A NOTE ON MORSE THEORY

MELINDA LANIUS

1. INTRODUCTION

Morse theory could be very well be called critical point theory. The idea is that by understanding the critical points of a smooth function on your manifold, you can recover the topology of your space. This basic idea has blossomed into many Morse theories. For instance, Robin Forman developed a combinatorial adaptation called discrete morse theory. We also have Morse-Bott theory, where we consider smooth functions on a manifold whose critical set is a closed submanifold. As a final example, Edward Witten used deformation of a differential and harmonic forms to produce a Morse homology.

In this note, we will provide the basic definitions and theorems of Morse theory. We’ll conclude with a discussion of Morse homology. The stress will be on examples and understanding the ideas, rather than approaching the details of proofs.

2. THE BASICS

Any introduction to the theory is hardly complete without the example of a torus provided by John Milnor in his excellent book “Morse theory”. Accordingly, we’ll interweave this “canonical” example throughout our exposition.

Let $M$ be a manifold and take $f \in C^\infty(\mathbb{R})$. Consider the set of critical points of $f$. That is, consider all the points $p \in M$, such that in a local coordinate system $x^1, \ldots, x^n$ all first partial derivatives vanish:

$$
\frac{df}{dx^1}(p) = \cdots = \frac{df}{dx^n}(p) = 0.
$$

Note that this does not depend on the choice of local coordinates, so we’ve a well-defined notion of critical set. A critical point is called non-degenerate if and only if the Hessian

$$
H(f(p)) = \begin{bmatrix}
\frac{\partial^2 f}{\partial x^i \partial x^j}(p)
\end{bmatrix}
$$

is invertible. We demand that the critical points of our function $f$ be non-degenerate and we call such a function Morse.
For the torus $\mathbb{T}$, we will consider a height function. In particular, if you consider the torus embedded into $\mathbb{R}^3$, take $f : \mathbb{T} \to \mathbb{R}$ by $f((x, y, z)) = z$. The points $a, b, c,$ and $d$ are the critical points of this function. These points are non-degenerate. Thus $f$ is an example of a Morse function on the torus.

Next, we’ll develop a notion of index for each non-degenerate critical point $p$ of our function. We define the index to be the dimension of the negative eigenspace of the Hessian of $f$ at the point $p$:

$$\lambda(p) := \text{Dim Eig}^{-} H(f(p)).$$

We can compute the index $\lambda$ in another way. There is a local coordinate system $(y^1, \ldots, y^n)$ in a neighborhood $U$ of $p$ with $y^i(p) = 0$ for all $i \in \{1, \ldots, n\}$. Further, for all $q \in U$,

$$f(q) = f(p) - (y^1(q))^2 - \cdots - (y^\lambda(q))^2 + (y^{\lambda+1}(q))^2 + \cdots + (y^n(q))^2.$$  

Note, this formulation of index in local coordinates is quite useful for proving general theorems, but is not so useful when actually doing a computation.

Let’s return to our example of the torus and find the index of each critical point.

<table>
<thead>
<tr>
<th>Critical point</th>
<th>Function $f$ in local coordinates</th>
<th>Index</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>$f = c - x^2 - y^2$</td>
<td>2</td>
</tr>
<tr>
<td>$b$</td>
<td>$f = c + x^2 - y^2$</td>
<td>1</td>
</tr>
<tr>
<td>$c$</td>
<td>$f = c + x^2 - y^2$</td>
<td>1</td>
</tr>
<tr>
<td>$d$</td>
<td>$f = c + x^2 + y^2$</td>
<td>0</td>
</tr>
</tbody>
</table>

We now have enough terminology to formulate one of the main theorems of Morse theory:

**Theorem**: Let $f : M \to \mathbb{R}$ be a smooth function, and let $p$ be a non-degenerate critical point with index $\lambda$. Setting $f(p) = c$, suppose that $f^{-1}([c - \epsilon, c + \epsilon])$ is compact, and contains no critical point of $f$ other than $p$, for some $\epsilon > 0$. Then, for all sufficiently small $\epsilon$, the set $f^{-1}((\infty, c + \epsilon])$ has the homotopy type of $f^{-1}((\infty, c - \epsilon])$ with a $\lambda$–cell attached.

