MODULE MAPS OVER LOCALLY COMPACT QUANTUM GROUPS

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Abstract. We study locally compact quantum groups $\mathbb{G}$ and their module maps through a general Banach algebra approach. As applications, we obtain various characterizations of compactness and discreteness, which in particular generalize a result by Lau (1978) and recover another one by Runde (2008). Properties of module maps on $L_\infty(\mathbb{G})$ are used to characterize strong Arens irregularity of $L_1(\mathbb{G})$ and are linked to commutation relations over $\mathbb{G}$ with several double commutant theorems established. We prove the quantum group version of the theorems by Young (1973), Lau (1981), and Forrest (1991) regarding Arens regularity of the group algebra $L_1(G)$ and the Fourier algebra $A(G)$. We also extend the classical Eberlein theorem on the inclusion $B(G) \subseteq WAP(G)$ to all locally compact quantum groups.

1. Introduction

Let $\mathbb{G} = (L_\infty(\mathbb{G}), \Gamma, \varphi, \psi)$ be a von Neumann algebraic locally compact quantum group and let $L_1(\mathbb{G})$ be the convolution quantum group algebra of $\mathbb{G}$. If we let $C_0(\mathbb{G})$ be the reduced C^*-algebra associated with $\mathbb{G}$, then its operator dual $M(\mathbb{G})$ is a faithful completely contractive Banach algebra containing $L_1(\mathbb{G})$ as an ideal. It has been shown in the recent work [26, 27, 29] that many important results in abstract harmonic analysis can be generalized to the locally compact quantum group setting, and thus we can develop a corresponding theory of quantum harmonic analysis. In this paper, we study $L_1(\mathbb{G})$-module maps and structures on $L_\infty(\mathbb{G})$. Through a general Banach algebra approach, we obtain in particular some interesting characterizations of compactness and discreteness of $\mathbb{G}$.

In Section 2, we recall some definitions for locally compact quantum groups and associated Banach algebras. We strengthen and extend the completely isometric embedding result $M(\mathbb{G}) \longrightarrow LUC(\mathbb{G})^*$ (cf. [29]) to a more general setting, where $LUC(\mathbb{G}) = \langle L_\infty(\mathbb{G}) \star L_1(\mathbb{G})\rangle$ is the space of left uniformly continuous functionals on $L_1(\mathbb{G})$. More precisely, we show that if X is any left introverted subspace of $L_\infty(\mathbb{G})$ with $C_0(\mathbb{G}) \subseteq X \subseteq M(C_0(\mathbb{G}))$, then there exists a completely isometric $C_0(\mathbb{G})$-bimodule and $L_1(\mathbb{G})$-bimodule algebra homomorphism $\pi : M(\mathbb{G}) \longrightarrow X^*$ such that $X^* = \pi(M(\mathbb{G})) \oplus C_0(\mathbb{G})^\perp$. Through the canonical inclusion $X \subseteq C_0(\mathbb{G})^{**}$, we take a natural approach to constructing such an embedding with some additional properties obtained (cf. Proposition 2.1 and Corollary 2.5). This approach, which is even new for the co-commutative case as considered in [42], yields more characterizations of compact and discrete quantum groups and an extension theorem for quantum group measure algebra homomorphisms.

In Section 3, we prove several characterizations of compact quantum groups, which in particular generalize a result by Lau [36] and recover another one by Runde [55]. We characterize compactness of $\mathbb{G}$ in terms of the space $WAP(\mathbb{G})$ of weakly almost periodic functionals on $L_1(\mathbb{G})$ and quotient strong Arens irregularity of $L_1(\mathbb{G})$. This shows that quotient strong Arens irregularity may have to be taken into account

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for answering the open question raised by Runde [56, Remark 4.5]. We also study when $L_1(\mathbb{G}) = M(\mathbb{G})$ holds and present characterizations for discreteness of $\mathbb{G}$. We show that compactness, discreteness, and finiteness of a quantum group $\mathbb{G}$ can be characterized simultaneously by comparing the right multiplier algebra $RM(L_1(\mathbb{G}))$ of $L_1(\mathbb{G})$ with the module product $L_1(\mathbb{G}) \star LUC(\mathbb{G})^*$. We obtain the quantum group version of the theorems by Young [65], Lau [38], and Forrest [17] regarding Arens regularity of the group algebra $L_1(G)$ and the Fourier algebra $A(G)$. Properties of module maps on $L_\infty(\mathbb{G})$ are further used to characterize strong Arens irregularity of $L_1(\mathbb{G})$ and are linked to commutation relations over $\mathbb{G}$. We establish several double commutant theorems, which in particular improve and extend the commutant theorem [18, Theorem 5.1] on $L_1(G)$ by Ghahramani and Lau. Many of the results obtained in this section are even new for the Fourier algebra $A(G)$.

In Section 4, we prove two more results on properties of module maps over $\mathbb{G}$, which characterize amenability and compactness of $\mathbb{G}$ in terms of weakly compact module maps on $L_\infty(\mathbb{G})$, generalizing and unifying some results on $L_1(G)$ and $A(G)$ by Akemann [1] and Lau [36, 37].

A classical Eberlein theorem says that every positive definite function on a locally compact group G is weakly almost periodic. In Section 5, we extend this result to all locally compact quantum groups $\mathbb{G}$. More precisely, we show that every bounded linear functional on the universal quantum group C^*-algebra $C_u(\widehat{\mathbb{G}})$ of $\widehat{\mathbb{G}}$ is canonically corresponding to a weakly almost periodic functional on $L_1(\mathbb{G})$. In this way, each $\mu \in C_u(\widehat{\mathbb{G}})^*$ defines a weakly compact $L_1(\mathbb{G})$-module map from $L_1(\mathbb{G})$ to $L_\infty(\mathbb{G})$.

2. Definitions and preliminary results

Let us start this section by recalling some notation related to locally compact quantum groups. The reader is referred to Kustermans and Vaes [34, 35], Runde [55, 56], van Daele [63], and [26, 27, 29] for more information. Let $\mathbb{G} = (M, \Gamma, \varphi, \psi)$ be a von Neumann algebraic locally compact quantum group. Then the pre-adjoint of the co-multiplication Γ induces on M_* an associative completely contractive multiplication $\star : M_* \widehat{\otimes} M_* \longrightarrow M_*$, where $\widehat{\otimes}$ is the operator space projective tensor product. In the case where M is $L_\infty(G)$ or $VN(G)$ with G a locally compact group, the algebra $(M_*, \star)$ is the usual convolution group algebra $L_1(G)$, respectively, the Fourier algebra $A(G)$.

As for locally compact groups, the von Neumann algebra M and the convolution algebra $(M_*, \star)$ are denoted by $L_\infty(\mathbb{G})$ and $L_1(\mathbb{G})$, respectively. Then $L_1(\mathbb{G})$ is a faithful completely contractive Banach algebra satisfying $\langle L_1(\mathbb{G}) \star L_1(\mathbb{G}) \rangle = L_1(\mathbb{G})$ (cf. [26, Fact 1] and [27, Proposition 1]). The multiplication on $L_1(\mathbb{G})$ induces canonically a completely contractive $L_1(\mathbb{G})$-bimodule structure on $L_\infty(\mathbb{G})$ satisfying

$$x \star f = (f \otimes \iota)\Gamma(x) \quad \text{and} \quad f \star x = (\iota \otimes f)\Gamma(x) \quad (x \in L_\infty(\mathbb{G}),\ f \in L_1(\mathbb{G})). \tag{2.1}$$

The quantum group $\mathbb{G}$ is said to be *co-amenable* if $L_1(\mathbb{G})$ has a bounded approximate identity.

It is known that there are two Banach algebra multiplications $\Box$ and $\Diamond$ on $L_1(\mathbb{G})^{**}$, each extending the multiplication $\star$ on $L_1(\mathbb{G})$. For $m,\ n \in L_1(\mathbb{G})^{**}$ and $x \in L_\infty(\mathbb{G})$, by definition, the *left Arens product* $m \Box n \in L_1(\mathbb{G})^{**}$ satisfies $\langle m \Box n, x \rangle = \langle m, n \Box x \rangle$, where $n \Box x = (\iota \otimes n)\Gamma(x) \in L_\infty(\mathbb{G})$ is given by $\langle n \Box x, f \rangle = \langle n, x \star f \rangle$ $(f \in L_1(\mathbb{G}))$. Similarly, the *right Arens product* $m \Diamond n \in L_1(\mathbb{G})^{**}$ satisfies $\langle x, m \Diamond n \rangle = \langle x \Diamond m, n \rangle$ with $x \Diamond m = (m \otimes \iota)\Gamma(x) \in L_\infty(\mathbb{G})$ given by $\langle f, x \Diamond m \rangle = \langle f \star x, m \rangle$ $(f \in L_1(\mathbb{G}))$. It can be shown by a matricial argument that both Arens products are completely contractive multiplications on $L_1(\mathbb{G})^{**}$. The algebra $L_1(\mathbb{G})$ is said to be *Arens regular* if $\Box$ and $\Diamond$ coincide on $L_1(\mathbb{G})^{**}$.

Clearly, the map $L_1(\mathbb{G})^{**} \longrightarrow L_1(\mathbb{G})^{**}$, $n \longmapsto n\Box m$ is w^*-w^* continuous for each $m \in L_1(\mathbb{G})^{**}$. The left topological centre of $L_1(\mathbb{G})^{**}$ is defined by

$$\mathfrak{Z}(L_1(\mathbb{G})^{**}, \Box) \; = \; \{m \in L_1(\mathbb{G})^{**} : \text{ the map } n \longmapsto m\Box n \text{ is } w^*\text{-}w^* \text{ continuous on } L_1(\mathbb{G})^{**}\}.$$

The right topological centre $\mathfrak{Z}(L_1(\mathbb{G})^{**}, \Diamond)$ of $L_1(\mathbb{G})^{**}$ is defined analogously. Then we have

$$L_1(\mathbb{G}) \; \subseteq \; \mathfrak{Z}(L_1(\mathbb{G})^{**}, \Box) \cap \mathfrak{Z}(L_1(\mathbb{G})^{**}, \Diamond) \quad \text{and} \quad \mathfrak{Z}(L_1(\mathbb{G})^{**}, \Box) \cup \mathfrak{Z}(L_1(\mathbb{G})^{**}, \Diamond) \; \subseteq \; L_1(\mathbb{G})^{**},$$

and $L_1(\mathbb{G})$ is Arens regular if and only if $\mathfrak{Z}(L_1(\mathbb{G})^{**}, \Box) = L_1(\mathbb{G})^{**} = \mathfrak{Z}(L_1(\mathbb{G})^{**}, \Diamond)$. The algebra $L_1(\mathbb{G})$ is said to be *strongly Arens irregular* (SAI) if $\mathfrak{Z}(L_1(\mathbb{G})^{**}, \Box) = L_1(\mathbb{G}) = \mathfrak{Z}(L_1(\mathbb{G})^{**}, \Diamond)$ (cf. [8]).

For an $L_1(\mathbb{G})$-submodule X of $L_\infty(\mathbb{G})$ and for $x \in X$ and $m \in X^*$, one can naturally define $m\Box x$, $x\Diamond m \in L_\infty(\mathbb{G})$. Then X is said to be *left introverted* in $L_\infty(\mathbb{G})$ if $X^*\Box X \subseteq X$. In this case, the canonical quotient map $L_1(\mathbb{G})^{**} \longrightarrow X^*$ yields a Banach algebra multiplication on X^* (also denoted by $\Box$) such that $(X^*, \Box) \cong (L_1(\mathbb{G})^{**}, \Box)/X^\perp$. The topological centre $\mathfrak{Z}_t(X^*)$ of X^* is defined analogously to $\mathfrak{Z}(L_1(\mathbb{G})^{**}, \Box)$. Right introverted subspaces of $L_\infty(\mathbb{G})$ and their topological centres are defined similarly.

Let $C_0(\mathbb{G})$ be the reduced C^*-algebra associated with $\mathbb{G}$ (cf. [35]) and let $M(C_0(\mathbb{G}))$ be the multiplier algebra of $C_0(\mathbb{G})$. Then

$$C_0(\mathbb{G}) \; \subseteq \; M(C_0(\mathbb{G})) \; \subseteq \; L_\infty(\mathbb{G}). \tag{2.2}$$

A quantum group $\mathbb{G}$ is *compact* if $1 \in C_0(\mathbb{G})$, and is *discrete* if the dual quantum group $\widehat{\mathbb{G}}$ of $\mathbb{G}$ is compact, which is equivalent to $L_1(\mathbb{G})$ being unital (cf. [14] and [55]). The co-multiplication Γ maps $C_0(\mathbb{G})$ into the multiplier algebra $M(C_0(\mathbb{G}) \otimes C_0(\mathbb{G}))$ of the minimal C^*-algebra tensor product $C_0(\mathbb{G}) \otimes C_0(\mathbb{G})$. Then $C_0(\mathbb{G})^*$ is a completely contractive Banach algebra under the multiplication (also denoted by $\star$) given by

$$\langle \mu \star \nu, x\rangle = \langle \mu \otimes \nu, \Gamma(x)\rangle = \langle \mu, (id \otimes \nu)\Gamma(x)\rangle = \langle \nu, (\mu \otimes id)\Gamma(x)\rangle \quad (\mu,\, \nu \in C_0(\mathbb{G})^*,\; x \in C_0(\mathbb{G})), \tag{2.3}$$

where $\mu \otimes \nu = \mu(id \otimes \nu) = \nu(\mu \otimes id) \in M(C_0(\mathbb{G}) \otimes C_0(\mathbb{G}))^*$. It is known that $L_1(\mathbb{G})$ is identified with a closed two-sided ideal in $(C_0(\mathbb{G})^*, \star)$ via $f \longmapsto f|_{C_0(\mathbb{G})}$ (cf. [35, pages 913–914]). If $\mathbb{G}$ is commutative (respectively, co-commutative), then $C_0(\mathbb{G}) = C_0(G)$ and $C_0(\mathbb{G})^* = M(G)$ (respectively, $C_0(\mathbb{G}) = C^*_\lambda(G)$ and $C_0(\mathbb{G})^* = B_\lambda(G)$) for some locally compact group G, where $C^*_\lambda(G)$ is the reduced group C^*-algebra of G and $B_\lambda(G)$ is the reduced Fourier-Stieltjes algebra of G. The C^*-algebra $C_0(\mathbb{G})$ is two-sided introverted in $L_\infty(\mathbb{G})$, and the Arens products $\Box$ and $\Diamond$ on $C_0(\mathbb{G})^*$ coincide; they are just the product $\star$ due to (2.1) and (2.3) (cf. [29, (2.10)]). We use $M(\mathbb{G})$ to denote the completely contractive Banach algebra $(C_0(\mathbb{G})^*, \star)$. Then $M(\mathbb{G})$ is a *dual Banach algebra* in the sense of [54, Definition 1.1]; that is, the multiplication on $M(\mathbb{G})$ is separately w^*-continuous. We showed in [29, Proposition 2.2] that the multiplication on $M(\mathbb{G})$ is also faithful.

According to [26, 56], the subspaces $LUC(\mathbb{G})$ and $RUC(\mathbb{G})$ of $L_\infty(\mathbb{G})$ are defined by

$$LUC(\mathbb{G}) \; = \; \langle L_\infty(\mathbb{G}) \star L_1(\mathbb{G})\rangle \quad \text{and} \quad RUC(\mathbb{G}) \; = \; \langle L_1(\mathbb{G}) \star L_\infty(\mathbb{G})\rangle. \tag{2.4}$$

Then $LUC(\mathbb{G})$ is left introverted in $L_\infty(\mathbb{G})$, and $RUC(\mathbb{G})$ is right introverted in $L_\infty(\mathbb{G})$. They are the usual spaces $LUC(G)$ and $RUC(G)$ if $\mathbb{G} = L_\infty(G)$ for a locally compact group G, where $LUC(G)$ (respectively, $RUC(G)$) is the space of bounded left (respectively, right) uniformly continuous functions on G. If $\mathbb{G} = VN(G)$, then $LUC(\mathbb{G}) = RUC(\mathbb{G})$ is the space $UCB(\widehat{G})$ of uniformly continuous functionals on $A(G)$ (cf. [20]). We say that $\mathbb{G}$ is an *SIN* quantum group if $LUC(\mathbb{G}) = RUC(\mathbb{G})$ (cf. [26]). In [56, Theorem 2.4], Runde showed that $LUC(\mathbb{G})$ is an operator system in $L_\infty(\mathbb{G})$ such that

$$C_0(\mathbb{G}) \subseteq LUC(\mathbb{G}) \subseteq M(C_0(\mathbb{G})). \tag{2.5}$$

It was proved in [29, Theorem 5.6] that if $\mathbb{G}$ is semi-regular, then $LUC(\mathbb{G})$ is a unital C^*-subalgebra of $M(C_0(\mathbb{G}))$. This is a quite general result, which covers all Kac algebras, though we still do not know whether it holds for all quantum groups. See [56, 58] for some co-amenable cases, where $LUC(\mathbb{G})$ was also shown to be a C^*-algebra.

Let $WAP(\mathbb{G})$ be the space of weakly almost periodic functionals on $L_1(\mathbb{G})$, i.e., the subspace of $L_\infty(\mathbb{G})$ consisting of $x \in L_\infty(\mathbb{G})$ such that $L_1(\mathbb{G}) \longrightarrow L_\infty(\mathbb{G})$, $f \longmapsto x \star f$ is weakly compact. Then $WAP(\mathbb{G})$ is an $L_1(\mathbb{G})$-submodule of $L_\infty(\mathbb{G})$ and is two-sided introverted in $L_\infty(\mathbb{G})$. It is known that if $\mathbb{G} = L_\infty(G)$, then $WAP(\mathbb{G}) = WAP(G)$, the space of weakly almost periodic functions on G (cf. [8, page 69]), and hence $WAP(\mathbb{G})$ is often denoted by $WAP(\widehat{G})$ when $\mathbb{G} = VN(G)$. We have $C_0(\mathbb{G}) \subseteq WAP(\mathbb{G})$ since the two Arens products on $C_0(\mathbb{G})^*$ coincide, and $WAP(\mathbb{G}) \subseteq LUC(\mathbb{G}) \cap RUC(\mathbb{G})$ if $\mathbb{G}$ is co-amenable (cf. [8, Propositions 3.11 and 3.12] and [56, Theorem 4.4]). The relation between $LUC(\mathbb{G})$ and $WAP(\mathbb{G})$ will be investigated in Section 3.

