} (2 \cdot 10^{1+3N} s_{i+1}^3 \alpha + 3 \cdot 10^{2N} s_{i+1}^3 \alpha) \\ \leq 10^{3-N} s_{i+1}^{-1} \alpha = 10^{3-N} s_{i+1}^{-1} \left(\frac{d(u_1, u_2)}{2s_{i+1}} \right)^{2/3}.$$

Thus $|(g \circ (\text{id} \otimes \gamma_1))'(x_1) - (g \circ (\text{id} \otimes \gamma_2))'(x_2)|$ is equal to

$$\begin{aligned} \left| E_1 \varphi' \left(\frac{A_1}{B_1} \right) - E_2 \varphi' \left(\frac{A_2}{B_2} \right) \right| &\leq |E_1 - E_2| \cdot \left| \varphi' \left(\frac{A_1}{B_1} \right) \right| + |E_2| \cdot \left| \varphi' \left(\frac{A_1}{B_1} \right) - \varphi' \left(\frac{A_2}{B_2} \right) \right| \\ &\leq |E_1 - E_2| \cdot \|\varphi'\|_\infty + |E_2| \cdot \|\varphi''\|_\infty \cdot \left| \frac{A_1}{B_1} - \frac{A_2}{B_2} \right| \\ &\leq 10^{4-N} s_{i+1}^{-1} \left(\frac{d(u_1, u_2)}{2s_{i+1}} \right)^{2/3} + 10^{1-2N} s_{i+1}^{-1} \left(\frac{d(u_1, u_2)}{2s_{i+1}} \right)^{2/3} \\ &\leq 10^{5-N} s_{i+1}^{-1} \left(\frac{d(u_1, u_2)}{2s_{i+1}} \right)^{2/3}. \end{aligned}$$

Putting all this together,

$$\begin{aligned} |\Delta C_3| &\leq |\tilde{\varphi}(x_1)| |(g \circ (\text{id} \otimes \gamma_1))'(x_1) - (g \circ (\text{id} \otimes \gamma_2))'(x_2)| \\ &\quad + |\tilde{\varphi}(x_1) - \tilde{\varphi}(x_2)| |(g \circ (\text{id} \otimes \gamma_2))'(x_2)| \\ (5.20) \quad &\leq 6s_{i+1} \cdot 10^{5-N} s_{i+1}^{-1} \left(\frac{d(u_1, u_2)}{2s_{i+1}} \right)^{2/3} + \left(\frac{d(u_1, u_2)}{2s_{i+1}} \right)^{2/3} \cdot 10^{-1-N} s_{i+1}^{-1} \\ &\leq 10^{7-N} \left(\frac{d(u_1, u_2)}{2s_{i+1}} \right)^{2/3}. \end{aligned}$$

We can now tie all these estimates together. Using (5.12) again, we estimate

$$\angle(\dot{\Gamma}_{i+1}(u_1), \dot{\Gamma}_{i+1}(u_2)) \leq \frac{\pi}{2} |\dot{\Gamma}_{i+1}(u_1) - \dot{\Gamma}_{i+1}(u_2)| \leq \pi |(h \circ (\text{id} \otimes \gamma_1))'(x_1) - (h \circ (\text{id} \otimes \gamma_2))'(x_2)|.$$

This follows from (5.11) upon noting that

$$\dot{\Gamma}_{i+1}(u_k) = (H \circ (\text{id} \otimes \gamma_k))'(x_k) / |(H \circ (\text{id} \otimes \gamma_k))'(x_k)|$$

and $(H \circ (\text{id} \otimes \gamma_k))'(x_k) = (1, (h \circ (\text{id} \otimes \gamma_k))'(x_k))$.

We combine (5.13), (5.14), (5.15), (5.19) and (5.20) to conclude that

$$\begin{aligned} \angle(\dot{\Gamma}_{i+1}(u_1), \dot{\Gamma}_{i+1}(u_2)) &\leq \pi (|\Delta C_1| + |\Delta C_2| + |\Delta C_3|) \\ &\leq \pi \left(3\delta_0 a_{i+1}^{2/3} + 10^{5-N} + 10^{7-N} \right) \left(\frac{d(u_1, u_2)}{2s_{i+1}} \right)^{2/3} \leq \delta_0 \left(\frac{d(u_1, u_2)}{2s_{i+1}} \right)^{2/3}. \end{aligned}$$

This last inequality holds provided that $N = N(\delta_0)$ is chosen large enough. \square

Lemma 5.13. *Condition (C) of Definition 5.6 is satisfied.*

Proof. Let us write $\|D_{\dot{\Gamma}_i} H(v)\|$ for the magnitude of the directional derivative of H in the direction of $\dot{\Gamma}_i$ at the point v . Since H is C^2 on an open set,

$$w_{i+1}(H(v)) JH(v) = \|D_{\dot{\Gamma}_i} H(v)\| w_i(v)$$

for every v in the domain of H . Thus, for $u \in \text{spt } \Gamma_{i+1}$,

$$(5.21) \quad w_{i+1}(u) = JH^{-1}(u) \|D_{\dot{\Gamma}_i} H(H^{-1}(u))\| w_i(H^{-1}(u)).$$

We want to show that, on any given ball of radius s_{i+1} , there is a constant A' so that w_{i+1} takes values in $[(1 + \delta_0)^{-1} A', (1 + \delta_0) A']$ and is $\frac{2}{3}$ -Hölder continuous with constant $\frac{A' \delta_0}{(2s_{i+1})^{2/3}}$.

Sublemma 5.14. *On scales below $2s_{i+1}$, JH^{-1} is $10^{5-2N}s_{i+1}^{-1}$ -Lipschitz with values in $[(1 + 10^{3-N})^{-1}, 1 + 10^{3-N}]$. In particular, JH^{-1} is $\frac{2}{3}$ -Hölder continuous with constant $\frac{10^{6-2N}}{(2s_{i+1})^{2/3}}$.*

Proof. First, we compute the differential of H :

$$DH(x, y) = \begin{pmatrix} 1 & 0 \\ \tilde{\varphi}'(x)g(x, y) + \tilde{\varphi}(x)g_x(x, y) & 1 + \tilde{\varphi}(x)g_y(x, y) \end{pmatrix}.$$

Thus

$$(5.22) \quad JH = 1 + \tilde{\varphi}(x)g_y(x, y).$$

Outside Q , near the edge of Q , or near γ_0 , we have $JH \equiv 1$, so $JH^{-1} \equiv 1$. It remains to consider the case when v_1, v_2 are in Q and above γ_0 . We see that

$$g_y(x, y) = \frac{1}{10^N s_{i+1} - \gamma_0(x)} \varphi' \left(\frac{y - \gamma_0(x)}{10^N s_{i+1} - \gamma_0(x)} \right).$$

Now $10^N s_{i+1} - \gamma_0$ is 10^{-2} -Lipschitz and takes values in $[0.99 \cdot 10^N s_{i+1}, 1.01 \cdot 10^N s_{i+1}]$, thus $(10^N s_{i+1} - \gamma_0)^{-1}$ is $10^{-1-2N}s_{i+1}^{-2}$ -Lipschitz and takes values in $[0.98 \cdot 10^{-N}s_{i+1}^{-1}, 1.02 \cdot 10^{-N}s_{i+1}^{-1}]$.