This theorem allows us to recover the homotopy type of our manifold. For our torus, If $a < f(d)$, then $f^{-1}((\infty, a])$ is the empty set.
Since the index of \( d \) is zero, the homotopy type of \( f^{-1}((\infty, f(d) + \epsilon)) \) is a single 0-cell, or point. Note \( f^{-1}((\infty, f(d) + \epsilon)) \) is homeomorphic to a 2-cell, or disk.

Because the index of \( c \) is one, the homotopy type of \( f^{-1}((\infty, f(c) + \epsilon)) \) is a disk with a one-cell, or handle, attached. \( f^{-1}((\infty, f(c) + \epsilon)) \) is homeomorphic to a cylinder.

The index of \( b \) is also one. Thus the homotopy type of \( f^{-1}((\infty, f(b) + \epsilon)) \) is a cylinder with a one-cell, or handle, attached. Note \( f^{-1}((\infty, f(b) + \epsilon)) \) is homeomorphic to the torus with a disk removed.

The index of \( a \) is two. The homotopy type of \( f^{-1}((\infty, f(a) + \epsilon)) \) is a torus minus a disk with a 2-cell attached. Thus \( f^{-1}((\infty, f(a) + \epsilon)) \) is homeomorphic to the torus, as you’d expect.

In this way, we have recovered the homotopy type of our torus: \( e_0 \cup e_1 \cup e_1 \cup e_2 \).

3. **Morse Inequalities**

In the example of the height function on the torus, our function had the minimal number of critical points needed to describe the homotopy type of the torus. We call such functions perfect. So, we can have functions that provide exactly the information we want. On the other hand, how “bad” can a Morse function be? That is, is it possible to have a Morse function on our manifold with an arbitrary number of critical points for each index?

This answer here is no. The Morse inequalities, which are actually how Morse first formulated his version of critical point theory, provide bounds on the number of critical points of a given index.

Let \( f \) be a Morse function on a manifold \( M \). Let \( C_\lambda \) denote the number of critical points of \( f \) of index \( \lambda \). Recall that the Betti number \( \beta_\lambda(M) \) is the rank of the \( \lambda \)-th homology group of \( M \). The Morse inequalities state that

\[
C_\lambda - C_{\lambda-1} + \cdots \pm C_0 \geq \beta_\lambda - \beta_{\lambda-1} + \cdots \pm \beta_0.
\]

Note that a perfect Morse function gives equality here.
Let’s consider the Morse inequalities for the specific case of the 2-sphere. Recall that $\beta_0 = \beta_2 = 1$ and $\beta_1 = 0$. Thus our Morse inequalities are

\[
\begin{align*}
C_0 & \geq 1 \\
C_1 - C_0 & \geq -1 \\
C_2 - C_1 + C_0 & \geq 2 
\end{align*}
\]

Intuitively, one can interpret these as dictating how much you can perturb a perfect Morse function. In order to pick up a critical point of index $\lambda$, you must compensate by picking up a critical point of index either $\lambda - 1$ or $\lambda + 1$.

Consider the sphere $S^2$ embedded in $\mathbb{R}^3$ and take the $z$-height function. This function has one critical point $b$ of index 0, and $a$ of index 2. Thus this is an example of a perfect Morse function.

Now say we’d like a Morse function on $S^2$ that has one critical point of index 1. We can do this, but we have to compensate in one of two ways.

Let’s take our sphere and smoothly punch down the top. Still using the $z$-height function, we now have one critical point $b$ of index 0, one critical point $c$ of index 1, and critical points $a$ and $d$ of index 2.

Our other option is to compensate with a critical point of index 0.

Let’s take $S^2$ and smoothly punch up from the bottom. Still using the $z$-height function, we now have critical points $b$ and $d$ of index 0, one critical point $c$ of index 1, and one critical point $a$ of index 2.

One way to prove the Morse inequalities is to develope a Morse homology. Note that there are several ways to extract the Morse inequalities, and proving a homology theory is indeed overkill. However, Morse homology is defined for any compact smooth manifold and in fact is isomorphic to singular homology.
Further, Morse homology serves as the model for some highly powerful infinite-dimensional homologies, such as Floer homology. Thus it is of great interest in its own right.