If $\mathbb{G}$ is co-amenable, then there exists a canonical completely isometric algebra homomorphism $M(\mathbb{G}) \cong RM_{cb}(L_1(\mathbb{G})) \longrightarrow LUC(\mathbb{G})^*$ (cf. [29, Propositions 3.1 and 6.5]). In general, $M(\mathbb{G})$ can be linked to $LUC(\mathbb{G})^*$ without going though $RM_{cb}(L_1(\mathbb{G}))$ (cf. (2.14) below). In fact, as shown in [29, Proposition 6.1], we can obtain a completely isometric embedding $\pi : M(\mathbb{G}) \longrightarrow LUC(\mathbb{G})^*$ via the existence of a unique strictly continuous extension of each $\mu \in C_0(\mathbb{G})^*$ to $LUC(\mathbb{G})$. This generalizes the corresponding result by Lau and Losert [42] on $A(G)$. We note that an isometric embedding of $M(\mathbb{G}) \longrightarrow LUC(\mathbb{G})^*$ was considered in [52, Lemma 4.1], but missing in its proof the above strictly continuous extension property.

In the following, we show that this embedding result can be strengthened and extended to more general left introverted subspaces X of $L_\infty(\mathbb{G})$ satisfying $C_0(\mathbb{G}) \subseteq X \subseteq M(C_0(\mathbb{G}))$. A related result for subspaces of $M(C_0(\mathbb{G}))$ with a stronger left introversion property can be found in [58]. To make the presentation clear, we use $M(\widetilde{C_0(\mathbb{G})})$ to denote the idealizer of $C_0(\mathbb{G})$ in $C_0(\mathbb{G})^{**}$. That is,

$$M(\widetilde{C_0(\mathbb{G})}) = \{x \in C_0(\mathbb{G})^{**} : \widetilde{a}x, x\widetilde{a} \in C_0(\mathbb{G}) \text{ for all } a \in C_0(\mathbb{G})\}. \tag{2.6}$$

Here, for $a \in C_0(\mathbb{G})$, we use $\widetilde{a}$ to denote the canonical image of a in $C_0(\mathbb{G})^{**}$. It is known that we have the C^*-algebra isomorphism $M(C_0(\mathbb{G})) \cong M(\widetilde{C_0(\mathbb{G})})$, which extends the identification $C_0(\mathbb{G}) \cong \widetilde{C_0(\mathbb{G})}$. We shall define the embedding $M(\mathbb{G}) \longrightarrow X^*$ through the map

$$\tau : X \subseteq M(C_0(\mathbb{G})) \cong M(\widetilde{C_0(\mathbb{G})}) \subseteq C_0(\mathbb{G})^{**}. \tag{2.7}$$

This approach to constructing the embedding $M(\mathbb{G}) \longrightarrow X^*$ is different from the one used in [29, 58] and also different from the one used in [42] for $\mathbb{G} = VN(G)$. Note that $\tau(C_0(\mathbb{G})) = \widetilde{C_0(\mathbb{G})}$ and X is a $C_0(\mathbb{G})$-submodule of $M(C_0(\mathbb{G}))$. Therefore, $\tau : X \longrightarrow C_0(\mathbb{G})^{**}$ is a $C_0(\mathbb{G})$-bimodule map. We use $\cdot$ to denote the canonical $C_0(\mathbb{G})$-bimodule actions on $M(\mathbb{G})$. Then $M(\mathbb{G}) = M(\mathbb{G}) \cdot C_0(\mathbb{G}) = C_0(\mathbb{G}) \cdot M(\mathbb{G})$ (cf. [50, Proposition 9.4.27]).

Proposition 2.1. *Let $\mathbb{G}$ be a locally compact quantum group and let X be a left introverted subspace of $L_\infty(\mathbb{G})$ such that $C_0(\mathbb{G}) \subseteq X \subseteq M(C_0(\mathbb{G}))$. Let $\tau : X \longrightarrow C_0(\mathbb{G})^{**}$ be the map given in* (2.7). *Then $\pi = \tau^*|_{M(\mathbb{G})} : M(\mathbb{G}) \longrightarrow X^*$ is a completely isometric algebra homomorphism such that*

$$X^* = \pi(M(\mathbb{G})) \oplus C_0(\mathbb{G})^\perp,$$

where $C_0(\mathbb{G})^\perp = \{m \in X^* : m|_{C_0(\mathbb{G})} = 0\}$ *is a* w^**-closed ideal in* X^*.

Furthermore, we have

(i) $\pi : M(\mathbb{G}) \longrightarrow X^*$ *is a* $C_0(\mathbb{G})$*-bimodule and* $L_1(\mathbb{G})$*-bimodule map satisfying*

$$\langle \pi(\mu), x \star f\rangle = \langle x, f \star \mu\rangle \quad \text{and} \quad \langle \pi(\mu), f \star x\rangle = \langle x, \mu \star f\rangle \quad (\mu \in M(\mathbb{G}),\ x \in X,\ f \in L_1(\mathbb{G}));$$

(ii) $\pi^*|_X = \tau$;

(iii) $\pi(M(\mathbb{G})) \subseteq \mathfrak{Z}_t(X^*)$ *if* $X \subseteq LUC(\mathbb{G})$.

Proof. Let $\mu \in M(\mathbb{G})$. By definition, we have $\pi(\mu) = \tilde{\mu} \circ \tau$, where $\tilde{\mu}$ is the canonical image of μ in $M(\mathbb{G})^{**} = [C_0(\mathbb{G})^{**}]^*$. Therefore, $\pi(\mu)|_{C_0(\mathbb{G})} = \mu$, and $\pi : M(\mathbb{G}) \longrightarrow X^*$ is a complete isometry.

On the other hand, since $M(\mathbb{G}) = M(\mathbb{G}) \cdot C_0(\mathbb{G})$ and $\tau : X \longrightarrow C_0(\mathbb{G})^{**}$ is a $C_0(\mathbb{G})$-bimodule map, the functional $\pi(\mu) \in X^*$ is continuous in the relative strict topology of X, and thus we also have $\pi(\mu) = \mu'|_X$, where $\mu' \in M(C_0(\mathbb{G}))^*$ is the unique strictly continuous extension of μ. Note that the co-multiplication Γ maps $M(C_0(\mathbb{G}))$ into $M(C_0(\mathbb{G}) \otimes C_0(\mathbb{G}))$, and is strictly continuous on the closed unit ball of $M(C_0(\mathbb{G}))$. Hence, we derive from (2.1) and (2.3) that

$$\pi(\mu)\Box x = (\iota \otimes \mu)\Gamma(x) \quad \text{and} \quad \pi(\mu)\Box\pi(\nu) = \pi(\mu \star \nu) \quad \text{for all } x \in X \text{ and } \mu, \nu \in M(\mathbb{G}). \tag{2.8}$$

It follows that $\pi : M(\mathbb{G}) \longrightarrow X^*$ is an algebra homomorphism. Clearly, $C_0(\mathbb{G})^\perp$ is a w^*-closed ideal in X^*, and we have $X^* = \pi(M(\mathbb{G})) \oplus C_0(\mathbb{G})^\perp$.

(i) Since τ is a $C_0(\mathbb{G})$-bimodule map, so is the map $\pi = \tau^*|_{M(\mathbb{G})}$. Let $f \in L_1(\mathbb{G})$. Then $\pi(f) = f|_X$. Thus $\pi(f \star \mu) = \pi(f)\Box\pi(\mu) = f \star \pi(\mu)$. Similarly, we have $\pi(\mu \star f) = \pi(\mu) \star f$. Therefore, $\pi : M(\mathbb{G}) \longrightarrow X^*$ is an $L_1(\mathbb{G})$-bimodule map, and the two equalities hold.

(ii) This is evident.

(iii) We first suppose that $X = \langle X \star L_1(\mathbb{G})\rangle$. It is seen from (i) that $(x \star f)\Diamond\pi(\mu) = x \star (f \star \mu)$ for all $x \in X$ and $f \in L_1(\mathbb{G})$. Then $X\Diamond\pi(\mu) \subseteq X$ since $X = \langle X \star L_1(\mathbb{G})\rangle$. Combining this with the first equality in (i), we have $\langle \pi(\mu)\Box n, x\rangle = \langle x\Diamond\pi(\mu), n\rangle$ for all $n \in X^*$ and $x \in X$. Therefore, $\pi(\mu) \in \mathfrak{Z}_t(X^*)$.

In general, we suppose that $X \subseteq LUC(\mathbb{G})$. By the above argument, we have $M(\mathbb{G}) \subseteq \mathfrak{Z}_t(LUC(\mathbb{G})^*)$ under the embedding $M(\mathbb{G}) \longrightarrow LUC(\mathbb{G})^*$, whose composition with the restriction map $LUC(\mathbb{G})^* \longrightarrow X^*$ is just the map $\pi : M(\mathbb{G}) \longrightarrow X^*$. Since the restriction map $LUC(\mathbb{G})^* \longrightarrow X^*$ is a surjective algebra homomorphism, we obtain that $M(\mathbb{G}) \subseteq \mathfrak{Z}_t(X^*)$. □

Remark 2.2. It is seen from Proposition 2.1(ii) that $\pi^*|_X$ is completely isometric, since it is exactly the canonical inclusion map $\tau : X \longrightarrow C_0(\mathbb{G})^{**}$. In particular, if $\mathbb{G} = VN(G)$ and $X = UCB(\widehat{G})$, then $\pi^*|_{UCB(\widehat{G})}$ is completely isometric, which was proved in [31, Proposition 3.3] under the hypothesis that $A(G)$ has an approximate identity of completely bounded multiplier norm 1. The map $\pi^*|_{UCB(\widehat{G})}$ was earlier considered in [42, Proposition 7.5], where Lau and Losert showed that G is compact if and only if $\pi^*(UCB(\widehat{G})) = B_\lambda(G)^*$. See Theorem 3.7 below for the quantum group version of their result.

Let W and V be the left and right fundamental unitaries of $\mathbb{G}$, respectively. Let

$$\lambda : L_1(\mathbb{G}) \longrightarrow C_0(\widehat{\mathbb{G}}) \subseteq L_\infty(\widehat{\mathbb{G}}),\ f \longmapsto (f \otimes \iota)(W)$$

be the left regular representation of $\mathbb{G}$. Then λ has a natural w^*-w^* continuous and completely contractive algebra extension $M(\mathbb{G}) \longrightarrow L_\infty(\widehat{\mathbb{G}})$, which is still denoted by λ and given by $\langle \lambda(\mu), \hat{f}\rangle = \langle \mu, \lambda_*(\hat{f})\rangle$, where $\lambda_* : L_1(\widehat{\mathbb{G}}) \longrightarrow C_0(\mathbb{G}) \subseteq L_\infty(\mathbb{G})$ is the completely contractive injection $\hat{f} \longmapsto (\iota \otimes \hat{f})(W)$. Since $C_0(\widehat{\mathbb{G}}) = \overline{\lambda(L_1(\mathbb{G}))}^{\|\cdot\|}$, we have $\lambda : M(\mathbb{G}) \longrightarrow M(C_0(\widehat{\mathbb{G}})) \subseteq L_\infty(\widehat{\mathbb{G}})$. Similarly, the right regular representation

$\rho : L_1(\mathbb{G}) \longrightarrow L_\infty(\widehat{\mathbb{G}}')$, $f \longmapsto (\iota \otimes f)(V)$ of $\mathbb{G}$ is extended naturally to a w^*-w^* continuous and completely contractive algebra homomorphism $\rho : M(\mathbb{G}) \longrightarrow M(C_0(\widehat{\mathbb{G}}')) \subseteq L_\infty(\widehat{\mathbb{G}}')$ satisfying $\langle \rho(\mu), \hat{f}' \rangle = \langle \mu, \rho_*(\hat{f}') \rangle$, where $\rho_* : L_1(\widehat{\mathbb{G}}') \longrightarrow C_0(\mathbb{G}) \subseteq L_\infty(\mathbb{G})$ is the completely contractive injection $\hat{f}' \longmapsto (\hat{f}' \otimes \iota)(V)$ (cf. [29]). The proof of Proposition 2.1 shows that for all $\mu \in M(\mathbb{G})$, $\hat{f} \in L_1(\widehat{\mathbb{G}})$, and $\hat{f}' \in L_1(\widehat{\mathbb{G}}')$, we have

$$\langle \pi(\mu), \lambda_*(\hat{f}) \rangle = \langle \lambda(\mu), \hat{f} \rangle \quad \text{and} \quad \langle \pi(\mu), \rho_*(\hat{f}') \rangle = \langle \rho(\mu), \hat{f}' \rangle. \tag{2.9}$$

When $\mathbb{G} = VN(G)$ and $X = LUC(\mathbb{G})$, this relation between the map π and the left and right regular representations of $\mathbb{G}$ was given in [42, Proposiiton 4.2(b)].

As in the situation above for the left and right regular representations of $\mathbb{G}$, the maps λ_* and ρ_* can also be extended naturally to w^*-w^* continuous complete contractions $\lambda_* : M(\widehat{\mathbb{G}}) \longrightarrow M(C_0(\mathbb{G})) \subseteq L_\infty(\mathbb{G})$ and $\rho_* : M(\widehat{\mathbb{G}}') \longrightarrow M(C_0(\mathbb{G})) \subseteq L_\infty(\mathbb{G})$, respectively. If $\varpi_A : A^* \longrightarrow M(A)^*$ denotes the (unique) strictly continuous extension map for a given C^*-algebra A, then, extending (2.9), we can further obtain

$$\langle \varpi_{C_0(\mathbb{G})}(\mu), \lambda_*(\hat{\mu}) \rangle = \langle \lambda(\mu), \varpi_{C_0(\widehat{\mathbb{G}})}(\hat{\mu}) \rangle \quad \text{and} \quad \langle \varpi_{C_0(\mathbb{G})}(\mu), \rho_*(\hat{\mu}') \rangle = \langle \rho(\mu), \varpi_{C_0(\widehat{\mathbb{G}}')}(\hat{\mu}') \rangle, \tag{2.10}$$

where $\mu \in M(\mathbb{G})$, $\hat{\mu} \in M(\widehat{\mathbb{G}})$, and $\hat{\mu}' \in M(\widehat{\mathbb{G}}')$.

Note that the left regular representation $\hat{\lambda} : L_1(\widehat{\mathbb{G}}) \longrightarrow L_\infty(\mathbb{G})$ of $\widehat{\mathbb{G}}$ is given by $\hat{\lambda}(\hat{f}) = (\hat{f} \otimes \iota)(\Sigma W^* \Sigma)$, and the right regular representation $\hat{\rho}' : L_1(\widehat{\mathbb{G}}') \longrightarrow L_\infty(\mathbb{G})$ of $\widehat{\mathbb{G}}'$ is given by $\hat{\rho}'(\hat{f}') = (\iota \otimes \hat{f}')(\Sigma V^* \Sigma)$, where Σ is the flip operator on $L_2(\mathbb{G}) \otimes L_2(\mathbb{G})$. It follows that

$$\lambda_*(\hat{f}) = \hat{\lambda}((\hat{f})^*)^* \quad \text{and} \quad \rho_*(\hat{f}') = \hat{\rho}'((\hat{f}')^*)^* \quad (\hat{f} \in L_1(\widehat{\mathbb{G}}), \hat{f}' \in L_1(\widehat{\mathbb{G}}')).$$

Therefore, the maps $\lambda_* : M(\widehat{\mathbb{G}}) \longrightarrow L_\infty(\mathbb{G})$ and $\rho_* : M(\widehat{\mathbb{G}}') \longrightarrow L_\infty(\mathbb{G})$ are anti-algebra homomorphisms.

It is known that for any locally compact group G, the Fourier-Stieltjes algebra $B(G)$ of G is contained in $WAP(G)$, and the left regular representation of G maps $M(G)$ into $WAP(\widehat{G})$ (cf. [13, Theorem 11.2], [5, Corollary 3.3], and [12, Theorem 2.8 and Chapter 8]). For a general locally compact quantum group $\mathbb{G}$, from the proof below, we see in particular how the embedding $M(\widehat{\mathbb{G}}) \subseteq WAP(\mathbb{G})$ can be obtained quickly via the pair (λ, λ_*) of maps. The proof also motivates the argument used in Section 5 in establishing the stronger embedding $C_u(\widehat{\mathbb{G}})^* \subseteq WAP(\mathbb{G})$, a quantum group version of the above results on $L_\infty(G)$ and $VN(G)$, where $C_u(\widehat{\mathbb{G}})$ is the universal quantum group C^*-algebra of $\widehat{\mathbb{G}}$.

Proposition 2.3. *Let $\mathbb{G}$ be a locally compact quantum group. Then*

$$\lambda(M(\mathbb{G})) \subseteq WAP(\widehat{\mathbb{G}}),\ \rho(M(\mathbb{G})) \subseteq WAP(\widehat{\mathbb{G}}'),\ \lambda_*(M(\widehat{\mathbb{G}})) \subseteq WAP(\mathbb{G}),\ \textit{and}\ \ \rho_*(M(\widehat{\mathbb{G}}')) \subseteq WAP(\mathbb{G}).$$

Therefore, if $\mathbb{G}$ and $\widehat{\mathbb{G}}$ are co-amenable, then we have

$$\langle \pi(\mu), \lambda_*(\hat{\mu}) \rangle = \langle \lambda(\mu), \hat{\pi}(\hat{\mu}) \rangle \quad \textit{and} \quad \langle \pi(\mu), \rho_*(\hat{\mu}') \rangle = \langle \rho(\mu), \hat{\pi}'(\hat{\mu}') \rangle \tag{2.11}$$

for all $\mu \in M(\mathbb{G})$, $\hat{\mu} \in M(\widehat{\mathbb{G}})$, and $\hat{\mu}' \in M(\widehat{\mathbb{G}}')$, where π, $\hat{\pi}$, and $\hat{\pi}'$ are the embeddings $M(\mathbb{H}) \longrightarrow LUC(\mathbb{H})^*$ given in Proposition 2.1 with $\mathbb{H} = \mathbb{G}$, $\widehat{\mathbb{G}}$, and $\widehat{\mathbb{G}}'$, respectively.