On the other hand, $\varphi'(\frac{y-10^N s_{i+1}}{10^N s_{i+1} - \gamma_0(x)})$ has size at most $\frac{5}{2}$ and is Lipschitz with constant

$$\|\varphi''\|_\infty (1.01 \cdot 10^N s_{i+1} \cdot 10^{-1-2N}s_{i+1}^{-2} + 1.02 \cdot 10^{-N}s_{i+1}^{-1}) \leq 10^{2-N}s_{i+1}^{-1}.$$

Thus g_y has size at most $10^{1-N}s_{i+1}^{-1}$ and is Lipschitz with constant

$$1.02 \cdot 10^{-N}s_{i+1}^{-1} \cdot 10^{2-N}s_{i+1}^{-1} + \frac{5}{2} \cdot 10^{-1-2N}s_{i+1}^{-2} \leq 10^{3-2N}s_{i+1}^{-2}.$$

Therefore, JH takes values in $[(1 + 10^{3-N})^{-1}, 1 + 10^{3-N}]$ and is Lipschitz with constant

$$6s_{i+1} \cdot 10^{3-2N}s_{i+1}^{-2} + 10^{1-N}s_{i+1}^{-1} \cdot 10^{2-N} \leq 10^{4-2N}s_{i+1}^{-1}.$$

Since H^{-1} is 1.01-Lipschitz, $JH^{-1} = (JH \circ H^{-1})^{-1}$ takes values in $[(1 + 10^{3-N})^{-1}, 1 + 10^{3-N}]$ and is Lipschitz with constant $10^{5-2N}s_{i+1}^{-1}$. \square

Sublemma 5.15. *On scales below $2s_{i+1}$, $\|D_{\Gamma_i} H \circ H^{-1}\|$ is $\frac{2}{3}$ -Hölder continuous with constant $\frac{1}{2}\delta_0(2s_{i+1})^{-2/3}$ and takes values in $[1/1.01, 1.01]$.*

Proof. Writing $v_k = (x_k, \gamma_k(x_k))$ for $\gamma_k \in \Gamma_i$, we calculate

$$\|D_{\Gamma_i} H(v_k)\| = \frac{\|DH_{v_k}(1, \gamma'_k(x_k))\|}{\|(1, \gamma'_k(x_k))\|} = \sqrt{\frac{1 + ((h \circ (\text{id} \otimes \gamma_k))'(x_k))^2}{1 + (\gamma'_k(x_k))^2}}.$$

Thus $\|D_{\Gamma_i} H(v_k)\| \in [1/1.01, 1.01]$. Near $z = 1$, \sqrt{z} is 1-Lipschitz and H^{-1} is 1.01-Lipschitz, so it suffices to show that $\frac{1 + ((h \circ (\text{id} \otimes \gamma_k))'(x_k))^2}{1 + (\gamma'_k(x_k))^2}$ is $\frac{2}{3}$ -Hölder continuous with constant $\frac{1}{3}\delta_0(2s_{i+1})^{-2/3}$.

To this end, consider the equality

$$(5.23) \quad \frac{1 + L_1^2}{1 + M_1^2} - \frac{1 + L_2^2}{1 + M_2^2} = \frac{(1 + M_1^2)(L_1^2 - L_2^2) + (1 + L_1^2)(M_2^2 - M_1^2)}{(1 + M_1^2)(1 + M_2^2)}.$$

In our case, $M_k = \gamma'_k$ is at most $\frac{1}{200}$ and is $\frac{2}{3}$ -Hölder continuous with constant $3\delta_0 a_{i+1}^{2/3} (2s_{i+1})^{-2/3}$, while $L_k = (h \circ (\text{id} \otimes \gamma_k))'$ is at most $\frac{1}{100}$ and is $\frac{2}{3}$ -Hölder continuous with constant $\frac{1}{10}\delta_0(2s_{i+1})^{-2/3}$.

(This follows from condition (B), for sufficiently large N .) Thus by (5.23) $\frac{1+((h \circ (\text{id} \otimes \gamma_k))')^2}{1+(\gamma_k')^2}$ is $\frac{2}{3}$ -Hölder continuous with constant

$$2\frac{1}{10}\delta_0(2s_{i+1})^{-2/3} + 2 \cdot 3\delta_0 a_{i+1}^{2/3}(2s_{i+1})^{-2/3} \leq \frac{1}{3}\delta_0(2s_{i+1})^{-2/3}. \quad \square$$

Sublemma 5.16. *On scales below $2s_{i+1}$, $A^{-1}w_i \circ H^{-1}$ is $\frac{2}{3}$ -Hölder continuous with constant $\frac{10^{-N/2}\delta_0}{(2s_{i+1})^{2/3}}$ and takes values in $[(1 + \delta_0)^{-1}, 1 + \delta_0]$.*

Proof. We already know that $A^{-1}w_i$ is $\frac{2}{3}$ -Hölder continuous with constant $\frac{\delta_0}{(2s_i)^{2/3}}$ and takes values in $[(1 + \delta_0)^{-1}, 1 + \delta_0]$. Since H^{-1} is 1.01-Lipschitz, we conclude that $A^{-1}w_i \circ H^{-1}$ is $\frac{2}{3}$ -Hölder with constant

$$1.01\frac{\delta_0}{(2s_i)^{2/3}} \leq \frac{1.01a_{i+1}^{2/3}\delta_0}{(2s_{i+1})^{2/3}} \leq \frac{10^{-N/2}\delta_0}{(2s_{i+1})^{2/3}}. \quad \square$$

The following lemma is trivial.

Sublemma 5.17. *If $f_i : X \rightarrow [M_i^{-1}, M_i]$ is α -Hölder continuous with constant L_i for $i \in \{1, 2, 3\}$, then $g : X \rightarrow [(M_1M_2M_3)^{-1}, M_1M_2M_3]$ given by $g = f_1f_2f_3$ is α -Hölder continuous with constant $L_1M_2M_3 + M_1L_2M_3 + M_1M_2L_3$.*

We are now ready for the proof of Proposition 5.9.

Proof of Proposition 5.9. Sublemmas 5.14 to 5.17 together with (5.21) combine to show that $A^{-1}w_{i+1}$ takes values in $[1/1.02, 1.02]$ and is $\frac{2}{3}$ -Hölder continuous with constant at most

$$(2s_{i+1})^{-2/3} \left(10^{6-2N}(1.01)(1 + \delta_0) + (1 + 10^{3-N})\frac{\delta_0}{2}(1 + \delta_0) + (1 + 10^{3-N})(1.01)10^{-N/2}\delta_0 \right),$$

which is bounded above by $\frac{3}{4}\delta_0(2s_{i+1})^{-2/3}$ provided N is large enough.

Thus on balls of radius s_{i+1} , the ratio of maximum to minimum values of w_{i+1} (which is the same as the ratio for $A^{-1}w_{i+1}$) is at most

$$\left(\frac{1}{1.02} \right)^{-1} \left(\frac{1}{1.02} + \frac{3}{4}\delta_0 \right) = 1 + 1.02\frac{3}{4}\delta_0 \leq (1 + \delta_0)^2.$$

We can choose $A' \in [A/1.02, 1.02A]$ as appropriate for the given ball to conclude that the Hölder constant for w_{i+1} on the ball is at most $A'\frac{3}{4}\delta_0(2s_{i+1})^{-2/3} \leq \frac{A'\delta_0}{(2s_{i+1})^{2/3}}$. \square

Conditions (A), (B) and (C) having been verified for the new curve family, the proof of Proposition 5.9 is now complete. \square

6. VALIDITY OF POINCARÉ INEQUALITIES

This section contains the proof of our main theorems. We show that $S_{\mathbf{a}}$, equipped with the Euclidean metric and the canonical measure described in subsection 3.1, admits a 1-Poincaré inequality provided $\mathbf{a} \in \ell^1$, or a p -Poincaré inequality for each $p > 1$ if $\mathbf{a} \in \ell^2$.

According to Theorem 2.1, the validity of a Poincaré inequality is equivalent to the existence of curve families of uniformly and quantitatively large weighted modulus joining arbitrary pairs of points. The desired curve family must diverge transversally as it escapes from the endpoints. This divergence is measured via transversal measures on the edges of squares in the precarpets.