4. Morse Homology

Begin by considering a compact manifold $M$ equipped with a Riemannian metric $g$. Let $f$ be a Morse function on $M$. Then we’ll define the negative gradient flow to be a map $\Psi : \mathbb{R} \times M \to M$ such that $\frac{d}{dt}\Psi(t, x) = -\nabla f(\Psi(t, x))$ and $\Psi(0, \cdot) = id_M$.

Using the negative gradient flow, we can decompose our manifold into a disjoint union of unstable (or stable) manifolds. Let $x$ be a critical point of $f$. We define the unstable manifold of $x$ to be

$$W^u(x) := \left\{ p \in M \mid \lim_{t \to -\infty} \Psi(t, p) = x \right\}.$$

Similarly, the stable manifold of $x$ is defined to be

$$W^s(x) := \left\{ p \in M \mid \lim_{t \to \infty} \Psi(t, p) = x \right\}.$$

In fact,

$$M = \bigcup_{x \in \text{crit}(f)} W^u(x)$$

defines naturally the homotopy equivalent cell decomposition of $M$ with $W^u(x)$ being a cell of dimension $\lambda(x)$. However, this cell decomposition does not always directly represent a CW-complex. And this is the case with our torus example from section two.

On the other hand, it is commonly believed among Morse theorists that if we demand our function be Morse-Smale, then we do have a CW-decomposition directly from the unstable manifolds. A function is said to be Morse-Smale if the unstable and stable manifolds intersect transversely for any two critical points of $f$. The good news is that this is a generic condition.

We can fix our torus example by simply tilting its $z$-axis slightly, and again considering the height function.

We need one more notion, before we can define the Morse complex. For ease, we’ll define the complex using $\mathbb{Z}_2$ coefficients. Let $x$ and $y$ be two critical points of $f$ such that $\lambda(x) - \lambda(y) = 1$. Let $a$ be a regular value of the interval $(f(x), f(y))$. 

Then the level set $f^{-1}(a)$ intersects transversely $W^u(x) \cap W^s(y)$. Furthermore, this intersection is a set of points. Define the intersection number $n(x, y)$ to be the number of points in $f^{-1}(a) \cap W^u(x) \cap W^s(y)$.

Now we have everything we need to define the Morse complex with $\mathbb{Z}_2$ coefficients. Let $A_k = \mathbb{Z}_2^{c_k}$ where $c_k$ is the number of critical points of index $k$. We define our boundary operator

$$
\delta_k : A_k \to A_{k-1}
$$

by

$$
\delta_k(x) = \sum_{y \in \text{crit}_{k-1} f} n(x, y) \cdot y
$$

for all $x \in \text{crit}_k f$ and we extend linearly. This is a chain complex and $H_*(C_*(f), \delta) \cong H_*(M)$, the singular homology of $M$.

Let’s return to our Morse-Smale function on the torus and compute the $\mathbb{Z}_2$ homology. We have the following chain complex.

$$
0 \to \mathbb{Z}_2 \xrightarrow{\delta_2} \mathbb{Z}_2 \times \mathbb{Z}_2 \xrightarrow{\delta_1} \mathbb{Z}_2 \to 0
$$

Let’s compute $\delta_2$. We have $n(a, b) = 2$ and $n(a, c) = 2$. So $\delta_2(a) = (2b, 2c) = 0$. Now, let’s compute $\delta_1$. We have $n(b, d) = 2$ and $n(c, d) = 2$. So $\delta_1(b, c) = 2d + 2d = 0$. Thus the homology of the sequence

$$
0 \to \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \times \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \to 0
$$

becomes obvious. Note that our boundary maps being zero is a consequence of our function $f$ being perfect. Thus, the existence of perfect morse functions is an area of interest and study.

5. Bibliography

For a more detailed discussion of the material covered in sections 2 and 3, see the book *Morse Theory* by John Milnor. A more complete explanation of Morse homology can be found in the book *Morse Homology* by Matthias Schwarz. Note this is a very technical treatment. If you are looking more for a quick and clear exposition, I highly recommend the article *Morse theory indomitable* by Raoul Bott.