Proof. To make the notation simple, we prove only the third inclusion; the proof of the rest inclusions is similar, noticing that λ and ρ are algebra homomorphisms and λ_* and ρ_* are anti-algebra homomorphisms.

Let $\hat{\mu} \in M(\widehat{\mathbb{G}})$ and $f \in L_1(\mathbb{G})$. Then we have

$$\lambda_*(\hat{\mu}) \star f \ = \ \lambda_*(\hat{\mu} \cdot \lambda(f)), \tag{2.12}$$

where for $\hat{m} \in M_0(\widehat{\mathbb{G}})^*$, $\hat{\mu} \cdot \hat{m} \in M(\widehat{\mathbb{G}})^{**}$ is given by $\langle \hat{\mu} \cdot \hat{m}, \hat{n} \rangle = \langle \hat{\mu}, \hat{m}\hat{n} \rangle$ $(\hat{n} \in M(\widehat{\mathbb{G}})^*)$. The functional $\hat{\mu} \cdot \hat{m}$ is indeed in $M(\widehat{\mathbb{G}})$, since the multiplication on the von Neumann algebra $M(\widehat{\mathbb{G}})^*$ is separately w^*-w^*

continuous. Then $\hat{\mu} \cdot \lambda(f_i) \longrightarrow \hat{\mu} \cdot \hat{m}$ weakly in $M(\widehat{\mathbb{G}})$ if $\lambda(f_i) \longrightarrow \hat{m} \in M(\widehat{\mathbb{G}})^*$ in the w^*-topology of $M(\widehat{\mathbb{G}})^*$. It follows that the set $\{\hat{\mu} \cdot \lambda(f) : f \in L_1(\mathbb{G}) \text{ and } \|f\| \leq 1\}$ is relatively weakly compact in $M(\widehat{\mathbb{G}})$. Therefore, the set $\{\lambda_*(\hat{\mu}) \star f : f \in L_1(\mathbb{G}) \text{ and } \|f\| \leq 1\}$ is relatively weakly compact in $L_\infty(\mathbb{G})$ (cf. (2.12)); that is, $\lambda_*(\hat{\mu}) \in WAP(\mathbb{G})$.

The final assertion follows from (2.10) and the fact that $WAP(\mathbb{H}) \subseteq LUC(\mathbb{H})$ if $\mathbb{H}$ is co-amenable. □

Theorem 2.4. *Let $\mathbb{G}$ be a locally compact quantum group and let X be a left introverted subspace of $L_\infty(\mathbb{G})$ containing $C_0(\mathbb{G})$. Then*

(i) *X^* is right unital $\Longleftrightarrow$ $\mathbb{G}$ is co-amenable;*

(ii) *X^* is left unital $\Longleftrightarrow$ [$\mathbb{G}$ is co-amenable and $X \subseteq LUC(\mathbb{G})$]. In this case, X^* is unital and $X = \langle X \star L_1(\mathbb{G})\rangle$.*

Proof. It is known from [3, Theorem 3.1] that $\mathbb{G}$ is co-amenable if and only if $M(\mathbb{G})$ is unital; the latter is also equivalent to $M(\mathbb{G})$ being right or left unital, since $C_0(\mathbb{G}) = \langle C_0(\mathbb{G}) \star M(\mathbb{G})\rangle = \langle M(\mathbb{G}) \star C_0(\mathbb{G})\rangle$ (cf. [29, Proposition 2.2]). Note that the restriction map $X^* \longrightarrow M(\mathbb{G})$ is a surjective algebra homomorphism, and any w^*-cluster point of a bounded approximate identity of $L_1(\mathbb{G})$ in X^* is a right identity of X^*. Hence, (i) is true, and (ii) follows from (i) and the fact that $L_1(\mathbb{G})$ is w^*-dense in X^*. □

Therefore, $(L_\infty(\mathbb{G})^*, \Box)$ is (left) unital $\Longleftrightarrow$ [$\mathbb{G}$ is co-amenable and $L_\infty(\mathbb{G}) = LUC(\mathbb{G})$], and for $X = C_0(\mathbb{G})$, $LUC(\mathbb{G})$, or $WAP(\mathbb{G})$, we have

$$X^* \text{ is one-sided (and hence two-sided) unital} \quad \Longleftrightarrow \quad \mathbb{G} \text{ is co-amenable.} \tag{2.13}$$

Clearly, we have the following corollary by (2.8), Proposition 2.1(i), and Theorem 2.4.

Corollary 2.5. *Let X and π be the same as in Proposition 2.1. Then X is a left $M(\mathbb{G})$-submodule of $L_\infty(\mathbb{G})$ and π is a right $M(\mathbb{G})$-module map. In addition, if X is an $M(\mathbb{G})$-submodule of $L_\infty(\mathbb{G})$ (e.g., $X = C_0(\mathbb{G})$, $LUC(\mathbb{G})$, $WAP(\mathbb{G})$), then π is an $M(\mathbb{G})$-bimodule map.*

In particular, if $\mathbb{G}$ is co-amenable with μ_0 the identity of $M(\mathbb{G})$, then $e_0 = \pi(\mu_0)$ is a right identity of X^*, and π is given by $\mu \longmapsto e_0 \star \mu$, which is equal to $\mu \longmapsto \mu \star e_0$ if X is an $M(\mathbb{G})$-submodule of $L_\infty(\mathbb{G})$.

The corollary below follows immediately from Proposition 2.1 and its proof. See [31] for results on extension of reduced Fourier-Stieltjes algebra homomorphisms.

Corollary 2.6. *Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be locally compact quantum groups and let $j : M(\mathbb{G}_1) \longrightarrow M(\mathbb{G}_2)$ be a bounded algebra homomorphism. Suppose that for $i = 1, 2$, X_i is a left introverted subspace of $L_\infty(\mathbb{G}_i)$ such that $C_0(\mathbb{G}_i) \subseteq X_i \subseteq M(C_0(\mathbb{G}_i))$ and $j^*(\tau_2(X_2)) \subseteq \tau_1(X_1)$, where $\tau_i : X_i \longrightarrow C_0(\mathbb{G}_i)^{**}$ is given as in (2.7). Then the bounded linear map $\kappa = \tau_1^{-1} \circ j^* \circ \tau_2 : X_2 \longrightarrow X_1$ satisfies the following:*

(i) *the adjoint map $\kappa^* : X_1^* \longrightarrow X_2^*$ is the unique w^*-w^* continuous extension of j and an algebra homomorphism with $\|\kappa^*\| = \|j\|$;*

(ii) *if j is completely bounded, then so is κ^* and we have $\|\kappa^*\|_{cb} = \|j\|_{cb}$.*

Let $B_{L_1(\mathbb{G})}(L_\infty(\mathbb{G}))$ be the Banach algebra of bounded right $L_1(\mathbb{G})$-module maps on $L_\infty(\mathbb{G})$ and let $RM(L_1(\mathbb{G}))$ be the right multiplier algebra of $L_1(\mathbb{G})$ (with opposite composition as the multiplication). Then $RM(L_1(\mathbb{G})) \cong B^\sigma_{L_1(\mathbb{G})}(L_\infty(\mathbb{G}))$ via $\mu \longmapsto \mu^*$, where $B^\sigma_{L_1(\mathbb{G})}(L_\infty(\mathbb{G}))$ is the Banach algebra consisting of w^*-w^* continuous maps in $B_{L_1(\mathbb{G})}(L_\infty(\mathbb{G}))$. For $m \in LUC(\mathbb{G})^*$, let $m_L(x) = m\Box x$ $(x \in L_\infty(\mathbb{G}))$, and we use the same notation when $m \in L_\infty(\mathbb{G})^*$. Then the map

$$LUC(\mathbb{G})^* \longrightarrow B_{L_1(\mathbb{G})}(L_\infty(\mathbb{G})),\ m \longmapsto m_L$$

is an injective, contractive, and w^*-w^* continuous algebra homomorphism (cf. [29]). In the sequel, whenever the algebras $RM(L_1(\mathbb{G}))$ and $LUC(\mathbb{G})^*$ are compared, they are identified with their canonical images in $B_{L_1(\mathbb{G})}(L_\infty(\mathbb{G}))$. Also, we write $L_1(\mathbb{G}) = M(\mathbb{G})$ if the canonical embedding $L_1(\mathbb{G}) \longrightarrow M(\mathbb{G})$ is onto. It is seen from Proposition 2.1(i) that we have the commutative diagram

$$\begin{array}{ccc} M(\mathbb{G}) & \xrightarrow{\ m^r\ } & RM(L_1(\mathbb{G})) \\ \pi \Big\downarrow & & \Big\downarrow\ T \longmapsto T^* \\ LUC(\mathbb{G})^* & \xrightarrow[n \longmapsto n_L]{} & B_{L_1(\mathbb{G})}(L_\infty(\mathbb{G})) \end{array} \tag{2.14}$$

of algebra homomorphisms, where $m^r(\mu)(f) = f \star \mu$ ($\mu \in M(\mathbb{G})$, $f \in L_1(\mathbb{G})$). Therefore, we always have $M(\mathbb{G}) \subseteq RM(L_1(\mathbb{G})) \cap LUC(\mathbb{G})^*$.

For general Banach algebras, we introduced in [26] the concept of *quotient strong Arens irregularity* (Q-SAI). It is known from Proposition 2.1(iii) and [26, Theorem 32] that

$$L_1(\mathbb{G}) \text{ is Q-SAI} \iff \mathfrak{Z}_t(LUC(\mathbb{G})^*) \subseteq RM(L_1(\mathbb{G})) \iff \mathfrak{Z}_t(LUC(\mathbb{G})^*) = M(\mathbb{G}). \tag{2.15}$$

It was also shown in [26, Theorem 15] that

$$\mathbb{G} \text{ is co-amenable} \iff RM(L_1(\mathbb{G})) \subseteq \mathfrak{Z}_t(LUC(\mathbb{G})^*) \iff LUC(\mathbb{G})^* = B_{L_1(\mathbb{G})}(L_\infty(\mathbb{G})). \tag{2.16}$$

Therefore, Q-SAI and co-amenability are in a sense opposite to each other, and every commutative locally compact quantum group happens to possess both properties.

Let $B^l_{L_1(\mathbb{G})}(L_\infty(\mathbb{G}))$ be the space of $T \in B_{L_1(\mathbb{G})}(L_\infty(\mathbb{G}))$ such that $T^*(L_1(\mathbb{G})) \subseteq \mathfrak{Z}_t(L_1(\mathbb{G})^{**}, \Box)$. Then

$B^l_{L_1(\mathbb{G})}(L_\infty(\mathbb{G})) = \{T \in B_{L_1(\mathbb{G})}(L_\infty(\mathbb{G})) : L_1(\mathbb{G})^{**} \to B_{L_1(\mathbb{G})}(L_\infty(\mathbb{G})), m \mapsto T \circ m_L \text{ is } w^*\text{-}w^* \text{ continuous}\}.$

The space $B^r_{L_1(\mathbb{G})}(L_\infty(\mathbb{G}))$ can be defined and obtained analogously by replacing $\mathfrak{Z}_t(L_1(\mathbb{G})^{**}, \Box)$ and m_L by $\mathfrak{Z}_t(L_1(\mathbb{G})^{**}, \diamondsuit)$ and m_R, respectively, where $m_R(x) = x \diamondsuit m$ ($x \in L_\infty(\mathbb{G})$). Then we have $B^r_{L_1(\mathbb{G})}(L_\infty(\mathbb{G})) = B_{L_1(\mathbb{G})^{**}}(L_\infty(\mathbb{G}))$, the algebra of bounded right $(L_1(\mathbb{G})^{**}, \diamondsuit)$-module maps on $L_\infty(\mathbb{G})$. It is evident that

$$B^\sigma_{L_1(\mathbb{G})}(L_\infty(\mathbb{G})) \subseteq B^l_{L_1(\mathbb{G})}(L_\infty(\mathbb{G})) \subseteq B_{L_1(\mathbb{G})}(L_\infty(\mathbb{G})).$$

Due to [26, Corollary 4(i) and Theorems 23 and 32], we have

$$L_1(\mathbb{G}) \text{ is Q-SAI} \iff B^l_{L_1(\mathbb{G})}(L_\infty(\mathbb{G})) = B^\sigma_{L_1(\mathbb{G})}(L_\infty(\mathbb{G})). \tag{2.17}$$

Remark 2.7. Suppose that $L_1(\mathbb{G})$ is Q-SAI and X is given as in Proposition 2.1(iii). Then we have $C_0(\mathbb{G}) \subseteq X \subseteq LUC(\mathbb{G})$ and $\mathfrak{Z}_t(LUC(\mathbb{G})^*) = \mathfrak{Z}_t(C_0(\mathbb{G})^*) = M(\mathbb{G}) \subseteq \mathfrak{Z}_t(X^*)$. In this situation, however, we may not have $\mathfrak{Z}_t(X^*) = M(\mathbb{G})$. For example, if $\mathbb{G} = L_\infty(G)$ with G a non-compact locally compact group, then $X = WAP(G)$ satisfies all the conditions in Proposition 2.1(iii), but $\mathfrak{Z}_t(X^*) = WAP(G)^* \neq M(G)$ since $C_0(G) \subsetneqq WAP(G)$.

Remark 2.8. Let $\mathbb{G}$ and $\mathbb{H}$ be locally compact quantum groups. Suppose that there exist bounded algebra homomorphisms $t : L_1(\mathbb{G}) \longrightarrow L_1(\mathbb{H})$ and $s : L_1(\mathbb{H}) \longrightarrow L_1(\mathbb{G})$ such that

$$t \circ s = id_{L_1(\mathbb{H})}, \ (s \circ t)(f) \star s(h) = f \star s(h), \text{ and } s(h) \star (s \circ t)(f) = s(h) \star f \quad (f \in L_1(\mathbb{G}), h \in L_1(\mathbb{H})).$$

It can be shown that if $L_1(\mathbb{G})$ is Q-SAI (respectively, SAI), then so is $L_1(\mathbb{H})$. See [24, 25] for such results in the co-commutative case. The above conditions are satisfied if $L_\infty(\mathbb{H})$ is a von Neumann subalgebra of $L_\infty(\mathbb{G})$ with $\Gamma_{\mathbb{H}} = \Gamma_{\mathbb{G}}|_{L_\infty(\mathbb{H})}$ and there is a normal conditional expectation P from $L_\infty(\mathbb{G})$ onto $L_\infty(\mathbb{H})$ such that $(id_{L_\infty(\mathbb{G})} \otimes P) \circ \Gamma_{\mathbb{G}} = (P \otimes id_{L_\infty(\mathbb{G})}) \circ \Gamma_{\mathbb{G}} = \Gamma_{\mathbb{G}} \circ P$. This is the case in particular if $\mathbb{G} = L_\infty(G)$

and $\mathbb{H} = L_\infty(G/K)$, or $\mathbb{G} = VN(G)$ and $\mathbb{H} = VN(G_0)$, where G is a locally compact group, K is a compact normal subgroup of G, and G_0 is an open subgroup of G. In other words, for commutative and co-commutative quantum groups, these conditions are satisfied if $\mathbb{H}$ is a *Kac subalgebra* of $\mathbb{G}$ (cf. [15, Theorem 4.5.10 and Corollary 4.3.6]). The reader is referred to [59, 62] for the related concepts of quantum subgroups and coideals of quotient type.

Finally, we recall that the class of Banach algebras of *type* (M) was introduced in [27]. Roughly speaking, a Banach algebra A is of type (M) if an algebraic form of the Kakutani-Kodaira theorem on locally compact groups holds for A (see [27] for the precise definition). It is known from [27] that every $L_1(G)$ is in this class, and so is $A(G)$ if G is amenable. Also, any separable quantum group algebra $L_1(\mathbb{G})$ with $\mathbb{G}$ co-amenable is of type (M). The reader is referred to [27] for more information on this class of Banach algebras.

3. Module maps over quantum groups, compactness, and discreteness

Theorem 3.1. *Let $\mathbb{G}$ be a locally compact quantum group. Then the following statements are equivalent:*

(i) $\mathbb{G}$ *is compact;*
(ii) $LUC(\mathbb{G}) = C_0(\mathbb{G})$*;*
(iii) *the embedding* $\pi : M(\mathbb{G}) \longrightarrow LUC(\mathbb{G})^*$ *is* w^*-w^* *continuous;*
(iv) $LUC(\mathbb{G})^* = M(\mathbb{G})$*;*
(v) $LUC(\mathbb{G})^* \subseteq RM(L_1(\mathbb{G}))$*;*
(vi) $L_1(\mathbb{G}) \star L_1(\mathbb{G})^{**} \subseteq L_1(\mathbb{G})$*;*
(vii) $L_1(\mathbb{G}) \star LUC(\mathbb{G})^* \subseteq M(\mathbb{G})$*;*
(viii) $L_1(\mathbb{G}) \star LUC(\mathbb{G})^* \subseteq RM(L_1(\mathbb{G}))$*;*
(ix) $B_{L_1(\mathbb{G})}(L_\infty(\mathbb{G})) = B^\sigma_{L_1(\mathbb{G})}(L_\infty(\mathbb{G}))$.

In addition, the inclusions in (v) *and* (vi) *can be replaced by the equalities if $\mathbb{G}$ is co-amenable.*

Proof. (i) $\Longleftrightarrow$ (ii). This follows from (2.5) and the facts that $1 \in LUC(\mathbb{G})$, and $\mathbb{G}$ is compact (i.e., $1 \in C_0(\mathbb{G})$) if and only if $C_0(\mathbb{G}) = M(C_0(\mathbb{G}))$.