We explicitly construct this family using the structure of the carpet $S_{\mathbf{a}}$. In subsection 6.1 we state and prove four lemmas providing the ‘building blocks’ of the construction. Each of these building blocks consists of families of disjoint curves joining edges of certain squares in the carpet. These families are each equipped with a natural transversal measure, and concatenated to produce the desired family connecting the given endpoints.

The proofs of the ‘building block’ lemmas in the cases $\mathbf{a} \in \ell^1$ and $\mathbf{a} \in \ell^2$ are very different.

In the case $\mathbf{a} \in \ell^1$ ($p = 1$ and $q = \infty$), the thinnest part of the carpet still has positive length, so we can use combinatorial constructions to build modular curve families which avoid all omitted squares in the precarpet $S_{\mathbf{a},N}$. The machinery from section 5 is unavailable to us in this case because the argument in section 5.2 required q to be finite.

In the case $\mathbf{a} \in \ell^2$ ($p > 1$), considering only curves which pass through a thin part of the carpet cannot suffice. We instead construct a larger family using the machinery of section 5. Ziemer’s duality principle is insufficient for our needs, as we require families of disjoint paths equipped with suitable transversal measures. It is far from clear if such families can be produced using the duality method.

Hence, as we wish to employ the machinery of section 5, we need to assume that

$$(6.1) \quad a_i < a_* \quad \forall i,$$

where a_* is the constant in Proposition 5.7. Here as always we have $\mathbf{a} = (a_1, a_2, \dots) \in \ell^2$. In the $p > 1$ case it is legitimate for us to assume that (6.1) holds, because any $S_{\mathbf{a}}$ with $\mathbf{a} \in \ell^2$ is the finite union of similar copies of some carpet $S_{\mathbf{a}'}$, where $\mathbf{a}' \in \ell^2$ and all entries of \mathbf{a}' are less than a_* , glued along their boundaries. By Theorem 2.2, if each of these smaller carpets supports a p -Poincaré inequality, then the original carpet will also. Note that the constants for the overall Poincaré inequality will depend on the number of copies which are glued together, which in turn depends on how far out in the sequence \mathbf{a} we must go to ensure condition (6.1). If \mathbf{a} is monotone decreasing, this data depends only on $\|\mathbf{a}\|_2$. (We remark in passing that the gluing procedure in Theorem 2.2 differs from the union considered here, however, the resulting metrics are bi-Lipschitz equivalent and the validity of Poincaré inequalities is unaltered by this change of metric.)

So throughout this section, in the $p > 1$ case, we impose assumption (6.1).

In the $p = 1$ case, we do not really need this assumption, though we impose it with $a_* = \frac{1}{20}$ *only* in order to simplify the proof. If (6.1) is not satisfied, the combinatorial argument in the proof of Lemma 6.10 becomes slightly more complicated, however, the rest of the argument is unchanged. We leave such modifications to the industrious reader.

By Theorem 2.5, to demonstrate that $S_{\mathbf{a}}$ admits a p -Poincaré inequality for the desired p , it suffices to prove that the precarpets $\{(S_{\mathbf{a},m}, d, \mu_m)\}_{m \in \mathbb{N}}$ support a p -Poincaré inequality with constants independent of m . In order to simplify the discussion, we work in a fixed precarpet $S_{\mathbf{a},M}$. In subsection 6.2 we use the ‘building block’ lemmas of subsection 6.1 to build the desired path family in $S_{\mathbf{a},M}$, and complete the proof of our main theorems.

6.1. Building block lemmas. Recall that \mathcal{T}_m denotes the collection of all level m squares in the construction of the carpet $S_{\mathbf{a}}$.

Definition 6.1. Fix $m \in \mathbb{N}$ and a square $T' \in \mathcal{T}_{m-1}$. For non-negative integers a, b, k and l , a set $\mathcal{C} = [as_m, (a+k)s_m] \times [bs_m, (b+l)s_m]$ is called a k by l block in T' if it is contained in T' and does not contain the removed central subsquare of T' . We will often choose a preferred edge L of a block \mathcal{C} that does not contain the boundary of the removed central subsquare of T' , and declare it to be the *leading edge* of \mathcal{C} . The pair (\mathcal{C}, L) is called a *directed block in T'* .

A directed 1 by 1 block in T' is called a *directed square in T'* . We will suppress reference to T' if the dependence is clear or unimportant.

Note that the choice of a leading edge of a block gives rise to an outward-pointing unit normal vector on the boundary of the block.

Directed blocks (possibly in different squares or even of different generations) are *coherent* if the corresponding outward-pointing unit normal vectors coincide.

We introduce a distinguished set π_M which will parameterize certain curve families. Its definition depends on p , specifically, on whether $p = 1$ or $p > 1$.

First suppose that $p = 1$. In this case, let $\pi_M \subset [0, 1]$ be the set of all $x \in [0, 1]$ with the property that the line $\{x\} \times \mathbb{R}$ does not meet the interior or left hand side of a peripheral square removed in the construction of $S_{\mathbf{a}, M}$. Let (\mathcal{C}, L) be a directed block in a square $T' \in \mathcal{T}_{m-1}$. There is a unique orientation preserving isometry $i : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ so that $(0, 0) \in i(\mathcal{C}) \subset [0, 1]^2$, and so that $i(L)$ is contained in the x -axis. We define $\pi_M(L)$ to be the union of $L \cap i^{-1}(\pi_M)$ with the endpoints of L .

Given two such sets $\pi_M(L_1)$ and $\pi_M(L_2)$ arising from isometries i_1 and i_2 , there is a unique bijection $h : \pi_M(L_1) \rightarrow \pi_M(L_2)$ so that $i_2 \circ h \circ i_1^{-1}$ is an order-preserving, piecewise linear bijection from $i_1(\pi_M(L_1)) \subset \mathbb{R}$ to $i_2(\pi_M(L_2)) \subset \mathbb{R}$ with a.e. constant derivative. We call h the natural *ordered bijection*.

Now suppose that $p > 1$. In this case we let $\pi_M(L) = L$ and we let h be the corresponding bijection, which now works out to be the restriction of an affine map.

In both cases, $\mathcal{H}^1(\pi_M(L)) \asymp \text{diam}(L)$.

Definition 6.2. Let E be a Borel subset of a side of a block \mathcal{C} of positive and finite \mathcal{H}^1 measure. A *path family on E (in \mathcal{C})* is a collection of disjoint curves $\Gamma = \{\gamma_z\}_{z \in E}$ in $\mathcal{C} \cap S_{\mathbf{a}, M}$ with the property that $\gamma_z(0) = z$ for all $z \in E$. We also require that the measure ν_Γ , defined as in (5.1) by

$$\nu_\Gamma(A) = \frac{1}{\mathcal{H}^1(E)} \int_E \mathcal{H}^1(A \cap \gamma_u) d\mathcal{H}^1(u),$$

is Borel. Note that $\nu_\Gamma = \nu_{(\Gamma, \sigma)}$ where σ is the measure $(\mathcal{H}^1(E))^{-1} \mathcal{H}^1|_E$.

Recall that $p \in [1, \infty)$ has Hölder conjugate $q \in (1, \infty]$ defined by $\frac{1}{p} + \frac{1}{q} = 1$. (The convention $1/\infty = 0$ is used frequently in this section.)

As previously discussed, we will construct curve families of uniformly and quantitatively large weighted modulus joining arbitrary pairs of points in $S_{\mathbf{a}}$. The following notion of q -connection quantifies the degree to which these curve families must diverge transversally as they escape from the endpoints, measured with respect to the L^q norm. The mysterious exponent on the right hand side of (6.2) can be justified by a dimensional analysis. Note that the measure ν_Γ is homogeneous of degree 1 relative to the scalings $x \mapsto \lambda x$, $\lambda > 0$, of \mathbb{R}^2 . Hence the Radon–Nikodym derivative $\frac{d\nu_\Gamma}{d\mathcal{H}^2}$ is homogeneous of degree -1 . Since the L^q norm is computed with respect to Lebesgue measure, it follows that the left hand side of (6.2) is homogeneous of degree $-1 + \frac{2}{q}$.