(ii) $\Longrightarrow$ (vi). Suppose that $C_0(\mathbb{G}) = LUC(\mathbb{G})$. Let $f,\, g \in L_1(\mathbb{G})$, $m \in L_1(\mathbb{G})^{**}$, and $p = m|_{LUC(\mathbb{G})}$. Then $p \in M(\mathbb{G})$ and $g \star p \in L_1(\mathbb{G})$. For all $x \in L_\infty(\mathbb{G})$, we have $x \star f \in C_0(\mathbb{G})$, and thus

$$\langle (f \star g) \star m, x\rangle = \langle p, (x \star f) \star g\rangle_{M(\mathbb{G}),C_0(\mathbb{G})} = \langle g \star p, x \star f\rangle_{L_1(\mathbb{G}),L_\infty(\mathbb{G})} = \langle f \star (g \star p), x\rangle.$$

Therefore, $(f \star g) \star m = f \star (g \star p) \in L_1(\mathbb{G})$, and hence (vi) holds since $\langle L_1(\mathbb{G}) \star L_1(\mathbb{G})\rangle = L_1(\mathbb{G})$.

(vi) $\Longrightarrow$ (ii). Assume that $L_1(\mathbb{G}) \star L_1(\mathbb{G})^{**} \subseteq L_1(\mathbb{G})$ but $C_0(\mathbb{G}) \subsetneqq LUC(\mathbb{G})$. Then there exists $m \in L_1(\mathbb{G})^{**}$ such that $m|_{LUC(\mathbb{G})} \neq 0$ but $m|_{C_0(\mathbb{G})} = 0$. For all $a \in C_0(\mathbb{G})$ and $f \in L_1(\mathbb{G})$, we have $\langle a, f \star m\rangle = \langle a \star f, m\rangle = 0$ since $a \star f \in C_0(\mathbb{G})$, and hence $f \star m = 0$, noticing that $f \star m \in L_1(\mathbb{G})$. Thus $\langle x \star f, m\rangle = \langle x, f \star m\rangle = 0$ for all $x \in L_\infty(\mathbb{G})$ and $f \in L_1(\mathbb{G})$; that is, $m|_{LUC(\mathbb{G})} = 0$, a contradiction.

(ii) $\Longrightarrow$ (iii). Note that for $\mu \in M(\mathbb{G})$, the functional $\pi(\mu)$ is an extension of μ to $LUC(\mathbb{G})$. Therefore, if $C_0(\mathbb{G}) = LUC(\mathbb{G})$, then $\pi : M(\mathbb{G}) \longrightarrow LUC(\mathbb{G})^*$ is just the identity map and hence is w^*-w^* continuous.

(iii) $\Longrightarrow$ (iv). Suppose that $\pi : M(\mathbb{G}) \longrightarrow LUC(\mathbb{G})^*$ is w^*-w^* continuous. Then $\pi(M(\mathbb{G}))$ is w^*-closed in $LUC(\mathbb{G})^*$, since π is isometric. Note that $\pi(L_1(\mathbb{G}))$ is w^*-dense in $LUC(\mathbb{G})^*$. Therefore, we have $LUC(\mathbb{G})^* = \pi(M(\mathbb{G}))$.

(vi) $\Longleftrightarrow$ (vii) $\Longleftrightarrow$ (viii). This can be shown by the same argument as used in the proof of (ii) $\Longrightarrow$ (vi), noticing that $LUC(\mathbb{G})^*$ is a canonical quotient algebra of $(L_1(\mathbb{G})^{**}, \Box)$ and $\langle L_1(\mathbb{G}) \star L_1(\mathbb{G})\rangle = L_1(\mathbb{G})$.

(iv) $\Longrightarrow$ (v) $\Longrightarrow$ (viii), and (ix) $\Longrightarrow$ (v). This is obvious.

(v) $\Longrightarrow$ (ix). This follows from [28, Theorem 3.2(V)] on Banach algebras A satisfying $\langle A^2 \rangle = A$. To make the proof self-contained, we give below a direct proof. Suppose that $LUC(\mathbb{G})^* \subseteq RM(L_1(\mathbb{G}))$. Let $T \in B_{L_1(\mathbb{G})}(L_\infty(\mathbb{G}))$. Then $S = T|_{LUC(\mathbb{G})} : LUC(\mathbb{G}) \longrightarrow LUC(\mathbb{G})$ and thus $S^*(LUC(\mathbb{G})^*) \subseteq RM(L_1(\mathbb{G}))$. Let $f,\, g \in L_1(\mathbb{G})$. Then $S^*(g)_L = \mu^*$ for some $\mu \in RM(L_1(\mathbb{G}))$. Hence, for all $x \in L_\infty(\mathbb{G})$, we have

$$\langle T^*(f \star g), x\rangle = \langle g, T(x) \star f\rangle = \langle g, S(x \star f)\rangle = \langle S^*(g), x \star f\rangle = \langle S^*(g)_L(x), f\rangle = \langle x, \mu(f)\rangle.$$

Therefore, $T^*(f \star g) = \mu(f) \in L_1(\mathbb{G})$. Since $\langle L_1(\mathbb{G}) \star L_1(\mathbb{G})\rangle = L_1(\mathbb{G})$, it follows that $T^*(L_1(\mathbb{G})) \subseteq L_1(\mathbb{G})$ and thus $T \in B^\sigma_{L_1(\mathbb{G})}(L_\infty(\mathbb{G}))$.

The final assertion holds since if $\mathbb{G}$ is co-amenable, then $RM(L_1(\mathbb{G})) \subseteq LUC(\mathbb{G})^*$ (cf. (2.16)) and $L_1(\mathbb{G}) \subseteq L_1(\mathbb{G}) \star L_1(\mathbb{G})^{**}$, noticing that now $(L_1(\mathbb{G})^{**}, \Box)$ is right unital (cf. Theorem 2.4). □

The above Banach algebra approach gives an elementary proof of (i) $\Longleftrightarrow$ (vi), which is one of the main results of [55] by Runde. The proof can even be quicker if the embedding $M(\mathbb{G}) \hookrightarrow LUC(\mathbb{G})^*$ is used. For $\mathbb{G} = L_\infty(G)$, (i) $\Longleftrightarrow$ (ix) was shown by Lau [36, Theorem 2]. For $\mathbb{G} = VN(G)$, (i) $\Longleftrightarrow$ (ii), (i) $\Longleftrightarrow$ (iii), (i) $\Longleftrightarrow$ (iv), and (i) $\Longleftrightarrow$ (vi) were shown by Lau [37, Proposition 4.5], Ilie and Stokke [31, Proposition 3.5], Lau and Losert [42, Theorem 4.12], and Lau [38, Theorem 3.7], respectively.

Obviously, (vi) is equivalent to $L_1(\mathbb{G})^{**} \star L_1(\mathbb{G}) \subseteq L_1(\mathbb{G})$, since $L_1(\mathbb{G})$ is an involutive Banach algebra. Note that $C_0(\mathbb{G})^\perp \cap RM(L_1(\mathbb{G})) = \{0\}$, where $C_0(\mathbb{G})^\perp$ is the annihilator of $C_0(\mathbb{G})$ in $LUC(\mathbb{G})^*$. Therefore, due to (i) $\Longleftrightarrow$ (ii), we can also replace the product $L_1(\mathbb{G}) \star LUC(\mathbb{G})^*$ in (vii) and (viii) by $LUC(\mathbb{G})^* \star L_1(\mathbb{G})$.

Recall that a locally compact quantum group $\mathbb{G}$ is called *amenable* if there exists a left invariant mean on $L_\infty(\mathbb{G})$; that is, there exists $m \in L_\infty(\mathbb{G})^*$ such that $\|m\| = \langle m, 1\rangle = 1$ and $f \star m = \langle 1, f\rangle m$ for all $f \in L_1(\mathbb{G})$. In this case, $m|_{LUC(\mathbb{G})}$ is a left invariant mean on $LUC(\mathbb{G})$ since $1 \in LUC(\mathbb{G})$. Right invariant means are defined similarly. It is known that the involution on $L_1(\mathbb{G})$ can be canonically extended to a linear involution $\circ$ on $L_1(\mathbb{G})^{**}$ satisfying $(m \Box n)^\circ = n^\circ \Diamond m^\circ$ (cf. [26, page 633]). Clearly, $m \in L_\infty(\mathbb{G})^*$ is a left invariant mean if and only if m° is a right invariant mean. Therefore, the existence of a right invariant mean on $L_\infty(\mathbb{G})$ is equivalent to $\mathbb{G}$ being amenable. It is also known that every co-commutative locally compact quantum group is amenable.

It is not clear whether $WAP(\mathbb{G})$ always has a left invariant mean, though this is the case when $\mathbb{G}$ is commutative or amenable (cf. [56, Remark 4.7]). Similar to the situation for $LUC(\mathbb{G})$, the restriction to $WAP(\mathbb{G})$ of any one-sided invariant mean on $L_\infty(\mathbb{G})$ is a mean on $WAP(\mathbb{G})$ with the same side of invariance. Furthermore, for any left (respectively, right) invariant mean m on $WAP(\mathbb{G})$ with $p \in L_\infty(\mathbb{G})^*$ a Hahn-Banach extension of m, it is seen that $n = p^\circ|_{WAP(\mathbb{G})}$ is a right (respectively, left) invariant mean on $WAP(\mathbb{G})$. As noted in [56, Remark 4.7], we can conclude that

(3.1) any one-sided invariant mean on $WAP(\mathbb{G})$ is unique and two-sided invariant whenever it exists.

According to [26], the norm closed subspace $LUC(\mathbb{G})^*_R$ of $LUC(\mathbb{G})^*$ is defined by

$$LUC(\mathbb{G})^*_R \;=\; \{m \in LUC(\mathbb{G})^* : x \Diamond m \in LUC(\mathbb{G}) \text{ for all } x \in LUC(\mathbb{G})\}.$$

For $m \in LUC(\mathbb{G})^*_R$ and $n \in LUC(\mathbb{G})^*$, we naturally define $m \Diamond n \in LUC(\mathbb{G})^*$. It is known from [26, Theorem 2] that

$$\mathfrak{Z}_t(LUC(\mathbb{G})^*) \;=\; \{m \in LUC(\mathbb{G})^*_R : m \Box n = m \Diamond n \text{ for all } n \in LUC(\mathbb{G})^*\}. \tag{3.2}$$

Therefore, we have

$$M(\mathbb{G}) \subseteq \mathfrak{Z}_t(LUC(\mathbb{G})^*) \subseteq LUC(\mathbb{G})^*_R \subseteq LUC(\mathbb{G})^*.$$

It is evident that $m \in LUC(\mathbb{G})^*_R$ if m is a right invariant mean on $LUC(\mathbb{G})$.

Theorem 3.2. *Let $\mathbb{G}$ be a locally compact quantum group. Then the following statements are equivalent:*

(i) $LUC(\mathbb{G}) \subseteq WAP(\mathbb{G})$;

(ii) $LUC(\mathbb{G})^* = \mathfrak{Z}_t(LUC(\mathbb{G})^*)$;

(iii) $L_1(\mathbb{G}) \star L_1(\mathbb{G})^{**} \subseteq \mathfrak{Z}_t(L_1(\mathbb{G})^{**}, \Box)$;

(iv) $L_1(\mathbb{G}) \star LUC(\mathbb{G})^* \subseteq \mathfrak{Z}_t(LUC(\mathbb{G})^*)$;

(v) $B_{L_1(\mathbb{G})}(L_\infty(\mathbb{G})) = B^l_{L_1(\mathbb{G})}(L_\infty(\mathbb{G}))$.

If $LUC(\mathbb{G})^*$ is left faithful (e.g., $\mathbb{G}$ is co-amenable or SIN), then (i) - (v) *are equivalent to*

(vi) $LUC(\mathbb{G})^*_R = \mathfrak{Z}_t(LUC(\mathbb{G})^*)$.

Furthermore, the above (i) - (vi) *are all equivalent to $\mathbb{G}$ being compact in the following two cases:*

(a) $L_1(\mathbb{G})$ *is Q-SAI;*

(b) $\mathbb{G}$ *is amenable with $L_1(\mathbb{G})$ separable.*

Proof. The first two assertions follow from [28, Theorem 5.4] on more general Banach algebras A satisfying $\langle A^2 \rangle = A$. To show the final assertion, we suppose that (vi) holds. We prove that $\mathbb{G}$ is compact in the cases (a) and (b), noticing that [$\mathbb{G}$ is compact] $\Longrightarrow$ (i) $\Longrightarrow$ (vi).

Case (a). In this case, by the equivalence (ii) $\Longleftrightarrow$ (vi) (cf. [28, Theorem 5.4]), we have $LUC(\mathbb{G})^* = \mathfrak{Z}_t(LUC(\mathbb{G})^*) = M(\mathbb{G})$ (cf. (2.15)). Therefore, $\mathbb{G}$ is compact (cf. Theorem 3.1).

Case (b). Let m_0 be a fixed right invariant mean on $LUC(\mathbb{G})$, which exists by taking restriction to $LUC(\mathbb{G})$ of a right invariant mean on $L_\infty(\mathbb{G})$. Let $\gamma \in L_\infty(\mathbb{G})^*$ be any left invariant mean and let $n = \gamma|_{LUC(\mathbb{G})}$. Then $m_0 \in LUC(\mathbb{G})^*_R = \mathfrak{Z}_t(LUC(\mathbb{G})^*)$ and hence $m_0 \Box n = m_0 \Diamond n$. Since $L_1(\mathbb{G})$ is w^*-dense in $LUC(\mathbb{G})^*$, n is a left invariant mean, and m_0 is a right invariant mean, we obtain

$$n = \langle m_0, 1 \rangle n = m_0 \Box n = m_0 \Diamond n = \langle n, 1 \rangle m_0 = m_0;$$

that is, $n = m_0$. Taking an $f_0 \in L_1(\mathbb{G})$ with $f_0(1) = 1$, we have

$$\langle \gamma, x \rangle = f_0(1) \langle \gamma, x \rangle = \langle \gamma, x \star f_0 \rangle = \langle n, x \star f_0 \rangle = \langle m_0, x \star f_0 \rangle \quad \text{for all } x \in L_\infty(\mathbb{G}).$$

Therefore, γ is the unique left invariant mean on $L_\infty(\mathbb{G})$.

To obtain that $\mathbb{G}$ is compact, by [3, Proposition 3.1], we need only show that γ is in $L_1(\mathbb{G})$. This is indeed true by applying [39, Proposition 4.15(b)] on F-algebras, a class of Banach algebras including all convolution quantum group algebras. In fact, as mentioned in [39, Proposition 4.15(b)] (with details omitted), this follows by an argument given in the proof of [20, Theorem 7] (more precisely, by using [19, Corollary 1.3]). For convenience, we include below details of this argument.

Let K be the set of normal states on $L_\infty(\mathbb{G})$. Then, by the Hahn-Banach theorem, we have $\gamma \in \overline{K}^{w^*}$ (the w^*-closure of K in $L_\infty(\mathbb{G})^*$). Let (u_i) be a norm dense sequence in K. For each i, let $s_i : L_1(\mathbb{G}) \longrightarrow L_1(\mathbb{G})$ be the bounded linear map $f \longmapsto (u_i \star f) - f$. Then the map $s_i^{**} : L_\infty(\mathbb{G})^* \longrightarrow L_\infty(\mathbb{G})^*$ is given by $p \longmapsto (u_i \star p) - p$. Let

$$F = \overline{K}^{w^*} \cap \{p \in L_\infty(\mathbb{G})^* : s_i^{**}(p) = 0 \ \text{ for } i = 1, 2, \cdots\}.$$

Clearly, if $p \in \overline{K}^{w^*}$, then $s_i^{**}(p) = 0$ for all i if and only if p is a left invariant mean on $L_\infty(\mathbb{G})$. Therefore, we have $F = \{\gamma\}$. By [19, Corollary 1.3], there exists a sequence (v_i) in K such that $v_i \longrightarrow \gamma$ in the w^*-topology of $L_\infty(\mathbb{G})^*$. Then (v_i) is a weak Cauchy sequence in $L_1(\mathbb{G})$. It follows from the weak sequential completeness of $L_1(\mathbb{G})$ that γ is indeed in $L_1(\mathbb{G})$. □

Remark 3.3. (i) An inspection of the proof shows that (b) can be replaced by the more general condition

(c) $\mathbb{G}$ is amenable, and is compact whenever $L_\infty(\mathbb{G})$ has a unique left invariant mean.

It is known from [42, Corollary 4.11] that (c) is satisfied if $\mathbb{G}$ is co-commutative; in this case, Lau and Losert [43, Proposition 5.1] showed that (ii) in Theorem 3.2 is equivalent to $\mathbb{G}$ being compact.

(ii) Obviously, $L_\infty(\mathbb{G})$ has a unique left invariant mean if $\mathbb{G}$ is compact. In fact, due to (3.1), we have

$$LUC(\mathbb{G}) \subseteq WAP(\mathbb{G}) \quad \Longrightarrow \quad L_\infty(\mathbb{G}) \text{ has at most one left invariant mean.} \tag{3.3}$$

The proof of Theorem 3.2 shows that if $\mathbb{G}$ is amenable with $L_1(\mathbb{G})$ separable, then

$$L_\infty(\mathbb{G}) \text{ has a unique left invariant mean} \quad \Longleftrightarrow \quad \mathbb{G} \text{ is compact.} \tag{3.4}$$

Corollary 3.4. *Let $\mathbb{G}$ be a locally compact quantum group. Then the following statements are equivalent:*

(i) $\mathbb{G}$ *is compact;*

(ii) $L_1(\mathbb{G})$ *is Q-SAI and* $LUC(\mathbb{G}) \subseteq WAP(\mathbb{G})$.

If $L_1(\mathbb{G})$ *is separable, then* (i) *and* (ii) *are equivalent to*

(iii) $\mathbb{G}$ *is amenable and* $LUC(\mathbb{G}) \subseteq WAP(\mathbb{G})$.

Therefore, if $\mathbb{G}$ is co-amenable, then $\mathbb{G}$ is compact $\Longleftrightarrow$ [$L_1(\mathbb{G})$ *is Q-SAI and* $LUC(\mathbb{G}) = WAP(\mathbb{G})$].