Definition 6.3. We say that the directed block (\mathcal{C}_2, L_2) *follows* the directed block (\mathcal{C}_1, L_1) if $\mathcal{C}_1 \cap \mathcal{C}_2 = L_1$ and $L_1 \not\subset L_2$. Let $h : \pi_M(L_1) \rightarrow \pi_M(L_2)$ be the natural ordered bijection. A path family Γ on $\pi_M(L_1)$ in \mathcal{C}_2 is called an q -*connection* (with constant C) if $\gamma(1) = h(\gamma(0))$

for each $\gamma \in \Gamma$, and if $\nu_\Gamma \ll \mathcal{H}^2 \llcorner \text{spt}(\Gamma)$ with

$$(6.2) \quad \left\| \frac{d\nu_\Gamma}{d\mathcal{H}^2} \right\|_{L^q(\mathcal{C}_2; \mathcal{H}^2)} \leq C (\mathcal{H}^1(\pi_M(L_1)))^{-1+2/q}.$$

For the remainder of this subsection, we fix $0 < m \leq M$, and directed blocks (\mathcal{C}_2, L_2) following (\mathcal{C}_1, L_1) in a directed square $(T', L') \in \mathcal{T}_{m-1}$. We declare the *central column* of T' to be the central row or column of T' that intersects L' .

In the case $\mathbf{a} \in \ell^1$ and $p = 1$, we let $p^* = 1$. When $\mathbf{a} \in \ell^2$ and $p > 1$, we let $p^* = 2$.

Lemma 6.4 (Expanding). *Suppose that*

- (\mathcal{C}_1, L_1) and (\mathcal{C}_2, L_2) are coherent with (T', L') ,
- neither \mathcal{C}_1 nor \mathcal{C}_2 intersects the central column of T' ,
- the sides of \mathcal{C}_2 perpendicular to L_2 have length equal to that of L_1 .

Then there is a q -connection Γ in \mathcal{C}_2 with constant $C = C(\|\mathbf{a}\|_{p^})$.*

Proof. We begin with the case $\mathbf{a} \in \ell^1$, $p = 1$ and $q = \infty$.

For the purposes of this proof, we will assume that L_1 contains an endpoint of the side of \mathcal{C}_2 in which it lies.

We may assume without loss of generality that $T' = [0, s_{m-1}]^2$ and $e' = [0, s_{m-1}] \times \{0\}$. We may further assume that $\mathcal{C}_2 = [0, as_m] \times [0, bs_m]$ and $\mathcal{C}_1 = [0, bs_m] \times [bs_m, cs_m]$. Here a, b , and c are positive integers with $a \geq c$ and $c > b$. For ease of notation, set $E = \pi_M(L_1)$ and $F = \pi_M(L_2)$. Note that the natural ordered bijection $h: E \rightarrow F$ satisfies

$$h'(z) = \frac{\mathcal{H}^1(F)}{\mathcal{H}^1(E)} \geq 1$$

for every interior point z of E , i.e., for all but finitely many points. In the case that $m = M$, the function h is affine.

We now define a path family Γ on E . Given $z = (u, ls_m) \in E$, let γ_z^1 be the vertical line segment connecting (u, ls_m) to (u, u) , let γ_z^2 be the horizontal line segment connecting (u, u) to $h(z) + (0, u)$, and let γ_z^3 be the vertical line segment connecting $h(z) + (0, u)$ to $h(z)$. Let γ_z be the concatenation of γ_z^1 , γ_z^2 and γ_z^3 ; then $\gamma_z \subseteq S_{\mathbf{a}, M} \cap \mathcal{C}_2$. Let $\Gamma = \{\gamma_z : z \in E\}$. See Figure 6.

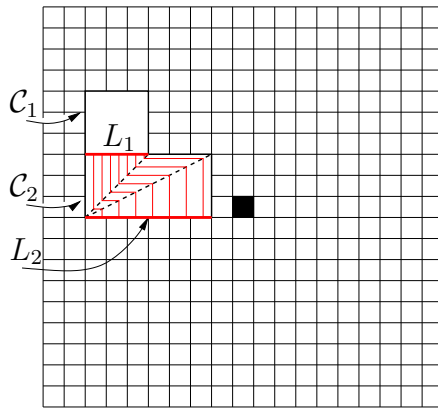


FIGURE 6. Expanding to a larger interval while avoiding the central column

We may write the support of Γ as the union of the supports of the curve families

$$\Gamma^i = \{\gamma_z^i\}_{z \in E}, \quad i = 1, 2, 3.$$

Given a set A contained in the support of Γ , we write $A^i = A \cap \text{spt } \Gamma^i$.

For $i = 1$ or 2 , Fubini's theorem yields $\mathcal{H}^2(A^i) = \mathcal{H}^1(E)\nu_\Gamma(A^i)$. For $i = 3$, a simple change of variables shows that

$$\mathcal{H}^2(A^3) = \mathcal{H}^1(F)\nu_\Gamma(A^3) \geq \mathcal{H}^1(E)\nu_\Gamma(A^3).$$

Together, this shows that Γ is an ∞ -connection with constant 1.

We now outline the proof in the $\mathbf{a} \in \ell^2$ and $p > 1$ case. The main difference here is that the Cantor set corresponding to the thinnest part of the carpet has zero length, which means that we cannot discard the curves that intersect the boundary of a square at point in the ‘shadow’ of a removed square.

To produce the desired q -connection, we use the bending machinery of subsection 5.2. The ‘input’ consists of the family of straight line segments connecting each point of L_1 to the corresponding point of L_2 under the natural ordered bijection, equipped with the measure σ induced from normalized linear measure on L_1 . Since a q -connection must lie in $\mathcal{C}_2 \cap S_{\mathbf{a},M}$, we ‘bend’ this initial family around the subsquares of \mathcal{C}_2 that were removed in the construction of $S_{\mathbf{a},M}$. This initial family satisfies the hypotheses of Proposition 5.7, and successive applications of that result produce a path family on E for which the q -norm of the Radon-Nikodym derivative is bounded quantitatively by that of the initial family. This implies that the output family is a q -connection. \square

Remark 6.5. When we use the above lemma, we will usually expand by a factor of two and always by a factor less than ten.

The proofs of the following three results are nearly identical to that given above. See Figures 7 and 8 for the $p = 1$ case, and Figure 9 for the $p > 1$ case.

Lemma 6.6 (Expanding to generation larger). *Suppose that*

- (\mathcal{C}_1, L_1) is coherent with (T', L')
- $L_2 = L'$,
- the length of L_1 is equal to the length of an edge of \mathcal{C}_2 perpendicular to L_2 ,
- L_1 intersects an edge of T' perpendicular to L' .

Then there is a q -connection Γ in \mathcal{C}_2 with constant $C = C(\|\mathbf{a}\|_{p^})$.*

Lemma 6.7 (Turning). *Suppose that*

- L_1 and L_2 are perpendicular, and
- all edges of \mathcal{C}_2 have length equal to the length of L_1 .

Then there is q -connection Γ in \mathcal{C}_2 with constant $C = C(\|\mathbf{a}\|_{p^})$.*

Lemma 6.8 (Straight). *Suppose that*

- (\mathcal{C}_1, L_1) and (\mathcal{C}_2, L_2) are coherent with (T', L') ,
- L_1 and L_2 are of equal length.