Proof. The first two assertions follow from Theorem 3.2. The final assertion holds, since $WAP(\mathbb{G}) \subseteq LUC(\mathbb{G})$ if $\mathbb{G}$ is co-amenable (cf. [8, Proposition 3.12]). □

Remark 3.5. (i) Since every group algebra $L_1(G)$ is Q-SAI (cf. [40]), for any commutative quantum group $\mathbb{G}$, we have that $\mathbb{G}$ is compact if and only if $LUC(\mathbb{G}) = WAP(\mathbb{G})$. In the co-commutative case, though $L_1(\mathbb{G})$ can be non Q-SAI (cf. [47]), we still have

$$\mathbb{G} \text{ is compact} \quad \Longleftrightarrow \quad LUC(\mathbb{G}) \subseteq WAP(\mathbb{G}). \tag{3.5}$$

This is true due to [20, Theorem 12] (see [23, Corollary 6.6] for an improvement of [20, Theorem 12]). The equivalence in (3.5) also follows from (3.3) and the fact that $VN(G)$ has a unique invariant mean precisely when G is discrete (cf. [42, Corollary 4.11]). Theorem 3.2 together with Remark 3.3 shows that (3.5) holds if $\mathbb{G}$ satisfies one of the above conditions (a), (b), and (c). It would be interesting to know whether we have (3.5) for general locally compact quantum groups.

(ii) We point out that, even for co-commutative compact quantum groups $\mathbb{G}$, it is still open whether $WAP(\mathbb{G}) \subseteq LUC(\mathbb{G})$ holds. If this is true, then there is no infinite group G with the Fourier algebra $A(G)$ Arens regular; that is known only for amenable groups G (cf. [17, Proposition 3.5] and [38, Proposition 3.3]). As noted in [43, Problem 3] and [46, Remark 7], it is possible that an Olshanskii group would provide a counterexample to this open question. Therefore, it is very difficult to give an affirmative answer to the question whether $LUC(\mathbb{G}) = WAP(\mathbb{G})$ is equivalent to $\mathbb{G}$ being compact for a general locally compact quantum group $\mathbb{G}$ (see the problem raised by Runde in [56, Remark 4.5]).

Note that the adjoint of the inclusion map $L_1(\mathbb{G}) \longrightarrow M(\mathbb{G})$ is the surjective and normal $*$-homomorphism $C_0(\mathbb{G})^{**} \longrightarrow L_\infty(\mathbb{G})$, $x \longmapsto x|_{L_1(\mathbb{G})}$, which extends the inclusion $C_0(\mathbb{G}) \subseteq L_\infty(\mathbb{G})$. Clearly, the kernel of this $*$-homomorphism is the w^*-closed ideal $L_1(\mathbb{G})^\perp$ in $C_0(\mathbb{G})^{**}$. Then there exists a central projection p in $C_0(\mathbb{G})^{**}$ such that $L_1(\mathbb{G})^\perp = (1-p)C_0(\mathbb{G})^{**}$, and thus we have

$$C_0(\mathbb{G})^{**} \;=\; pC_0(\mathbb{G})^{**} \oplus_\infty L_1(\mathbb{G})^\perp \;\cong\; L_\infty(\mathbb{G}) \oplus_\infty L_1(\mathbb{G})^\perp \quad \text{via} \;\; x \oplus y \longmapsto x|_{L_1(\mathbb{G})} \oplus y. \tag{3.6}$$

Let $\kappa : L_\infty(\mathbb{G}) \longrightarrow C_0(\mathbb{G})^{**}$ be the injective and normal $*$-homomorphism induced from (3.6). Recall that

$$C_0(\mathbb{G}) \;\subseteq\; M(C_0(\mathbb{G})) \;\subseteq\; L_\infty(\mathbb{G}) \quad \text{and} \quad M(C_0(\mathbb{G})) \;\cong\; M(\widetilde{C_0(\mathbb{G})}) \;\subseteq\; C_0(\mathbb{G})^{**}.$$

However, as shown below, we do not have $\kappa(M(C_0(\mathbb{G}))) = M(\widetilde{C_0(\mathbb{G})})$ in general. On the other hand, comparing $\kappa : L_\infty(\mathbb{G}) \longrightarrow C_0(\mathbb{G})^{**}$ with the map $\tau = \pi^*|_{LUC(\mathbb{G})} : LUC(\mathbb{G}) \longrightarrow C_0(\mathbb{G})^{**}$ as given in Section 2, we see that κ is always w^*-w^* continuous and τ is always an $M(\mathbb{G})$-bimodule map (cf. Corollary 2.5). It turns out that κ is an $M(\mathbb{G})$-bimodule map if and only if τ is relatively w^*-w^* continuous.

Proposition 3.6. *For any locally compact quantum group $\mathbb{G}$, the following statements are equivalent:*

(i) $L_1(\mathbb{G}) = M(\mathbb{G})$;
(ii) $\kappa(C_0(\mathbb{G})) = \widetilde{C_0(\mathbb{G})}$ (*respectively,* $\kappa(M(C_0(\mathbb{G}))) = M(\widetilde{C_0(\mathbb{G})})$);
(iii) $\kappa|_{LUC(\mathbb{G})} = \tau$ (*respectively,* $\kappa(1_{L_\infty(\mathbb{G})}) = 1_{C_0(\mathbb{G})^{**}}$);
(iv) $\kappa : L_\infty(\mathbb{G}) \longrightarrow C_0(\mathbb{G})^{**}$ *is an $M(\mathbb{G})$-bimodule (respectively, $L_1(\mathbb{G})$-bimodule) map;*
(v) $\tau : LUC(\mathbb{G}) \longrightarrow M(\mathbb{G})^*$ *is $\sigma(LUC(\mathbb{G}), L_1(\mathbb{G}))$-$w^*$ continuous.*

Proof. Note that $\kappa(1_{L_\infty(\mathbb{G})}) = p$, and $\tau : LUC(\mathbb{G}) \longrightarrow M(\mathbb{G})^*$ is the inclusion map $LUC(\mathbb{G}) \longrightarrow L_\infty(\mathbb{G})$ when $L_1(\mathbb{G}) = M(\mathbb{G})$ canonically. Thus we have (iii) $\Longrightarrow$ (i) $\Longrightarrow$ each of (ii) - (v).

[(ii) $\Longrightarrow$ (i)] and [(v) $\Longrightarrow$ (i)]. This follows from the Hahn-Banach theorem and the facts that $\widetilde{C_0(\mathbb{G})}$ is w^*-dense in $C_0(\mathbb{G})^{**}$ and $pC_0(\mathbb{G})^{**}$ is w^*-closed in $C_0(\mathbb{G})^{**}$.

(iv) $\Longrightarrow$ (i). Suppose that $\kappa : L_\infty(\mathbb{G}) \longrightarrow C_0(\mathbb{G})^{**}$ is an $L_1(\mathbb{G})$-bimodule map. Since $(L_1(\mathbb{G})^\perp)\star L_1(\mathbb{G}) = \{0\}$, by (3.6), we have

$$C_0(\mathbb{G})^{**} \star L_1(\mathbb{G}) \;=\; (pC_0(\mathbb{G})^{**}) \star L_1(\mathbb{G}) \;=\; \kappa(L_\infty(\mathbb{G})) \star L_1(\mathbb{G}) \;=\; \kappa(L_\infty(\mathbb{G}) \star L_1(\mathbb{G})) \;\subseteq\; pC_0(\mathbb{G})^{**}.$$

Note that $1_{C_0(\mathbb{G})^{**}} \in C_0(\mathbb{G})^{**} \star L_1(\mathbb{G})$. Therefore, $1_{C_0(\mathbb{G})^{**}} = p$ and hence $L_1(\mathbb{G})^\perp = \{0\}$; that is, $L_1(\mathbb{G}) = M(\mathbb{G})$. □

It is known from [55, Theorem 4.4] that $\mathbb{G}$ is discrete if and only if $M(\widetilde{C_0(\mathbb{G})}) = C_0(\mathbb{G})^{**}$. We shall see from the following theorem that $\mathbb{G}$ is discrete if and only if [$M(C_0(\mathbb{G})) = L_\infty(\mathbb{G})$ and $L_1(\mathbb{G}) = M(\mathbb{G})$].

For co-amenable quantum groups $\mathbb{G}$, unlike the situation in Theorem 3.1(v) and (vi), the inclusions in Theorem 3.1(vii) and (viii) usually cannot be replaced by the equalities. In fact, we show below that the reversion of the inclusion in Theorem 3.1(viii) characterizes discreteness (the "reversion" of compactness). This is also the case with Theorem 3.1(vii) if $\mathbb{G}$ satisfies the following condition. We say that

$$\mathbb{G} \text{ satisfies } \textit{Condition } (*) \text{ if } [L_1(\mathbb{G}) \star L_1(\mathbb{G}) = L_1(\mathbb{G}) \implies \mathbb{G} \text{ is co-amenable}]. \tag{3.7}$$

Note that $L_1(\mathbb{G}) \star L_1(\mathbb{G}) = L_1(\mathbb{G})$ if $\mathbb{G}$ is co-amenable. It is known from [44, Proposition 2] that every co-commutative locally compact quantum group satisfies Condition $(*)$.

Theorem 3.7. *Let $\mathbb{G}$ be a locally compact quantum group. Consider the following statements:*

(i) $\mathbb{G}$ *is discrete;*

(ii) $L_1(\mathbb{G}) = M(\mathbb{G})$, *and* $LUC(\mathbb{G}) = L_\infty(\mathbb{G})$ (*respectively,* $M(C_0(\mathbb{G})) = L_\infty(\mathbb{G})$)*;*
(iii) $\pi^*|_{LUC(\mathbb{G})} : LUC(\mathbb{G}) \longrightarrow M(\mathbb{G})^*$ *is surjective;*
(iv) $RM(L_1(\mathbb{G})) \subseteq L_1(\mathbb{G}) \star LUC(\mathbb{G})^*$*;*
(v) $\mathfrak{Z}_t(LUC(\mathbb{G})^*) \subseteq L_1(\mathbb{G}) \star LUC(\mathbb{G})^*$*;*
(vi) $M(\mathbb{G}) \subseteq L_1(\mathbb{G}) \star LUC(\mathbb{G})^*$*;*
(vii) $L_1(\mathbb{G}) = M(\mathbb{G})$.

Then (i) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (iii) $\Longleftrightarrow$ (iv) $\Longrightarrow$ (v) $\Longrightarrow$ (vi) $\Longrightarrow$ (vii), (i) - (vi) *are equivalent if* $\mathbb{G}$ *satisfies Condition* $(*)$, *and* (i) - (vii) *are equivalent if* $\mathbb{G}$ *is co-amenable.*

Furthermore, if $\mathbb{G}$ *is co-amenable with* $L_1(\mathbb{G})$ *of type* (M) (*e.g.,* $L_1(\mathbb{G})$ *is separable*), *then* (i) - (vii) *are all equivalent to*

(viii) $LUC(\mathbb{G}) = L_\infty(\mathbb{G})$.

Proof. It is obvious that (i) $\Longrightarrow$ each of (ii) - (v), and (v) $\Longrightarrow$ (vi). We have (ii) $\Longleftrightarrow$ (i) by Proposition 3.6 and [55, Theorem 4.4], and (vi) $\Longrightarrow$ (vii) holds due to Proposition 2.1.

(iii) $\Longrightarrow$ (i). Suppose that $\pi^*(LUC(\mathbb{G})) = M(\mathbb{G})^*$. By Proposition 2.1(ii), we have $\widetilde{M(C_0(\mathbb{G}))} = C_0(\mathbb{G})^{**}$. Therefore, $\mathbb{G}$ is discrete (cf. [55, Theorem 4.4]).

(iv) $\Longrightarrow$ (i). Suppose that $RM(L_1(\mathbb{G})) \subseteq L_1(\mathbb{G}) \star LUC(\mathbb{G})^*$. Let $id \in RM(L_1(\mathbb{G}))$ be the identity map on $L_1(\mathbb{G})$. Then there exist $f \in L_1(\mathbb{G})$ and $n \in LUC(\mathbb{G})^*$ such that $id^* = (f \star n)_L$. By the decomposition $LUC(\mathbb{G})^* = M(\mathbb{G}) \oplus C_0(\mathbb{G})^\perp$ in Proposition 2.1, we have $n = \mu + p$ for some $\mu \in M(\mathbb{G})$ and $p \in C_0(\mathbb{G})^\perp$. Let $f_0 = f \star \mu$ and $m = f \star p$. Then $f_0 \in L_1(\mathbb{G})$, $m \in C_0(\mathbb{G})^\perp$, and $id^* = (f_0 + m)_L = (f_0)_L + m_L$. For $x \in C_0(\mathbb{G})$ and $g, h \in L_1(\mathbb{G})$, we have $\langle m_L(h \star x), g\rangle = \langle m, h \star x \star g\rangle = 0$, and hence

$$\langle x, g \star h\rangle = \langle h \star x, g\rangle = \langle id^*(h \star x), g\rangle = \langle (f_0)_L(h \star x), g\rangle + \langle m_L(h \star x), g\rangle = \langle f_0 \star h \star x, g\rangle = \langle x, g \star f_0 \star h\rangle.$$

Thus $g \star h = g \star f_0 \star h$ for all $g, h \in L_1(\mathbb{G})$. Therefore, f_0 is an identity of $L_1(\mathbb{G})$ since $L_1(\mathbb{G})$ is faithful. It follows that $L_1(\mathbb{G})$ is unital, and hence $\mathbb{G}$ is discrete.

The second and third assertions hold since $\mathbb{G}$ is co-amenable if and only if $M(\mathbb{G})$ is unital (cf. [3, Theorem 3.1]), and the final assertion follows from [26, Theorem 22]. □

Remark 3.8. (a) It is known from [4, Theorem 3.6] that if a locally compact group G contains an almost connected open normal subgroup, then G is compact whenever $A(G) = B_\lambda(G)$. It is open whether this is true for all groups G. Therefore, it is unknown whether (i) and (vii) are equivalent for general co-commutative quantum groups. Also, it is not clear for us whether (i) $\Longleftrightarrow$ (vii) for all non co-amenable compact quantum groups $\mathbb{G}$, though this holds if in addition $\mathbb{G}$ is co-commutative (cf. [4, Lemma 3.5]).

(b) Obviously, we have $\mathbb{G}$ is discrete $\Longrightarrow$ $[LUC(\mathbb{G}) = L_\infty(\mathbb{G})]$ $\Longrightarrow$ $[M(C_0(\mathbb{G})) = L_\infty(\mathbb{G})]$. The reverse implications hold if $\mathbb{G}$ is commutative or co-commutative (cf. [21, Theorem 3]). As seen in Theorem 3.7, we have "$[LUC(\mathbb{G}) = L_\infty(\mathbb{G})]$ $\Longrightarrow$ $\mathbb{G}$ is discrete" if $L_1(\mathbb{G})$ is of type (M).

On the other hand, there exists a non-discrete quantum group $\mathbb{G}$ such that $L_1(\mathbb{G}) \cong M(\mathbb{G})$ as Banach spaces and $M(C_0(\mathbb{G})) \cong L_\infty(\mathbb{G})$ as C^*-algebras. In fact, Baaj and Skandalis showed that there exists a quantum group $\mathbb{G}$ such that $C_0(\mathbb{G}) \cong K(H) \subseteq B(H) \cong L_\infty(\mathbb{G})$ (as C^*-algebras) for some Hilbert space H of infinite dimension (cf. [2, Section 8]). It is clear that this quantum group $\mathbb{G}$ is non-discrete (nor compact). Then the $*$-homomorphic embedding $C_0(\mathbb{G}) \longrightarrow L_\infty(\mathbb{G})$ obtained from the above $*$-isomorphisms is not the canonical inclusion $C_0(\mathbb{G}) \subseteq L_\infty(\mathbb{G})$. Therefore, both induced identifications $L_1(\mathbb{G}) \cong M(\mathbb{G})$ and $M(C_0(\mathbb{G})) \cong L_\infty(\mathbb{G})$ are not the canonical equalities. The reader is referred to [30] for conditions which are equivalent to $C_0(\mathbb{G}) \subseteq K(L_2(\mathbb{G}))$.

For convenience, a locally compact quantum group $\mathbb{G}$ is said to be *finite* if $L_\infty(\mathbb{G})$ is finite dimensional. It is clear that

$$\mathbb{G} \text{ is finite} \iff \mathbb{G} \text{ is compact and discrete.} \tag{3.8}$$

In fact, if $\mathbb{G}$ is compact and discrete, then $L_\infty(\mathbb{G})$ must be a finite direct sum of full matrix algebras.

The result below is immediate by Theorem 3.1, Theorem 3.7, and (3.8), which shows that compactness, discreteness, and finiteness of $\mathbb{G}$ can be characterized simultaneously by comparing $RM(L_1(\mathbb{G}))$ with the module product $L_1(\mathbb{G}) \star LUC(\mathbb{G})^*$. It is interesting to compare this result with those characterizations of Q-SAI and co-amenability given in terms of $RM(L_1(\mathbb{G}))$ and $\mathfrak{Z}_t(LUC(\mathbb{G})^*)$ (cf. (2.15) and (2.16)). Note that $\mathfrak{Z}_t(LUC(\mathbb{G})^*)$ and $L_1(\mathbb{G}) \star LUC(\mathbb{G})^*$ are not related in general (cf. Theorems 3.2 and 3.7).

Proposition 3.9. *Let $\mathbb{G}$ be a locally compact quantum group. Then the following assertions hold.*

(i) $\mathbb{G}$ *is compact* $\iff$ $L_1(\mathbb{G}) \star LUC(\mathbb{G})^* \subseteq RM(L_1(\mathbb{G}))$.

(ii) $\mathbb{G}$ *is discrete* $\iff$ $L_1(\mathbb{G}) \star LUC(\mathbb{G})^* \supseteq RM(L_1(\mathbb{G}))$.

(iii) $\mathbb{G}$ *is finite* $\iff$ $L_1(\mathbb{G}) \star LUC(\mathbb{G})^* = RM(L_1(\mathbb{G}))$.