Then there is a q -connection Γ in \mathcal{C}_2 with constant $C = C(\|\mathbf{a}\|_{p^})$.*

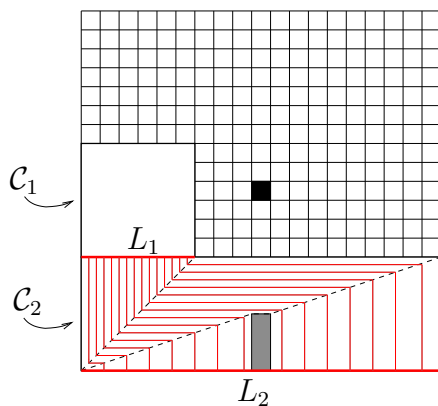


FIGURE 7. Expanding to the parent generation

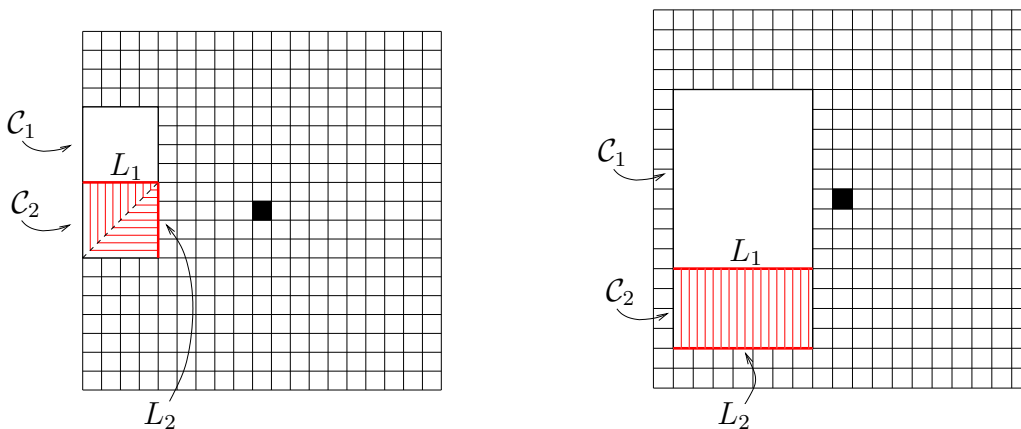


FIGURE 8. (a) Turning; (b) Going straight

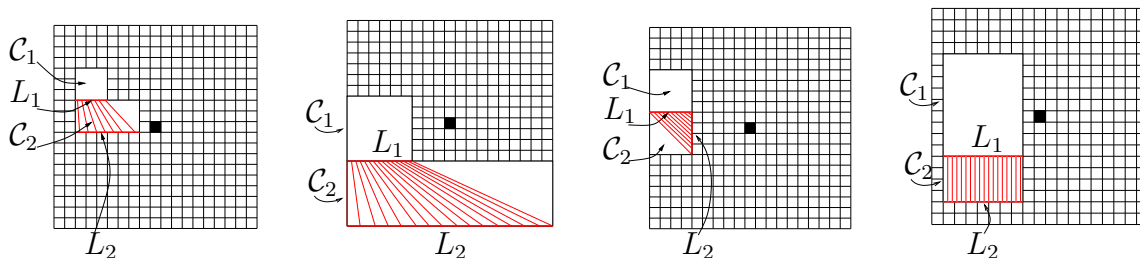


FIGURE 9. Initial curve families for the $p > 1$ case of Lemmas 6.4 - 6.8

6.2. **Poincaré inequality.** In this subsection we prove the following:

Proposition 6.9. *Suppose either $p = 1$ and $\mathbf{a} \in \ell^1$, or $p > 1$ and $\mathbf{a} \in \ell^2$. Then $S_{\mathbf{a}}$ admits a p -Poincaré inequality.*

Proof. As mentioned in the introduction to this section, it is enough to prove that for fixed M , the precarpet $(S_{\mathbf{a},M}, d, \mu_M)$ supports a p -Poincaré inequality with constant independent of M . In order to do this, we take advantage of Theorem 2.1. The constant C_2 in Keith's condition (2.1) will be an absolute value which could in principle be computed explicitly as a fixed multiple of the implicit multiplicative constant in conditions (5) and (6) of Lemma 6.10

below. On the other hand, the constant C_1 in (2.1) depends on C_0 in Lemma 6.10, which in turn depends on the constants in Lemmas 6.4–6.8 above. In particular, C_1 will depend heavily on $\|\mathbf{a}\|_{p^*}$.

In order to verify the condition in Theorem 2.1, let us fix $x, y \in S_{\mathbf{a}, M}$ with $x \neq y$. If $d(x, y) < 10s_M$, then we are in the Euclidean situation with possibly a square removed nearby, so (2.1) holds with uniform constants. Let us assume that for some $m \leq M$ we have $10s_m \leq d(x, y) < 10s_{m-1}$.

The implicit multiplicative constants in conditions (5) and (6) below are fixed, universal quantities which could be explicitly computed; to simplify the story we have spared the reader any explicit calculation. For instance, both of these multiplicative constants can be chosen to be 100.

Lemma 6.10. *There exist integers $K_- < 0 < K_+$, a sequence of directed blocks $\{(\mathcal{C}_i, L_i)\}_{i=K_-}^{K_+}$, and path families Γ_i each supported on \mathcal{C}_i , with the following properties, for some uniform constant C_0 .*

- (1) \mathcal{C}_{K_-} and \mathcal{C}_{K_+} are 2 by 1 blocks (or 1 by 2 blocks) on scale s_M containing x and y respectively.
- (2) $d(x, L_{K_-}) \geq s_M/2$ and $d(y, L_{K_+}) \geq s_M/2$.
- (3) Γ_{K_-} and Γ_{K_+} consist of the collection of straight lines joining x to L_{K_-} and y to L_{K_+} respectively.
- (4) For each $i = (K_- + 1), \dots, -1$, (\mathcal{C}_i, L_i) follows $(\mathcal{C}_{i-1}, L_{i-1})$; for each $i = 1, \dots, K_+ - 1$, (\mathcal{C}_i, L_i) follows $(\mathcal{C}_{i+1}, L_{i+1})$; (\mathcal{C}_0, L_1) follows $(\mathcal{C}_{-1}, L_{-1})$, and (\mathcal{C}_0, L_{-1}) follows (\mathcal{C}_1, L_1) . In each case, Γ_i is a q -connection with constant C_0 .
- (5) For each $i = (K_- + 1), \dots, (K_+ - 1)$,

$$\min\{d(x, \mathcal{C}_i), d(y, \mathcal{C}_i)\} \asymp \text{diam}(\mathcal{C}_i) \asymp \text{diam}(L_i).$$

- (6) $\sum_{i=K_-}^{K_+} \text{diam}(\mathcal{C}_i) \asymp d(x, y)$.
- (7) The blocks $\mathcal{C}_{K_-}, \dots, \mathcal{C}_{K_+}$ are essentially disjoint.

We postpone the proof of this lemma.

The path families $\Gamma_{K_-}, \dots, \Gamma_{K_+}$ concatenate together by gluing paths using the natural ordered bijection on each block. This gives a path family Γ consisting of pairwise disjoint, rectifiable curves joining x to y , carrying a probability measure $\sigma = \sigma_\Gamma$ on Γ which agrees with σ_{Γ_i} for each i on \mathcal{C}_i . The measure $\nu = \nu_\Gamma$ on the support of Γ defined by

$$(6.3) \quad \nu(A) = \int_{\Gamma} \mathcal{H}^1(A \cap \gamma) d\sigma$$

restricts to ν_{Γ_i} on each \mathcal{C}_i .

This measure ν is absolutely continuous with respect to μ_{xy} . For $i = K_-, \dots, K_+$, we have the following bound on the Radon–Nikodym derivative $\frac{d\nu}{d\mu_{xy}}$. We have $\text{diam}(\mathcal{C}_i) \asymp \min\{d(x, \mathcal{C}_i), d(y, \mathcal{C}_i)\}$, and so $\mu_{xy} \asymp \frac{1}{\text{diam}(\mathcal{C}_i)} \mathcal{H}^2$ on \mathcal{C}_i . Therefore

$$\begin{aligned} \left\| \frac{d\nu}{d\mu_{xy}} \right\|_{L^q(\mathcal{C}_i; \mu_{xy})} &\asymp \left\| \frac{d\nu}{d\mathcal{H}^2} \right\|_{L^q(\mathcal{C}_i; \mathcal{H}^2)} \text{diam}(\mathcal{C}_i)^{1-1/q} \\ &\lesssim (\mathcal{H}^1(\pi_M(L_i)))^{-1+2/q} \text{diam}(\mathcal{C}_i)^{1-1/q} \\ &\lesssim \text{diam}(\mathcal{C}_i)^{-1+2/q} \text{diam}(\mathcal{C}_i)^{1-1/q} = \text{diam}(\mathcal{C}_i)^{1/q}. \end{aligned}$$

This bound also holds on \mathcal{C}_{K_-} and \mathcal{C}_{K_+} : note that on $\text{spt}(\Gamma_{K_-}) \subset \mathcal{C}_{K_-}$, both ν and μ_{xy} are $\asymp \mathcal{H}^2/d(x, \cdot)$. An elementary calculation gives

$$\left\| \frac{d\nu}{d\mu_{xy}} \right\|_{L^q(\mathcal{C}_{K_-}; \mu_{xy})} \lesssim s_M^{1/q} \asymp \text{diam}(\mathcal{C}_{K_-})^{1/q}.$$

An analogous argument proves the bound for \mathcal{C}_{K_+} .

We claim that

$$\left\| \frac{d\nu}{d\mu_{xy}} \right\|_{L^q(\mu_{xy})} \lesssim d(x, y)^{1/q}.$$

If $q = \infty$ this is immediate from the above bounds. If $q < \infty$, we see that

$$\left\| \frac{d\nu}{d\mu_{xy}} \right\|_{L^q(\mu_{xy})}^q = \sum_{i=K_-}^{K_+} \left\| \frac{d\nu}{d\mu_{xy}} \right\|_{L^q(\mathcal{C}_i; \mu_{xy})}^q \lesssim \sum_{i=K_-}^{K_+} \text{diam}(\mathcal{C}_i) \lesssim d(x, y).$$

Let ρ be admissible for Γ . Then

$$\begin{aligned} 1 &\leq \int_{\Gamma} \int_{\gamma} \rho \, ds \, d\sigma(\gamma) = \int_{\text{spt} \Gamma} \rho \, d\nu \\ &= \int_{S_{\mathbf{a}, M}} \rho \frac{d\nu}{d\mu_{xy}} \, d\mu_{xy} \leq \|\rho\|_{L^p(\mu_{xy})} \left\| \frac{d\nu}{d\mu_{xy}} \right\|_{L^q(\mu_{xy})} \lesssim \|\rho\|_{L^p(\mu_{xy})} d(x, y)^{1/q}. \end{aligned}$$

Consequently,

$$\int_{S_{\mathbf{a}, M}} \rho^p \, d\mu_{xy} \gtrsim d(x, y)^{-p/q} = d(x, y)^{1-p}.$$

Thus (2.1) holds. This completes the proof that $S_{\mathbf{a}}$ admits a p -Poincaré inequality. \square

It remains to construct the block family described in the statement of Lemma 6.10.

Proof of Lemma 6.10. Recall that $m \leq M$ is chosen so that $10s_m \leq d(x, y) < 10s_{m-1}$.

We construct the curve family by induction. To make the proof more readable, we outline the basic steps, and leave the details to the reader. The basic idea is as follows: we use the expanding and turning lemmas (Lemmas 6.4–6.8) to build a sequence of blocks which grow in size at a linear rate as they travel away from x until reaching size $\sim d(x, y)/100$. We do the same for y , and then join up the two sequences using the same lemmas.

We now describe the construction in more detail.

First, $x \in T \subset T'$ for some $T \in \mathcal{T}_M$ and $T' \in \mathcal{T}_{M-1}$, and we can find a 1 by 2 (or 2 by 1) directed block (\mathcal{C}_0, L_0) so that $T \subset \mathcal{C}_0 \subset T'$, and L_0 is the short edge of \mathcal{C}_0 furthest from x , and L_0 does not meet the boundary of T' or the square of side s_M removed from T' . This gives us our first directed block (\mathcal{C}_0, L_0) , which satisfies conditions (1) and (2), and we define Γ_0 according to condition (3).

The induction step is as follows. We assume that we have a sequence of blocks contained in a 1 by 2 (or 2 by 1) directed block (\mathcal{C}_-, L_-) on scale s_n , which is contained in some $T' \in \mathcal{T}_{n-1}$ in such a way that the short edge L_- does not meet the the boundary of T' , or of the central removed square of T' .

We choose a 1 by 2 (or 2 by 1) directed block (\mathcal{C}_+, L_+) on scale s_{n-1} so that $T' \subset \mathcal{C}_+$, and L_+ is the short edge of \mathcal{C}_+ furthest from \mathcal{C}_- , and L_+ does not meet the boundary or centrally removed square of the square $T'' \in \mathcal{T}_{s_{n-2}}$ with $T' \subset T''$.

We now build a sequence of directed blocks

$$(\mathcal{C}_-, L_-) = (\mathcal{C}'_0, L'_0), (\mathcal{C}'_1, L'_1), \dots, (\mathcal{C}'_t, L'_t)$$

inside \mathcal{C}_+ , where $L'_t = L_+$, and where for $i = 1, \dots, t$, (\mathcal{C}'_i, L'_i) follows $(\mathcal{C}'_{i-1}, L'_{i-1})$. Moreover, these blocks satisfy conditions (4),(5) and (7), and their diameters sum to $\asymp d(\mathcal{C}_-, L_+)$.

The sequence of directed blocks is constructed using the following algorithm. See figures 10 and 11 for an illustration.

- (1) Use turning Lemma 6.7 and straight Lemma 6.8 between zero and six times to build a chain of blocks so that the last block is coherent with L_+ , closer to L_+ than \mathcal{C}_- , and is not contained in the central column of (\mathcal{C}_+, L_+) .
- (2) Use expanding Lemma 6.4 repeatedly to double away from the central column until with an expansion by a factor between two and four the long edge of \mathcal{C}_+ is reached.
- (3) Use expanding Lemma 6.4 repeatedly to double towards the central column, until of size $\geq \text{diam}(L_+)/5$.
- (4) Go straight (Lemma 6.8) until past the last removed square of the central column.
- (5) Expand by a factor less than five (Lemma 6.6), with the last edge L_+ .

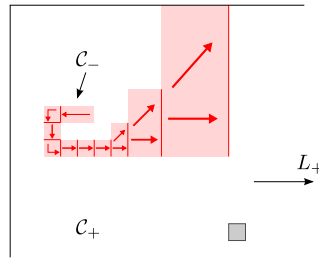


FIGURE 10. Steps (1) and (2) of the algorithm

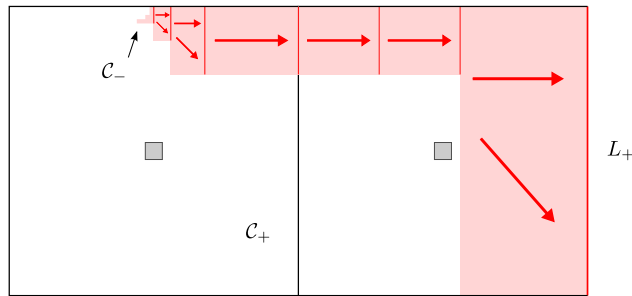


FIGURE 11. Steps (3)–(5) of the algorithm

We repeat this construction, growing in scale each time, until we are on scale s_m , then again until we have a block of size $d(x, y)/100$, at a distance less than $d(x, y)/10$ from x . This gives us most of the sequence of blocks with negative index.