Furthermore, the algebra $RM(L_1(\mathbb{G}))$ in (i) - (iii) *can be replaced by $M(\mathbb{G})$ if $\mathbb{G}$ satisfies Condition $(*)$.*

It is known that a group algebra $L_1(G)$ is Arens regular if and only if G is finite (cf. [65]). As mentioned in Remark 3.5(ii), the above $L_1(G)$ can be replaced by $A(G)$ if G is amenable. These two results can be seen dual of each other, noticing that $L_\infty(G)$ is always co-amenable, and $VN(G)$ is co-amenable precisely when G is amenable. Also, for a general locally compact group G, Arens regularity of $A(G)$ implies discreteness of G (cf. [17, Theorem 3.2]). We have the following quantum group version of these results.

Theorem 3.10. *Let $\mathbb{G}$ be a locally compact quantum group such that $L_1(\mathbb{G})$ is Arens regular. Then*

(i) $\mathbb{G}$ *is discrete if* $\mathbb{G}$ *is co-amenable;*

(ii) $\mathbb{G}$ *is compact if one of the conditions* (a), (b), *and* (c) *in Theorem 3.2 and Remark 3.3 is satisfied.*

Therefore, $\mathbb{G}$ *is finite if* $\mathbb{G}$ *is co-amenable satisfying one of the above* (a), (b), *and* (c).

Proof. (i) If $\mathbb{G}$ is co-amenable, then $L_1(\mathbb{G})$ is unital by [60, Theorem 3.3], and hence $\mathbb{G}$ is discrete.

(ii) Since $L_1(\mathbb{G})$ is Arens regular, we have $\mathfrak{Z}_t(L_1(\mathbb{G})^{**}, \Box) = L_1(\mathbb{G})^{**}$. Then the assertion holds by Theorem 3.2 and Remark 3.3.

The final assertion follows from (i), (ii), and (3.8). □

Let $LM(L_1(\mathbb{G}))$ be the left multiplier algebra of $L_1(\mathbb{G})$. Then $LM(L_1(\mathbb{G})) \longrightarrow B(L_\infty(\mathbb{G}))$, $\mu \longmapsto \mu^*$ is an injective anti-algebra homomorphism. In the theorem below, $RM(L_1(\mathbb{G}))$ and $LM(L_1(\mathbb{G}))$ are identified with their canonical images in $B(L_\infty(\mathbb{G}))$ and the commutants are taken in $B(L_\infty(\mathbb{G}))$. This result in particular improves and extends [18, Theorem 5.1], which says that $B_{L_1(G)}(L_\infty(G))^c = M(G)$ (or equivalently, $L_1(G)^{cc} = M(G)$) holds in $B(L_\infty(G))$ for every locally compact group G, noticing that $L_1(G)$ is always SAI and of type (M) (cf. [41] and [27]).

Theorem 3.11. *Let $\mathbb{G}$ be a locally compact quantum group. Then we have*

(I) $L_1(\mathbb{G})^c = M(\mathbb{G})^c = RM(L_1(\mathbb{G}))^c$ *under* $L_1(\mathbb{G}) \subseteq M(\mathbb{G}) \hookrightarrow RM(L_1(\mathbb{G}))$*;*

(II) $L_1(\mathbb{G})^c = M(\mathbb{G})^c = LM(L_1(\mathbb{G}))^c$ *under* $L_1(\mathbb{G}) \subseteq M(\mathbb{G}) \hookrightarrow LM(L_1(\mathbb{G}))$*;*

(III) $LM(L_1(\mathbb{G}))^c = RM(L_1(\mathbb{G}))$ $\iff$ $\mathbb{G}$ *is compact* $\iff$ $RM(L_1(\mathbb{G}))^c = LM(L_1(\mathbb{G}))$.

Furthermore, the following statements are equivalent if $L_1(\mathbb{G})$ is of type (M):

(i) $B_{L_1(\mathbb{G})^{**}}(L_\infty(\mathbb{G})) = B^\sigma_{L_1(\mathbb{G})}(L_\infty(\mathbb{G}))$;
(ii) $L_1(\mathbb{G})$ *is SAI;*
(iii) $M(\mathbb{G})^{cc} = M(\mathbb{G})$ *via* $M(\mathbb{G}) \cong RM(L_1(\mathbb{G}))$ *or* $M(\mathbb{G}) \cong LM(L_1(\mathbb{G}))$.

Proof. (I) Clearly, we have $RM(L_1(\mathbb{G}))^c \subseteq M(\mathbb{G})^c \subseteq L_1(\mathbb{G})^c$. Conversely, let $T \in L_1(\mathbb{G})^c$. Then $T(f\star x) = f \star T(x)$ ($f \in L_1(\mathbb{G})$, $x \in L_\infty(\mathbb{G})$). For all f, $g \in L_1(\mathbb{G})$, $\mu \in RM(L_1(\mathbb{G}))$, and $x \in L_\infty(\mathbb{G})$, since $g \star \mu^*(x) = \mu(g) \star x$, we have

$$\langle T(\mu^*(x)), f \star g\rangle = \langle T(g \star \mu^*(x)), f\rangle = \langle T(\mu(g) \star x), f\rangle = \langle Tx, f \star \mu(g)\rangle = \langle \mu^*(Tx), f \star g\rangle.$$

Thus $T \circ \mu^* = \mu^* \circ T$ for all $\mu \in RM(L_1(\mathbb{G}))$, since $\langle L_1(\mathbb{G})\star L_1(\mathbb{G})\rangle = L_1(\mathbb{G})$. Therefore, $T \in RM(L_1(\mathbb{G}))^c$.

(II) This follows by a similar argument as given above.

(III) It is easy to see that $L_1(\mathbb{G})^c = B_{L_1(\mathbb{G})}(L_\infty(\mathbb{G}))$ under $L_1(\mathbb{G}) \hookrightarrow LM(L_1(\mathbb{G}))$. The equivalences then follow from (I), (II), and Theorem 3.1 and its left-sided version.

Suppose now that $L_1(\mathbb{G})$ is of type (M). Since $\mathbb{G}$ is co-amenable, it is seen from Section 2 that

$$B^\sigma_{L_1(\mathbb{G})}(L_\infty(\mathbb{G})) \;=\; \{m_L : m \in L_1(\mathbb{G})^{**} \text{ and } L_1(\mathbb{G}) \star m \subseteq L_1(\mathbb{G})\}$$

and

$$B_{L_1(\mathbb{G})^{**}}(L_\infty(\mathbb{G})) \;=\; \{m_L : m \in L_1(\mathbb{G})^{**} \text{ and } L_1(\mathbb{G}) \star m \subseteq \mathfrak{Z}_t(L_1(\mathbb{G})^{**}, \diamondsuit)\}.$$

Recall from [27, Theorem 32(ii)] that $L_1(\mathbb{G})$ is SAI if and only if $L_1(\mathbb{G}) \star \mathfrak{Z}_t(L_1(\mathbb{G})^{**}, \diamondsuit) \subseteq L_1(\mathbb{G})$. Therefore, we obtain (i) $\iff$ (ii). On the other hand, under $L_1(\mathbb{G}) \subseteq M(\mathbb{G}) \cong RM(L_1(\mathbb{G}))$, we have $B^\sigma_{L_1(\mathbb{G})}(L_\infty(\mathbb{G})) = M(\mathbb{G})$, $L_1(\mathbb{G})^c = \{m_R : m \in L_1(\mathbb{G})^{**}\}$, and $B_{L_1(\mathbb{G})^{**}}(L_\infty(\mathbb{G})) = \{m_R : m \in L_1(\mathbb{G})^{**}\}^c$. The corresponding equalities hold for $L_1(\mathbb{G}) \subseteq M(\mathbb{G}) \cong LM(L_1(\mathbb{G}))$. It follows from (I) and (II) that we have (i) $\iff$ (iii). □

In the immediate corollary below, (ii) is the quantum group version of [42, Theorem 6.5(i)] on $VN(G)$.

Corollary 3.12. *Let* $\mathbb{G}$ *be a locally compact quantum group. Then we have*

(i) *if* $L_1(\mathbb{G})$ *is separable, then* $M(\mathbb{G})^{cc} = M(\mathbb{G}) \iff$ [$\mathbb{G}$ *is co-amenable and* $L_1(\mathbb{G})$ *is SAI*];
(ii) *if* $\mathbb{G}$ *is compact with* $L_1(\mathbb{G})$ *of type* (M), *then* $L_1(\mathbb{G})$ *is SAI.*

Let G be a locally compact group. Then $B_{L_1(G)}(L_\infty(G)) = LUC(G)^*$,

$$B^\sigma_{L_1(G)}(L_\infty(G)) \;=\; B_{L_1(G)^{**}}(L_\infty(G)) \;=\; B^l_{L_1(G)}(L_\infty(G)) \;\cong\; \mathfrak{Z}_t(LUC(G)^*) \;=\; M(G),$$

and $L_1(G)$ is SAI (cf. [40, 41]). In particular, Theorem 3.2 strengthens [36, Theorem 2] on $\mathbb{G} = L_\infty(G)$. The situation for $A(G)$ is very different. Firstly, the topological centres of $A(G)^{**}$ (with either Arens product) and $UCB(\widehat{G})^*$ are just their algebraic centres, since $A(G)$ is commutative. Secondly, on the one hand, $A(G)$ is SAI for many amenable groups G (cf. [16, 24, 25, 42, 43]). On the other hand, as shown by Losert [46, 47], both $A(\mathbb{F}_2)$ and $A(SU(3))$ are non SAI, though $A(\mathbb{F}_2)$ is Q-SAI (cf. Corollary 3.4) and $SU(3)$ is compact. Finally, we have $UCB(\widehat{G})^* \subseteq B_{A(G)}(VN(G))$, $B_\lambda(G) \subseteq B^\sigma_{A(G)}(VN(G))$, and $\mathfrak{Z}(UCB(\widehat{G})^*) \subseteq B_{A(G)^{**}}(VN(G)) = B^l_{A(G)}(VN(G))$, and any (and hence all) of these three equalities holds precisely when G is amenable. In this case, (iv) $\iff$ (ix) in Theorem 3.1, (ii) $\iff$ (v) in Theorem 3.2, and the equivalence in (2.17) are indeed non-trivial.

We close this section with some applications to $A(G)$. Obviously, we always have

$$[\mathfrak{Z}(A(G)^{**})\Box A(G)^{**} \subseteq A(G)] \Longrightarrow [\mathfrak{Z}(A(G)^{**})\Box A(G)^{**} \subseteq \mathfrak{Z}(A(G)^{**})] \Longrightarrow [A(G) \cdot A(G)^{**} \subseteq \mathfrak{Z}(A(G)^{**})].$$

By Theorem 3.1 and [61, Theorem 2.2], we have $\mathfrak{Z}(A(G)^{**})\Box A(G)^{**} \subseteq A(G) \Longleftrightarrow A(G)\cdot A(G)^{**} \subseteq A(G)$. Combining the above with Theorem 3.2 and Remark 3.3, we obtain the following result on Fourier algebras.

Corollary 3.13. *Let G be a locally compact group and $\mathbb{G} = VN(G)$. Then* (i) - (ix) *in Theorem 3.1 and* (i) - (vi) *in Theorem 3.2 are all equivalent, which are also equivalent to each of the following statements:*

(i) *G is discrete;*
(ii) $\mathfrak{Z}(A(G)^{**})\Box A(G)^{**} \subseteq A(G)$*;*
(iii) $\mathfrak{Z}(A(G)^{**})\Box A(G)^{**} \subseteq \mathfrak{Z}(A(G)^{**})$.

Let $B(G)$ be the Fourier-Stieltjes algebra of G, let $MA(G)$ be the multiplier algebra of $A(G)$, and let $M_{cb}A(G)$ be the completely bounded multiplier algebra of $A(G)$. Then we have

$$A(G) \subseteq B_\lambda(G) \subseteq B(G) \subseteq M_{cb}A(G) \subseteq MA(G) \subseteq B(VN(G)).$$

As pointed out earlier, the quantum group $\mathbb{G} = VN(G)$ satisfies Condition $(*)$ defined in (3.7). It is known from [45, Theorem 1] that G is amenable if and only if $B(G) = MA(G)$. Also, we have $MA(G) \subseteq MA(G)^c \cap MA(G)^{cc}$. Therefore, we obtain the corollary below from Theorems 3.7 and 3.11, and Proposition 3.9 and Corollary 3.13.

Corollary 3.14. *Let G be a locally compact group. Then Proposition 3.9 holds for $\mathbb{G} = VN(G)$ with $RM(L_1(\mathbb{G}))$ replaced by any of $B_\lambda(G)$, $B(G)$, $M_{cb}A(G)$, $MA(G)$, and $\mathfrak{Z}(UCB(\widehat{G})^*)$.*

Furthermore, we have

(i) $A(G)^c = A(G) \iff$ *G is finite;*
(ii) $A(G)^{cc} = A(G) \iff$ *G is compact and $A(G)$ is SAI;*
(iii) $B(G)^c = B(G)$ *(respectively, $B_\lambda(G)^c = B_\lambda(G)$)* $\iff$ *G is amenable and discrete;*
(iv) $B(G)^{cc} = B(G)$ *(respectively, $B_\lambda(G)^{cc} = B_\lambda(G)$)* $\iff$ *G is amenable and $A(G)$ is SAI.*

Example 3.15. Let $G = SU(3)$. Since $A(G)$ is non SAI as mentioned above, by Theorem 3.11 and Corollary 3.13, we obtain

$$B^\sigma_{A(G)}(VN(G)) \subsetneqq B_{A(G)^{**}}(VN(G)) \subsetneqq B_{A(G)}(VN(G)).$$

On the other hand, we can see from Theorem 3.11 and its proof that

$$B^\sigma_{A(G)}(VN(G))^{cc} = B_{A(G)^{**}}(VN(G)) = B_{A(G)}(VN(G))^c = A(G)^{cc}.$$

Note that $B_{A(G)^{**}}(VN(G))$ is also a commutative Banach algebra, since $B_{A(G)^{**}}(VN(G)) \cong \mathfrak{Z}(A(G)^{**})$.

Remark 3.16. The canonical representation $LUC(\mathbb{G})^* \longrightarrow B_{L_1(\mathbb{G})}(L_\infty(\mathbb{G}))$ in fact induces an injective and completely contractive algebra homomorphism $\Phi_L : LUC(\mathbb{G})^* \longrightarrow CB_{L_1(\mathbb{G})}(L_\infty(\mathbb{G}))$, $m \longmapsto m_L$, which is just the adjoint map of the completely contractive module product $L_\infty(\mathbb{G})\widehat{\otimes}L_1(\mathbb{G}) \longrightarrow LUC(\mathbb{G})$, $x \otimes f \longmapsto x \star f$ (cf. [29, Section 6]). It is easy to see that the algebras $B(L_\infty(\mathbb{G}))$, $RM(L_1(\mathbb{G}))$, and $LM(L_1(\mathbb{G}))$ can be replaced by $CB(L_\infty(\mathbb{G}))$, $RM_{cb}(L_1(\mathbb{G}))$, and $LM_{cb}(L_1(\mathbb{G}))$, respectively, in the results presented in this section, and each of these results (as well as those in Section 4) has its left-sided and right-sided versions.

4. Weakly compact module maps over quantum groups

As mentioned in Section 3, for a general non-commutative and non-amenable locally compact quantum group $\mathbb{G}$, it is not clear whether $WAP(\mathbb{G})$ has a left invariant mean, whose existence is thus assumed in the proposition below. This proposition generalizes [36, Theorem 4] on $L_\infty(G)$ with $X = LUC(G)$ or $L_\infty(G)$, noticing that the map T there satisfies $T(1) = 1$. The proof of [36, Theorem 4] is modified for the present quantum group setting.

Proposition 4.1. *Let $\mathbb{G}$ be a locally compact quantum group such that $WAP(\mathbb{G})$ has a left invariant mean. Then the following statements are equivalent:*

- (i) *$\mathbb{G}$ is amenable;*
- (ii) *there exists a unital, completely positive, and weakly compact map $S \in B_{L_1(\mathbb{G})}(L_\infty(\mathbb{G}))$;*
- (iii) *there exists a weakly compact map $S \in B_{L_1(\mathbb{G})}(L_\infty(\mathbb{G}))$ such that $1 \in S(L_\infty(\mathbb{G}))$.*

Furthermore, if $\mathbb{G}$ is co-amenable, then (i) - (iii) *are equivalent to*

- (iv) *there exists a weakly compact map $S \in B_{L_1(\mathbb{G})}(LUC(\mathbb{G}))$ such that $1 \in S(LUC(\mathbb{G}))$,*

where $B_{L_1(\mathbb{G})}(LUC(\mathbb{G}))$ is the space of bounded right $L_1(\mathbb{G})$-module maps on $LUC(\mathbb{G})$.

Proof. (i) $\Longrightarrow$ (ii). Suppose that $m \in L_\infty(\mathbb{G})^*$ is a left invariant mean. We define $S : L_\infty(\mathbb{G}) \longrightarrow L_\infty(\mathbb{G})$ by $S(x) = \langle m, x\rangle 1$. Then $S \in B_{L_1(\mathbb{G})}(L_\infty(\mathbb{G}))$ is unital, completely positive, and weakly compact.

(ii) $\Longrightarrow$ (iii). This is trivial.

(iii) $\Longrightarrow$ (i). Suppose that S is a weakly compact operator in $B_{L_1(\mathbb{G})}(L_\infty(\mathbb{G}))$ such that $1 \in S(L_\infty(\mathbb{G}))$. For $x \in L_\infty(\mathbb{G})$, since the set $\{S(x) \star f : f \in L_1(\mathbb{G}) \text{ and } \|f\| \le 1\} = \{S(x \star f) : f \in L_1(\mathbb{G}) \text{ and } \|f\| \le 1\}$ is relatively weakly compact in $L_\infty(\mathbb{G})$, we have $S(x) \in WAP(\mathbb{G})$. Let β be a left invariant mean on $WAP(\mathbb{G})$ and let $p(x) = \langle \beta, S(x)\rangle$ ($x \in L_\infty(\mathbb{G})$). Then p is a left invariant bounded linear functional on $L_\infty(\mathbb{G})$, and $p \neq 0$ since $1 \in S(L_\infty(\mathbb{G}))$. Thus $L_\infty(\mathbb{G})^*$ has a non-zero left invariant element. Therefore, $L_\infty(\mathbb{G})$ has a left invariant mean (cf. [53]); that is, the quantum group $\mathbb{G}$ is amenable.