We do the same for y , getting most of the (reverse) sequence of blocks with positive index, then join the two chains together using Lemmas 6.7 and 6.8.

It is straightforward to check that this construction is valid, and that the resulting chain of blocks satisfies conditions (1)–(7). We leave the details to the reader. \square

7. UNIFORMIZATION BY ROUND CARPETS AND SLIT CARPETS

In this section we prove Theorems 1.9 and 1.10 from the introduction, on the existence of round and slit carpets supporting Poincaré inequalities.

The proof of Theorem 1.9 relies on the following uniformization theorem of Bonk [3, Theorem 1.1 and Corollary 1.2].

Theorem 7.1 (Bonk). *Let $\{D_i : i \in I\}$ be a family of pairwise disjoint domains in \mathbb{R}^2 with Jordan curve boundaries $S_i = \partial D_i$. Assume that the curves $\{S_i\}$ are uniformly relatively separated uniform quasicircles. Then there exists a quasiconformal map $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ so that $f(D_i)$ is a disc for all $i \in I$. In particular, if T is a planar carpet whose peripheral circles are uniformly relatively separated uniform quasicircles, then T can be mapped to a round carpet T' by a quasimetric homeomorphism f . Furthermore, if T has measure zero, then f is unique up to post-composition with a Möbius transformation.*

A Jordan curve $S \subset \mathbb{R}^2$ is a *quasicircle* if it is the image of \mathbb{S}^1 under a quasiconformal map of \mathbb{R}^2 . A family of Jordan curves $\{S_i\}_{i \in I}$ consists of *uniform quasicircles* if there exists $K \geq 1$ so that each S_i is the image of \mathbb{S}^1 under a K -quasiconformal map of \mathbb{R}^2 . Finally, a family of Jordan curves $\{S_i\}_{i \in I}$ is *uniformly relatively separated* if there exists $c > 0$ so that

$$(7.1) \quad \frac{\text{dist}(S_i, S_j)}{\min\{\text{diam } S_i, \text{diam } S_j\}} \geq c \quad \forall i, j \in I, i \neq j.$$

Theorem 7.1 is stated in [3] for the extended complex plane $\widehat{\mathbb{C}}$ endowed with the spherical metric. However, an application of the (conformal) stereographic projection mapping converts the statement in [3] to our formulation.

7.1. Quasisymmetric uniformizability of $S_{\mathbf{a}}$ by round carpets. In this subsection we prove Theorem 1.9.

For any \mathbf{a} , the peripheral circles of the carpet $S_{\mathbf{a}}$ are uniformly separated. Indeed, when $\mathbf{a} \in c_0$, the uniform relative separation condition (7.1) holds in the following stronger form:

$$(7.2) \quad \lim_{\max\{\text{diam } S_i, \text{diam } S_j\} \rightarrow 0} \max_{i, j \in I, i \neq j} \frac{\text{dist}(S_i, S_j)}{\min\{\text{diam } S_i, \text{diam } S_j\}} \rightarrow \infty.$$

Since the peripheral circles are rigid squares, they are uniform quasicircles. By Theorem 7.1, $S_{\mathbf{a}}$ is quasisymmetrically uniformized by a round carpet T' whenever $\mathbf{a} \in c_0$.

When $\mathbf{a} \notin \ell^2$ (and so $S_{\mathbf{a}}$ has zero Lebesgue measure), T' is rigid, i.e., unique up to the application of a Möbius transformation.

When $\mathbf{a} \in \ell^2$ (and so $S_{\mathbf{a}}$ has positive Lebesgue measure), we observe that $T' = f(S_{\mathbf{a}})$ is an Ahlfors 2-regular subset of \mathbb{R}^2 (the volume upper bound is trivial; the lower bound follows from the quasisymmetry of f as in Corollary 3.10 and Remark 3.6(1) of [35]). It now follows from Theorem 1.6 and a result of Koskela and MacManus [28, Theorem 2.3] that the round carpet T' supports a p -Poincaré inequality for some $p < 2$.

7.2. Quasisymmetric uniformizability of $S_{\mathbf{a}}$ by slit carpets. We now turn to the proof of Theorem 1.10.

Let $S_{\mathbf{a}}$ be a carpet with $\mathbf{a} \in \ell^2$. For each m , the interior $S_{\mathbf{a},m}^o$ of the precarpet $S_{\mathbf{a},m}$ is a finitely connected domain in the plane. By Koebe’s uniformization theorem (see for instance [14, V§2]), $S_{\mathbf{a},m}^o$ can be conformally uniformized to a parallel slit domain D_m . Now the

identity map between the Euclidean metric and the internal metric δ on D_m is conformal, hence $S_{\mathbf{a},m}^o$ is conformally equivalent to (D_m, δ) .

By Proposition 3.1 and the proof of Theorem 1.6, the precarpets $S_{\mathbf{a},m}$ are Ahlfors 2-regular and support a p -Poincaré inequality for any $p > 1$ with constants independent of m . In particular, such precarpets are 2-Loewner in the Euclidean metric; since they are quasiconvex (with constant independent of m) they are also 2-Loewner in the internal metric.

It is straightforward to check that the domains (D_m, δ) are LLC. We now appeal to a theorem of Heinonen [18, Theorem 6.1] which asserts that any quasiconformal map from a bounded domain which is Loewner in the internal metric to a bounded LLC domain is quasisymmetric. The preceding result is quantitative in the usual sense; in our situation this implies that the relevant quasisymmetric distortion function is independent of m . Moreover, the target domains (D_m, δ) are uniformly Ahlfors 2-regular by an argument similar to that used in subsection 7.1.

Passing to the limit as $m \rightarrow \infty$, we obtain a quasisymmetric map from $S_{\mathbf{a}}$ onto an Ahlfors 2-regular parallel slit carpet. To complete the proof, we again use [28, Theorem 2.3] to conclude that the target carpet supports a p -Poincaré inequality for some $p < 2$.

8. REMARKS AND QUESTIONS

Remark 8.1. Suppose that $\mathbf{a} \in \ell^2$. By Theorem 1.6 and the main result of [10], the metric measure space $(S_{\mathbf{a}}, d, \mu)$ admits a strong measurable differentiable structure in the sense of Cheeger. It is easy to see what this structure is. Indeed, when $\mathbf{a} \in \ell^2$, the set $S_{\mathbf{a}}$ has positive Lebesgue measure and μ coincides with Lebesgue measure restricted to $S_{\mathbf{a}}$ (up to a constant multiple). The differentiable structure on $S_{\mathbf{a}}$ is then the restriction of the standard differentiable structure from \mathbb{R}^2 .

In [36], Weaver considers an abstract notion of derivation on metric measure spaces. According to Theorem 43 of [36], Cheeger and Weaver derivations coincide whenever both are defined. In the context of our carpets $S_{\mathbf{a}}$, this is precisely the case $\mathbf{a} \in \ell^2$, as discussed above. One may inquire about the structure of the module of Weaver derivations on general carpets $S_{\mathbf{a}}$. Theorem 41 of [36] indicates that the module of Weaver derivations on $S_{1/3}$ is trivial; an adaptation of the argument extends this fact to all carpets $S_{\mathbf{a}}$ with $\mathbf{a} \notin c_0$. At present, we do not have a precise description of the module of Weaver derivations on $S_{\mathbf{a}}$ when $\mathbf{a} \in c_0 \setminus \ell^2$. Further information on the module of Weaver derivations on planar sets can be found in [15, Chapter 5].

Remark 8.2. Remarkably, the ℓ^2 summability condition on the defining sequence \mathbf{a} has recently arisen in a different (although related) context. Doré and Maleva [11] show that when $\mathbf{a} \in c_0 \setminus \ell^2$, the compact set $S_{\mathbf{a}}$ is a *universal differentiability set*, i.e., it contains a differentiability point for every real-valued Lipschitz function on \mathbb{R}^2 .