The final assertion holds, since $B_{L_1(\mathbb{G})}(L_\infty(\mathbb{G})) \cong LUC(\mathbb{G})^* \cong B_{L_1(\mathbb{G})}(LUC(\mathbb{G}))$ canonically when $\mathbb{G}$ is co-amenable (cf. [29, Proposition 6.5 and Remark 6.7]). □

Let $RM^{wc}(L_1(\mathbb{G}))$ be the Banach algebra of weakly compact right multipliers of $L_1(\mathbb{G})$ and let $B^{wc}(L_\infty(\mathbb{G}))$ be the Banach algebra of weakly compact maps in $B(L_\infty(\mathbb{G}))$. Then

$$RM^{wc}(L_1(\mathbb{G})) \;\cong\; B^\sigma_{L_1(\mathbb{G})}(L_\infty(\mathbb{G})) \cap B^{wc}(L_\infty(\mathbb{G})) \quad \text{via} \quad RM(L_1(\mathbb{G})) \hookrightarrow B(L_\infty(\mathbb{G})). \tag{4.1}$$

The theorem below generalizes and unifies [1, Theorem 4] on $L_1(G)$ and [37, Proposition 6.11] on $A(G)$.

Theorem 4.2. *Let $\mathbb{G}$ be a locally compact quantum group. Then*

$$\mathbb{G} \textit{ is compact} \quad \Longleftrightarrow \quad L_1(\mathbb{G}) \subseteq RM^{wc}(L_1(\mathbb{G})) \textit{ canonically.}$$

Furthermore, we have $L_1(\mathbb{G}) \cong RM^{wc}(L_1(\mathbb{G}))$ canonically if $\mathbb{G}$ is compact and co-amenable.

Proof. Let $f \longmapsto r_f$ be the canonical embedding $L_1(\mathbb{G}) \longrightarrow RM(L_1(\mathbb{G}))$ given by $r_f(g) = g \star f$. According to Theorem 3.1, $\mathbb{G}$ is compact if and only if $L_1(\mathbb{G}) \star L_1(\mathbb{G})^{**} \subseteq L_1(\mathbb{G})$, which is true if and only if the map $r_f : L_1(\mathbb{G}) \longrightarrow L_1(\mathbb{G})$ is weakly compact for all $f \in L_1(\mathbb{G})$ (cf. [50, Proposition 1.4.13]). Therefore, the equivalence holds.

Suppose now that $\mathbb{G}$ is compact and co-amenable. Let $\mu \in RM^{wc}(L_1(\mathbb{G}))$ and let (e_α) be a bounded approximate identity of $L_1(\mathbb{G})$ such that $\mu(e_\alpha) \longrightarrow f_0 \in L_1(\mathbb{G})$ weakly. Then $\mu(f) = \lim_\alpha \mu(f \star e_\alpha) = \lim_\alpha f \star \mu(e_\alpha) = f \star f_0 = r_{f_0}(f)$ for all $f \in L_1(\mathbb{G})$; that is, $\mu = r_{f_0} \in L_1(\mathbb{G})$. □

The converse of the second assertion in Theorem 4.2 holds in the two classical cases.

Corollary 4.3. *Let $\mathbb{G}$ be a commutative or co-commutative locally compact quantum group. Then*

$$\mathbb{G} \text{ is compact and co-amenable} \quad \Longleftrightarrow \quad L_1(\mathbb{G}) \cong RM^{wc}(L_1(\mathbb{G})) \text{ canonically.}$$

Proof. By Theorem 4.2, we need only show "$\Longleftarrow$", which is obvious if $\mathbb{G}$ is commutative. Suppose that $\mathbb{G}$ is co-commutative and $L_1(\mathbb{G}) \cong RM^{wc}(L_1(\mathbb{G}))$ canonically. Then the embedding $L_1(\mathbb{G}) \longrightarrow RM(L_1(\mathbb{G}))$, $f \longmapsto r_f$ is bounded from below. It follows from [45, Theorem 1] that $\mathbb{G}$ is co-amenable. □

Remark 4.4. It is interesting to know whether we have $RM^{wc}(L_1(\mathbb{G})) \neq \{0\} \Longleftrightarrow \mathbb{G}$ is compact. This is true for $L_1(G)$ and $A(G)$ (cf. [57, Theorem 1] and [37, Proposition 6.9]).

For $x \in L_\infty(\mathbb{G})$, let $x_\ell(f) = x \star f$ ($f \in L_1(\mathbb{G})$). Then $x_\ell \in B_{L_1(\mathbb{G})}(L_1(\mathbb{G}), L_\infty(\mathbb{G}))$, the space of bounded right $L_1(\mathbb{G})$-module maps from $L_1(\mathbb{G})$ to $L_\infty(\mathbb{G})$. In fact, we have $x_\ell = \Gamma(x)$ under the identification $CB(L_1(\mathbb{G}), L_\infty(\mathbb{G})) \cong L_\infty(\mathbb{G}) \bar{\otimes} L_\infty(\mathbb{G})$. Therefore, the map

$$L_\infty(\mathbb{G}) \longrightarrow CB_{L_1(\mathbb{G})}(L_1(\mathbb{G}), L_\infty(\mathbb{G})),\ x \longmapsto x_\ell \tag{4.2}$$

is completely isometric, and we obtain

$$WAP(\mathbb{G}) \ = \ \{x \in L_\infty(\mathbb{G}) : x_\ell \in CB^{wc}_{L_1(\mathbb{G})}(L_1(\mathbb{G}), L_\infty(\mathbb{G}))\}, \tag{4.3}$$

where $CB^{wc}_{L_1(\mathbb{G})}(L_1(\mathbb{G}), L_\infty(\mathbb{G}))$ is the space of weakly compact maps in $CB_{L_1(\mathbb{G})}(L_1(\mathbb{G}), L_\infty(\mathbb{G}))$. We show in Section 5 that the space $WAP(\mathbb{G})$ always contains a canonical copy of $C_u(\widehat{\mathbb{G}})^*$ (cf. Proposition 2.3 and the paragraph before it).

Remark 4.5. It is also interesting to compare Corollary 4.3 with the following characterization of Arens regularity (cf. (4.4)). Suppose that $\mathbb{G}$ is co-amenable. Then $L_\infty(\mathbb{G}) \cong CB_{L_1(\mathbb{G})}(L_1(\mathbb{G}), L_\infty(\mathbb{G})) = B_{L_1(\mathbb{G})}(L_1(\mathbb{G}), L_\infty(\mathbb{G}))$ and $WAP(\mathbb{G}) \cong CB^{wc}_{L_1(\mathbb{G})}(L_1(\mathbb{G}), L_\infty(\mathbb{G})) = B^{wc}_{L_1(\mathbb{G})}(L_1(\mathbb{G}), L_\infty(\mathbb{G}))$, since we have $T = T^*(E)_L$ for all $T \in B_{L_1(\mathbb{G})}(L_1(\mathbb{G}), L_\infty(\mathbb{G}))$, where E is a right identity of $(L_1(\mathbb{G})^{**}, \Box)$. Therefore, we have

$$L_1(\mathbb{G}) \text{ is Arens regular} \quad \Longleftrightarrow \quad B_{L_1(\mathbb{G})}(L_1(\mathbb{G}), L_\infty(\mathbb{G})) = B^{wc}_{L_1(\mathbb{G})}(L_1(\mathbb{G}), L_\infty(\mathbb{G})). \tag{4.4}$$

Note that when $\mathbb{G}$ is co-amenable, the space $B^{wc}_{L_1(\mathbb{G})}(L_1(\mathbb{G}), L_\infty(\mathbb{G}))$ consists precisely of all maps in $B_{L_1(\mathbb{G})}(L_1(\mathbb{G}), L_\infty(\mathbb{G}))$ factoring through reflexive Banach spaces, which is also equal to the space of all maps in $CB_{L_1(\mathbb{G})}(L_1(\mathbb{G}), L_\infty(\mathbb{G}))$ factoring through reflexive operator spaces. This fact can be derived by combining [8, Proposition 3.13] with [9, Corollary 1] and its cb-version [51, Theorem 2.1]. It is also seen from [10, Theorem 4.4] that these weakly compact right $L_1(\mathbb{G})$-module maps from $L_1(\mathbb{G})$ to $L_\infty(\mathbb{G})$ can factor through reflexive completely contractive $L_1(\mathbb{G})$-bimodules.

5. An Eberlein theorem over quantum groups

A theorem by Eberlein (cf. [13, Theorem 11.2] and [5, Corollary 3.3]) shows that if G is a locally compact group, then every positive definite function on G is weakly almost periodic. Therefore, we have

$$B(G) \subseteq WAP(G).$$

As shown by Dunkl and Ramirez [12, Theorem 2.8 and Chapter 8], the dual version of this classical Eberlein theorem holds; that is, the left regular representation of G defines a homomorphic embedding

$$M(G) \longrightarrow VN(G) \text{ with range contained in } WAP(\widehat{G}).$$

In the setting of locally compact quantum groups, these two results can be unified and stated as follows:

If $\mathbb{G}$ is a commutative or co-commutative locally compact quantum group, then the left regular representation $\hat{\lambda} : M(\widehat{\mathbb{G}}) \longrightarrow L_\infty(\mathbb{G})$ of $\widehat{\mathbb{G}}$ extends to an injective homomorphism

$$\hat{\lambda}_u : M_u(\widehat{\mathbb{G}}) \longrightarrow L_\infty(\mathbb{G})$$

with $\hat{\lambda}_u(M_u(\widehat{\mathbb{G}})) \subseteq WAP(\mathbb{G})$ (see below for the definition of the algebra $M_u(\widehat{\mathbb{G}})$).

We show in this section that the above assertion in fact holds for all locally compact quantum groups. In this way, by (4.2) and (4.3), we obtain canonically a completely isometric embedding

$$M_u(\widehat{\mathbb{G}}) \subseteq CB^{wc}_{L_1(\mathbb{G})}(L_1(\mathbb{G}), L_\infty(\mathbb{G})). \tag{5.1}$$

To be consistent with the notation used in [33], the C^*-algebra $C_0(\mathbb{G})$ will be denoted by A. Let

$$A_u = C_u(\mathbb{G})$$

be the *universal quantum group C^*-algebra* of $\mathbb{G}$, and let $\pi : A_u \longrightarrow A$ be the canonical surjective $*$-homomorphism, whose unique $*$-homomorphic extension $M(A_u) \longrightarrow M(A)$ is also denoted by π. For the dual quantum group $\widehat{\mathbb{G}}$ of $\mathbb{G}$, they are denoted by $\widehat{A}$, $\widehat{A}_u$, and $\hat{\pi}$, respectively. It is known from [33] that there exist unitaries $\mathcal{U} \in M(A_u \otimes \widehat{A}_u)$ and $\mathcal{V} \in M(A_u \otimes \widehat{A})$, and co-associative non-degenerate $*$-homomorphisms $\Delta_u : A_u \longrightarrow M(A_u \otimes A_u)$ and $\widehat{\Delta}_u : \widehat{A}_u \longrightarrow M(\widehat{A}_u \otimes \widehat{A}_u)$ such that

$$(\iota \otimes \hat{\pi})(\mathcal{U}) = \mathcal{V}, \quad (\Delta_u \otimes \iota)(\mathcal{U}) = \mathcal{U}_{13}\mathcal{U}_{23}, \quad \text{and} \quad (\iota \otimes \widehat{\Delta}_u)(\mathcal{U}) = \mathcal{U}_{13}\mathcal{U}_{12}. \tag{5.2}$$

Then $(A_u)^*$ is a Banach algebra with the multiplication $\star_u$ induced by Δ_u, and $\pi^* : A^* \longrightarrow (A_u)^*$ is an isometric algebra homomorphism. See [33] for more information on (A_u, Δ_u).

Using the unitary $\mathcal{U}$, we define the maps

$$\Phi_u : (A_u)^* \longrightarrow M(\widehat{A}_u),\ \mu \longmapsto (\mu \otimes \iota)(\mathcal{U}) \quad \text{and} \quad \Psi_u : (\widehat{A}_u)^* \longrightarrow M(A_u),\ \hat{\mu} \longmapsto (\iota \otimes \hat{\mu})(\mathcal{U}). \tag{5.3}$$

Let

$$\lambda_u = \hat{\pi} \circ \Phi_u : (A_u)^* \longrightarrow M(A).$$

Due to the first equality in (5.2), we see that the map λ_u is given by the formula

$$\lambda_u : (A_u)^* \longrightarrow M(\widehat{A}), \quad \mu \longmapsto (\mu \otimes \iota)(\mathcal{V}). \tag{5.4}$$

Proposition 5.1. *The following statements hold.*

(i) $\Phi_u : (A_u)^* \longrightarrow M(\widehat{A}_u)$ *and* $\Psi_u : (\widehat{A}_u)^* \longrightarrow M(A_u)$ *are homomorphisms satisfying*

$$\langle \Phi_u(\mu), \hat{\mu} \rangle = \langle \mu, \Psi_u(\hat{\mu}) \rangle \quad (\mu \in (A_u)^*,\ \hat{\mu} \in (\widehat{A}_u)^*).$$

(ii) $\lambda_u : (A_u)^* \longrightarrow M(\widehat{A})$ *is an injective homomorphism, and* λ_u *extends the left regular representation* $\lambda : A^* \longrightarrow M(\widehat{A})$ *of* $\mathbb{G}$ *in the sense that* $\lambda_u \circ \pi^* = \lambda$.

(iii) $(\Psi_u \circ (\hat{\pi})^*)(L_1(\widehat{\mathbb{G}})) \subseteq A_u$.

Proof. (i) The maps Φ_u and Ψ_u are homomorphisms due to (5.2), and the equality $\langle \Phi_u(\mu), \hat{\mu}\rangle = \langle \mu, \Psi_u(\hat{\mu})\rangle$ follows from the definition of Φ_u and Ψ_u.

(ii) Clearly, $\lambda_u = \hat{\pi} \circ \Phi_u : (A_u)^* \longrightarrow M(\widehat{A})$ is a homomorphism. Also, the map λ_u is injective by (5.4) and [33, (5.2)], which asserts that

$$A_u \;=\; \overline{\text{span}}^{\|\cdot\|}\{(\iota \otimes \hat{f})(\mathcal{V}) : \hat{f} \in L_1(\widehat{\mathbb{G}})\}. \tag{5.5}$$

Finally, we have $\lambda_u \circ \pi^* = \lambda$, since $(\pi \otimes \iota)(\mathcal{V}) = W$ (cf. [33, Proposition 5.1]).

(iii) Let $\hat{f} \in L_1(\widehat{\mathbb{G}})$. Then, by (5.5) and the first equality in (5.2), we have

$$\Psi_u((\hat{\pi})^*(\hat{f})) = (\iota \otimes (\hat{\pi})^*(\hat{f}))(\mathcal{U}) = (\iota \otimes \hat{f})((\iota \otimes \hat{\pi})(\mathcal{U})) = (\iota \otimes \hat{f})(\mathcal{V}) \in A_u.$$

Therefore, the inclusion $(\Psi_u \circ (\hat{\pi})^*)(L_1(\widehat{\mathbb{G}})) \subseteq A_u$ holds. □

Note that $L_1(\mathbb{G})$ is an ideal in A^*, and $\pi^*(A^*)$ is an ideal in $(A_u)^*$ (cf. [33, Proposition 8.3]). Since $L_1(\mathbb{G}) = \langle L_1(\mathbb{G}) \star L_1(\mathbb{G})\rangle$, it follows immediately that

$$\pi^*(L_1(\mathbb{G})) \text{ is a two-sided ideal in } (A_u)^*. \tag{5.6}$$

Note that $\lambda(L_1(\mathbb{G}))$ is norm dense in $\widehat{A}$. Together with (5.4) - (5.6), we can obtain some further properties of the map λ_u as given in the proposition below.

Proposition 5.2. *The following statements hold.*

(i) $\lambda_u : (A_u)^* \longrightarrow M(\widehat{A})$ *is the unique homomorphic extension of* $\lambda : A^* \longrightarrow M(\widehat{A})$.

(ii) $\lambda_u : (A_u)^* \longrightarrow L_\infty(\widehat{\mathbb{G}})$ *is* w^*-w^* *continuous.*

(iii) $\lambda_u : (A_u)^* \longrightarrow M(\widehat{A})$ *is strictly continuous on bounded subsets of* $(A_u)^*$. *That is, if* $\omega \in (A_u)^*$ *and* (ω_i) *is a bounded net in* $(A_u)^*$ *such that* $\omega_i \star_u \pi^*(f) \longrightarrow \omega \star_u \pi^*(f)$ *and* $\pi^*(f) \star_u \omega_i \longrightarrow \pi^*(f) \star_u \omega$ *for all* $f \in L_1(\mathbb{G})$, *then* $\lambda_u(\omega_i) \longrightarrow \lambda_u(\omega)$ *strictly in* $M(\widehat{A})$.

Remark 5.3. It follows from (5.6) that there exists a canonical algebra homomorphism

$$(A_u)^* \longrightarrow M_{cb}(L_1(\mathbb{G})). \tag{5.7}$$

On the other hand, as observed in [29, Section 3], using the representation theorem established in [32] and the relation between λ and ρ, we can obtain an algebra embedding

$$M_{cb}(L_1(\mathbb{G})) \longrightarrow M(\widehat{A}). \tag{5.8}$$

In fact, the M_{cb}^{ℓ}-version of [32, Corollary 4.4] shows that each $\mu \in LM_{cb}(L_1(\mathbb{G}))$ is determined uniquely by an element $\hat{b}$ of $L_\infty(\widehat{\mathbb{G}})$ satisfying $\lambda(\mu(f)) = \hat{b}\lambda(f)$ $(f \in L_1(\mathbb{G}))$, and hence we have

$$\hat{b}\, C_0(\widehat{\mathbb{G}}) \;\subseteq\; C_0(\widehat{\mathbb{G}}).$$

Through a Hilbert C^*-module approach, Daws proved recently in [11, Theorem 4.2] that $\hat{b} \in M(C_0(\widehat{\mathbb{G}}))$. It is seen from Propositions 5.1(ii) and 5.2(i) that $\lambda_u : (A_u)^* \longrightarrow M(\widehat{A})$ is exactly the composition of the two maps in (5.7) and (5.8), and thus the homomorphism in (5.7) is also injective.