Question 8.3. Can the techniques developed in this paper be used in other settings? We have already indicated (see the introduction) some of the advantages and flexibility of the methods which we employ. In particular, we are interested to know whether our methods can be used to verify that the spaces constructed by Koskela and Wildrick [29] support Poincaré inequalities for the sharp range of exponents as described in that paper. Note that the construction of the spaces in [29] differs from that of the carpets $S_{\mathbf{a}}$: the latter are naturally obtained as decreasing limits of sequences of closures of domains in \mathbb{R}^2 , while the former are given as the closure of an increasing limit of a sequence of domains.

REFERENCES

- [1] BISHOP, C. J. An A_1 weight not comparable with any quasiconformal Jacobian. In *In the tradition of Ahlfors-Bers. IV*, vol. 432 of *Contemp. Math.* Amer. Math. Soc., Providence, RI, 2007, pp. 7–18.
- [2] BISHOP, C. J., AND TYSON, J. T. Locally minimal sets for conformal dimension. *Ann. Acad. Sci. Fenn. Ser. A I Math.* 26 (2001), 361–373.
- [3] BONK, M. Uniformization of Sierpiński carpets in the plane. *Invent. Math.* 186, 3 (2011), 559–665.
- [4] BONK, M., KLEINER, B., AND MERENKOV, S. Rigidity of Schottky sets. *Amer. J. Math.* 131, 2 (2009), 409–443.
- [5] BONK, M., AND MERENKOV, S. Quasisymmetric rigidity of Sierpiński carpets. Preprint, 2011.
- [6] BOURDON, M., AND PAJOT, H. Poincaré inequalities and quasiconformal structure on the boundaries of some hyperbolic buildings. *Proc. Amer. Math. Soc.* 127, 8 (1999), 2315–2324.
- [7] BOURDON, M., AND PAJOT, H. Quasi-conformal geometry and hyperbolic geometry. In *Rigidity in dynamics and geometry (Cambridge, 2000)*. Springer, Berlin, 2002, pp. 1–17.
- [8] BURAGO, D., BURAGO, Y., AND IVANOV, S. *A course in metric geometry*, vol. 33 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001.
- [9] BUSER, P. A note on the isoperimetric constant. *Ann. Sci. École Norm. Sup. (4)* 15 (1982), 213–230.
- [10] CHEEGER, J. Differentiability of Lipschitz functions on metric measure spaces. *Geom. Funct. Anal.* 9 (1999), 428–517.
- [11] DORÉ, M., AND MALEVA, O. A compact null set containing a differentiability point of every Lipschitz function. *Math. Ann.* 351, 3 (2011), 633–663.
- [12] DURAND CARTAGENA, E., JARAMILLO, J. A., AND SHANMUGALINGAM, N. The ∞ -Poincaré inequality in metric measure spaces. *Michigan Math. J.* to appear.
- [13] DURAND CARTAGENA, E., AND TYSON, J. T. Rectifiable curves in Sierpiński carpets. *Indiana Univ. Math. J.* to appear.
- [14] GOLUZIN, G. M. *Geometric theory of functions of a complex variable*. Translations of Mathematical Monographs, Vol. 26. American Mathematical Society, Providence, R.I., 1969.
- [15] GONG, J. *Derivations on metric measure spaces*. PhD thesis, University of Michigan, 2008.
- [16] HAJLASZ, P. Sobolev spaces on an arbitrary metric space. *Potential Anal.* 5 (1996), 403–415.
- [17] HANSON, B., AND HEINONEN, J. An n -dimensional space that admits a Poincaré inequality but has no manifold points. *Proc. Amer. Math. Soc.* 128, 11 (2000), 3379–3390.
- [18] HEINONEN, J. Quasiconformal mappings onto John domains. *Rev. Mat. Iberoamericana* 5 (1989), 97–123.
- [19] HEINONEN, J. A capacity estimate on Carnot groups. *Bull. Sci. Math.* 119 (1995), 475–484.
- [20] HEINONEN, J., AND KOSKELA, P. Quasiconformal maps in metric spaces with controlled geometry. *Acta Math.* 181 (1998), 1–61.
- [21] HEINONEN, J., KOSKELA, P., SHANMUGALINGAM, N., AND TYSON, J. T. Sobolev classes of Banach space-valued functions and quasiconformal mappings. *J. Anal. Math.* 85 (2001), 87–139.
- [22] JERISON, D. The Poincaré inequality for vector fields satisfying Hörmander’s condition. *Duke Math. J.* 53 (1986), 503–523.
- [23] KEITH, S. Modulus and the Poincaré inequality on metric measure spaces. *Math. Z.* 245, 2 (2003), 255–292.
- [24] KEITH, S. A differentiable structure for metric measure spaces. *Adv. Math.* 183, 2 (2004), 271–315.
- [25] KEITH, S., AND LAAKSO, T. Conformal Assouad dimension and modulus. *Geom. Funct. Anal.* 14, 6 (2004), 1278–1321.
- [26] KEITH, S., AND ZHONG, X. The Poincaré inequality is an open ended condition. *Annals of Math.* 167, 2 (2008), 575–599.
- [27] KOSKELA, P. Upper gradients and Poincaré inequalities. In *Lecture notes on analysis in metric spaces (Trento, 1999)*, Appunti Corsi Tenuti Docenti Sc. Scuola Norm. Sup., Pisa, 2000, pp. 55–69.
- [28] KOSKELA, P., AND MACMANUS, P. Quasiconformal mappings and Sobolev spaces. *Studia Math.* 131 (1998), 1–17.
- [29] KOSKELA, P., AND WILDRICK, K. Exceptional sets for quasiconformal mappings in general metric spaces. *International Mathematics Research Notices* 2008 (2008). doi:10.1093/imrn/rnn020.

- [30] LAAKSO, T. Ahlfors Q -regular spaces with arbitrary Q admitting weak Poincaré inequalities. *Geom. Funct. Anal.* 10 (2000), 111–123.
- [31] MACKAY, J., AND TYSON, J. T. *Conformal dimension: theory and application*, vol. 54 of *University Lecture Series*. American Mathematical Society, 2010.
- [32] MERENKOV, S. Planar relative Schottky sets and quasisymmetric maps. *Proc. London Math. Soc.*
- [33] MERENKOV, S. A Sierpiński carpet with the co-Hopfian property. *Invent. Math.* 180, 2 (2010), 361–388.
- [34] SEMMES, S. Finding curves on general spaces through quantitative topology, with applications to Sobolev and Poincaré inequalities. *Selecta Math.* 2 (1996), 155–295.
- [35] TYSON, J. T. Analytic properties of locally quasisymmetric mappings from Euclidean domains. *Indiana Univ. Math. J.* 49, 3 (2000), 995–1016.
- [36] WEAVER, N. Lipschitz algebras and derivations. II. Exterior differentiation. *J. Funct. Anal.* 178, 1 (2000), 64–112.
- [37] ZIEMER, W. P. Extremal length and conformal capacity. *Trans. Amer. Math. Soc.* 126 (1967), 460–473.
- [38] ZIEMER, W. P. Extremal length and p -capacity. *Michigan Math. J.* 16 (1969), 43–51.

MATHEMATICAL INSTITUTE, UNIVERSITY OF OXFORD, 24-29 ST GILES, OXFORD, OX1 3LB, UK
E-mail address: John.Mackay@maths.ox.ac.uk

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS, 1409 WEST GREEN ST., URBANA, IL 61801 USA
E-mail address: tyson@math.uiuc.edu

MATHEMATISCHES INSTITUT, UNIVERSITÄT BERN, SIDLERSTRASSE 5, 3012 BERN, SWITZERLAND
E-mail address: kevin.wildrick@math.unibe.ch