Remark 5.4. Note that $L_1(\widehat{\mathbb{G}}) = C_0(\widehat{\mathbb{G}}) \cdot L_1(\widehat{\mathbb{G}})$, where $\cdot$ denotes the canonical $C_0(\widehat{\mathbb{G}})$-bimodule action on $L_1(\widehat{\mathbb{G}})$ (cf. [29, Proposition 2.1]). By (5.5) and Proposition 5.2(iii), we obtain that *if $\omega_i \longrightarrow \omega$ strictly on a bounded subset of $(A_u)^*$, then $\omega_i \longrightarrow \omega$ in the w^*-topology on $(A_u)^*$.*

It is interesting to know whether the converse holds on the unit sphere of $(A_u)^*$, which is the case when $\mathbb{G}$ is commutative or co-commutative (cf. [49] and [22]).

Due to Proposition 5.1(iii), we can define the contraction

$$\gamma_u : L_1(\widehat{\mathbb{G}}) \longrightarrow A_u\,,\ \ \hat{f} \longmapsto \Psi_u((\hat{\pi})^*(\hat{f}))\ =\ (\iota \otimes \hat{f})(\mathcal{V}). \tag{5.9}$$

Then the lemma below holds by Proposition 5.1(i) and the definition of the maps λ_u and γ_u.

Lemma 5.5. *For all $\mu \in (A_u)^*$ and $\hat{f} \in L_1(\widehat{\mathbb{G}})$, we have*

$$\lambda_u(\mu)\hat{\star}\hat{f}\ =\ \lambda_u(\mu \cdot \gamma_u(\hat{f})) \quad \text{and} \quad \hat{f}\hat{\star}\lambda_u(\mu)\ =\ \lambda_u(\gamma_u(\hat{f}) \cdot \mu), \tag{5.10}$$

where $\hat{\star}$ and $\cdot$ denote the canonical module actions of $L_1(\widehat{\mathbb{G}})$ on $L_\infty(\widehat{\mathbb{G}})$ and A_u on $(A_u)^*$, respectively.

Let

$$M_u(\mathbb{G})\ =\ ((A_u)^*, \star_u) \tag{5.11}$$

be the *universal quantum measure algebra* of $\mathbb{G}$. Following a similar argument as used in the proof of Proposition 2.3 (comparing (5.10) with (2.12)), we show below that λ_u maps $M_u(\mathbb{G})$ into $WAP(\widehat{\mathbb{G}})$. The following theorem unifies the corresponding results in [13, 12] for $L_\infty(G)$ and $VN(G)$, and our approach is new even for these two classical cases.

Theorem 5.6. *Let $\mathbb{G}$ be a locally compact quantum group and let $C_u(\mathbb{G})$ be the universal quantum group C^*-algebra of $\mathbb{G}$. Then the injective complete contraction*

$$\lambda_u : M_u(\mathbb{G}) \longrightarrow M(C_0(\widehat{\mathbb{G}})) \subseteq L_\infty(\widehat{\mathbb{G}}),\ \mu \longmapsto (\mu \otimes \iota)(\mathcal{V})$$

is the unique homomorphic extension of the left regular representation $\lambda : M(\mathbb{G}) \longrightarrow M(C_0(\widehat{\mathbb{G}}))$ of $\mathbb{G}$, and we have

$$\lambda_u(M_u(\mathbb{G}))\ \subseteq\ WAP(\widehat{\mathbb{G}}). \tag{5.12}$$

Proof. We only need to show the inclusion $\lambda_u(M_u(\mathbb{G})) \subseteq WAP(\widehat{\mathbb{G}})$. Let $\mu \in M_u(\mathbb{G})$. If $(\hat{f}_i)$ is a net in $L_1(\widehat{\mathbb{G}})$ such that $\gamma_u(\hat{f}_i) \longrightarrow m \in M_u(\mathbb{G})^*$ in the w^*-topology of $M_u(\mathbb{G})^*$, then $\mu \cdot \gamma_u(\hat{f}_i) \longrightarrow \mu \cdot m$ weakly in $M_u(\mathbb{G})$, since $M_u(\mathbb{G})^*$ is a von Neumann algebra and thus $\mu \cdot m \in M_u(\mathbb{G})^{**}$ is actually in $M_u(\mathbb{G})$. Note that the map γ_u is contractive. It follows that the set $\{\mu \cdot \gamma_u(\hat{f}) : \hat{f} \in L_1(\widehat{\mathbb{G}}) \text{ and } \|\hat{f}\| \leq 1\}$ is relatively weakly compact in $M_u(\mathbb{G})$. By Lemma 5.5, we obtain that the set $\{\lambda_u(\mu)\hat{\star}\hat{f} : \hat{f} \in L_1(\widehat{\mathbb{G}}) \text{ and } \|\hat{f}\| \leq 1\}$ is relatively weakly compact in $L_\infty(\widehat{\mathbb{G}})$. Therefore, $\lambda_u(\mu) \in WAP(\widehat{\mathbb{G}})$. □

Remark 5.7. Clearly, when $\mathbb{G}$ is co-amenable, the embedding $M_{cb}(L_1(\mathbb{G})) \longrightarrow M(C_0(\widehat{\mathbb{G}}))$ in (5.8) has a range in $WAP(\widehat{\mathbb{G}})$ (cf. Proposition 2.3). This is also the case when $\mathbb{G}$ is co-commutative (cf. [64]). It is interesting to know whether this is true for all locally compact quantum groups $\mathbb{G}$.

Replacing $\mathbb{G}$ in Theorem 5.6 by $\widehat{\mathbb{G}}$, we can define the *quantum Eberlein algebra* of $\mathbb{G}$ by

$$E(\mathbb{G})\ =\ \overline{\hat{\lambda}_u(M_u(\widehat{\mathbb{G}}))}^{\|\cdot\|}\ \subseteq\ WAP(\mathbb{G}). \tag{5.13}$$

See, for example, [6, 7, 48] for information on the classical Eberlein algebra $E(G)$ of a locally compact group G. The proposition below is clear by Theorem 5.6 and Proposition 2.1, noticing that it is still open whether the inclusion $WAP(\mathbb{G}) \subseteq M(C_0(\mathbb{G}))$ always holds (cf. Remark 3.5(ii)).

Proposition 5.8. *Let $\mathbb{G}$ be a locally compact quantum group. Then the quantum Eberlein algebra $E(\mathbb{G})$ of $\mathbb{G}$ is an $M(\mathbb{G})$-submodule of $L_\infty(\mathbb{G})$ and is two-sided introverted in $L_\infty(\mathbb{G})$ satisfying*

$$C_0(\mathbb{G}) \;\subseteq\; E(\mathbb{G}) \;\subseteq\; WAP(\mathbb{G}) \cap M(C_0(\mathbb{G})).$$

Therefore, the space $E(\mathbb{G})^*$ is a dual Banach algebra (with the identical Arens products), and each of Proposition 2.1, Corollary 2.5, and the statement in (2.13) holds for $X = E(\mathbb{G})$.

References

1. C. A. Akemann, *Some mapping properties of the group algebras of a compact group*, Pacific J. Math. **22** (1967), 1–8.
2. S. Baaj and G. Skandalis, *Unitaires multiplicatifs et dualité pour les produits croisés de C^*-algèbres*, Ann. Sci. École Norm. Sup. (4) **26** (1993), 425–488.
3. E. Bédos and L. Tuset, *Amenability and co-amenability for locally compact quantum groups*, Internat. J. Math. **14** (2003), 865–884.
4. M. B. Bekka, E. Kaniuth, A. T.-M. Lau, and G. Schlichting, *Weak*-closedness of subspaces of Fourier-Stieltjes algebras and weak*-continuity of the restriction map*, Trans. Amer. Math. Soc. **350** (1998), 2277–2296.
5. R. B. Burckel, *Weakly almost periodic functions on semigroups*, Gordon and Breach Science Publishers, New York, 1970.
6. C. Chou, *Uniform closures of Fourier-Stieltjes algebras*, Proc. Amer. Math. Soc. **77** (1979), 99–102.
7. C. Chou, *Weakly almost periodic functions and Fourier-Stieltjes algebras of locally compact groups*, Trans. Amer. Math. Soc. **274** (1982), 141–157.
8. H. G. Dales and A. T.-M. Lau, *The second duals of Beurling algebras*, Mem. Amer. Math. Soc. **177** (2005), no. 836.
9. W. J. Davis, T. Figiel, W. B. Johnson, and A. Pelczyński, *Factoring weakly compact operators*, J. Funct. Anal. **17** (1974), 311–327.
10. M. Daws, *Weakly almost periodic functionals, representations, and operator spaces*, preprint, 2007, http://front.math.ucdavis.edu/0701.935.
11. M. Daws, *Multipliers of locally compact quantum groups via Hilbert C^*-modules*, J. London Math. Soc. (to appear).
12. C. F. Dunkl and D. E. Ramirez, *Weakly almost periodic functionals on the Fourier algebra*, Trans. Amer. Math. Soc. **185** (1973), 501-514.
13. W. F. Eberlein, *Abstract ergodic theorems and weak almost periodic functions*, Trans. Amer. Math. Soc. **67** (1949), 217-240.
14. E. Effros and Z.-J. Ruan, *Discrete quantum groups I, the Haar measure*, Internat. J. Math. **5** (1994), 681–723.
15. M. Enock and J.-M. Schwartz, *Kac Algebras and Duality of Locally Compact Groups*, Springer-Verlag, New York, 1992.
16. M. Filali, M. Neufang, and M. Sangani Monfared, *Weak factorizations of operators in the group von Neumann algebras of certain amenable groups and applications*, Math. Ann. (to appear).
17. B. Forrest, *Arens regularity and discrete groups*, Pacific J. Math. **151** (1991), 217–227.
18. F. Ghahramani and A. T.-M. Lau, *Multipliers and modulus on Banach algebras related to locally compact groups*, J. Funct. Anal. **150** (1997), 478–497.
19. E. E. Granirer, *Exposed points of convex sets and weak sequential convergence*, Mem. Amer. Math. Soc. no. 123 (1972).
20. E. E. Granirer, *Weakly almost periodic and uniformly continuous functionals on the Fourier algebra of any locally compact group*, Trans. Amer. Math. Soc. **189** (1974), 371–382.
21. E. E. Granirer, *On group representations whose C^*-algebra is an ideal in its von Neumann algebra*, Ann. Inst. Fourier (Grenoble) **29** (1979), 37–52.
22. E. E. Granirer and M. Leinert, *On some topologies which coincide on the unit sphere of the Fourier-Stieltjes algebra $B(G)$ and of the measure algebra $M(G)$*, Rocky Mountain J. Math. **11** (1981), 459-472.
23. Z. Hu, *The von Neumann algebra $VN(G)$ of a locally compact group and quotients of its subspaces*, Canad. J. Math. **49** (1997), 1117–1138.
24. Z. Hu, *Open subgroups and centre problem for the Fourier algebra*, Proc. Amer. Math. Soc. **134** (2006), 3085–3095.
25. Z. Hu and M. Neufang, *Decomposability of von Neumann algebras and the Mazur property of higher level*, Canad. J. Math. **59** (2006), 768–795.
26. Z. Hu, M. Neufang, and Z.-J. Ruan, *On topological centre problems and SIN quantum groups*, J. Funct. Anal. **257** (2009), 610–640.
27. Z. Hu, M. Neufang, and Z.-J. Ruan, *Multipliers on a new class of Banach algebras, locally compact quantum groups, and topological centres*, Proc. London Math. Soc. **100** (2010), 429–458.
28. Z. Hu, M. Neufang, and Z.-J. Ruan, *Module maps on duals of Banach algebras and topological centre problems*, J. Funct. Anal. **260** (2011), 1188–1218.
29. Z. Hu, M. Neufang, and Z.-J. Ruan, *Completely bounded multipliers over locally compact quantum groups*, Proc. London Math. Soc. **103** (2011), 1–39.
30. Z. Hu, M. Neufang, and Z.-J. Ruan, *Convolutions of trace class operators and a Stone-von Neumann theorem over locally compact quantum groups*, preprint, 2011.
31. M. Ilie and R. Stokke, *Extension of Fourier algebra homomorphisms to duals of algebras of uniformly continuous functionals*, J. Funct. Anal. **256** (2009), 2518–2541.
32. M. Junge, M. Neufang, and Z.-J. Ruan, *A representation theorem for locally compact quantum groups*, Internat. J. Math. **20** (2009), 377-400.
33. J. Kustermans, *Locally compact quantum groups in the universal setting*, International J. Math. **12** (2001), 289–338.
34. J. Kustermans and S. Vaes, *Locally compact quantum groups*, Ann. Sci. Éole Norm. Sup. **33** (2000), 837–934.

35. J. Kustermans and S. Vaes, *Locally compact quantum groups in the von Neumann algebraic setting*, Math. Scand. **92** (2003), 68–92.
36. A. T.-M. Lau, *Operators which commute with convolutions on subspaces of $L_\infty(G)$*, Colloq. Math. **39** (1978), 351–359.
37. A. T.-M. Lau, *Uniformly continuous functionals on the Fourier algebra of any locally compact group*, Trans. Amer. Math. Soc. **251** (1979), 39–59.
38. A. T.-M. Lau, *The second conjugate algebra of the Fourier algebra of a locally compact group*, Trans. Amer. Math. Soc. **267** (1981), 53–63
39. A. T.-M. Lau, *Analysis on a class of Banach algebras with applications to harmonic analysis on locally compact groups and semigroups*, Fund. Math. **118** (1983), 161–175.
40. A. T.-M. Lau, *Continuity of Arens multiplication on the dual space of bounded uniformly continuous functions on locally compact groups and topological semigroups*, Math. Proc. Cambridge Philos. Soc. **99** (1986), 273–283.
41. A. T.-M. Lau and V. Losert, *On the second conjugate algebra of $L_1(G)$ of a locally compact group*, J. London Math. Soc. (2) **37** (1988), 464–470.
42. A. T.-M. Lau and V. Losert, *The C^*-algebra generated by operators with compact support on a locally compact group*, J. Funct. Anal. **112** (1993), 1–30.
43. A. T.-M. Lau and V. Losert, *The centre of the second conjugate algebra of the Fourier algebra for infinite products of groups*, Math. Proc. Cambridge Philos. Soc. **138** (2005), 27–39.
44. V. Losert, *Some properties of groups without the property P_1*, Comment. Math. Helv. **54** (1979), 133–139.
45. V. Losert, *Properties of the Fourier algebra that are equivalent to amenability*, Proc. Amer. Math. Soc. **92** (1984), 347–354.
46. V. Losert, *The centre of the bidual of Fourier algebras (discrete groups)*, preprint, 2002.
47. V. Losert, *On the centre of the bidual of Fourier algebras (the compact case)*, presentation at the 2004 Istanbul International Conference on Abstract Harmonic Analysis, 2004.
48. M. Mayer, *Asymptotics of matrix coefficients and closures of Fourier-Stieltjes algebras*, J. Funct. Anal. **143** (1997), 42–54.
49. K. McKennon, *Multipliers, positive functionals, positive-definite functions, and Fourier-Stieltjes transforms*, Mem. Amer. Math. Soc. (1971), no. 111.
50. T. W. Palmer, *Banach Algebras and General Theory of $*$-Algebras, Volume 1*, Cambridge University Press, Cambridge, 1994.
51. H. Pfitzner and G. Schlüechtermann, *Factorization of completely bounded weakly compact operators*, preprint, 1997, http://front.math.ucdavis.edu/9706.210.
52. M. Ramezanpour and H. R. E. Vishki, *Module homomorphisms and multipliers on locally compact quantum groups*, J. Math. Anal. Appl. **359** (2009), 581–587.
53. Z.-J. Ruan, *Amenability of Hopf von Neumann algebras and Kac algebras*, J. Funct. Anal. **139** (1996), 466–499.
54. V. Runde, *Amenability for dual Banach algebras*, Studia Math. **148** (2002), 47–66.
55. V. Runde, *Characterizations of compact and discrete quantum groups through second duals*, J. Operator Theory **60** (2008), 415–428.
56. V. Runde, *Uniform continuity over locally compact quantum groups*, J. London Math. Soc. **80** (2009), 55–71.
57. S. Sakai, *Weakly compact operators on operator algebras*, Pacific J. Math. **14** (1964), 659–664.
58. P. Salmi, *Quantum semigroup compactifications and left uniformly continuous functionals*, Illinois J. Math. (to appear).
59. R. Tomatsu, *A characterization of right coideals of quotient type and its application to classification of Poisson boundaries*, Comm. Math. Phys. **275** (2007), 271–296.
60. A. Ülger, *Arens regularity of weakly sequentially complete Banach algebras*, Proc. Amer. Math. Soc. **127** (1999), 3221–3227.
61. A. Ülger, *Central elements of A^{**} for certain Banach algebras A without bounded approximate identities*, Glasg. Math. J. **41** (1999), 369–377.
62. S. Vaes, *A new approach to induction and imprimitivity results*, J. Funct. Anal. **229** (2005), 317–374.
63. A. van Daele, *Locally compact quantum groups. A von Neumann algebra approach*, preprint, 2006, http://front.math.ucdavis.edu/0602.5212.
64. G. Xu, *Herz-Schur multipliers and weakly almost periodic functions on locally compact groups*, Trans. Amer. Math. Soc. **349** (1997), 2525–2536.
65. N. J. Young, *The irregularity of multiplication in group algebras*, Quart J. Math. Oxford Ser. **24** (1973), 59–62.

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