A HILBERT MODULE APPROACH TO THE HAAGERUP PROPERTY

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Abstract. We develop a Hilbert module version of Haagerup property for general C*-algebras $\mathcal{A} \subseteq \mathcal{B}$. We show that if $\alpha : \Gamma \curvearrowright \mathcal{A}$ is an action of a countable discrete group Γ acting on a unital C*-algebra $\mathcal{A}$, then the reduced C*-algebra crossed product $\Gamma \ltimes_{\alpha,r} \mathcal{A}$ has the Hilbert $\mathcal{A}$-module Haagerup property if and only if the action α has the Haagerup property. We are particularly interested in the case when $\mathcal{A} = C(X)$ is a unital commutative C*-algebra. We compare the Haagerup property of such an action $\alpha : \Gamma \curvearrowright C(X)$ with the two special cases when (1) Γ has the Haagerup property and (2) Γ is coarsely embeddable into a Hilbert space. We also prove a contractive Schur mutiplier characterization for groups coarsely embeddable into a Hilbert space, and a uniformly bounded Schur multiplier characterization for exact groups.

1. Introduction

Let Γ be a countable discrete group. We say that Γ has the Haagerup property if there exists a sequence of positive definite functions $\{\phi_n\}$ in $C_0(\Gamma, \mathbb{C})$ such that $\{\phi_n\}$ converges to the constant function 1 pointwisely on Γ. This definition is motivated by the work of Haagerup [10] where he proved that free groups have such a property. The Haagerup property has been intensively studied in the literature. It forms a very interesting class of groups, including amenable groups, Coxeter groups, and groups acting properly on trees or on spaces with walls. It is also known that the Haagerup property is equivalent to several other important properties, such as the a-T-menability introduced by Gromov [8]. On the other hand, the Haagerup property is a strong negation of property (T), or relative property (T). Therefore, $SL(3,\mathbb{Z})$, $Sp(1,n)$, and $SL(2,\mathbb{Z}) \ltimes \mathbb{Z}^2$ do not have the Haagerup property. The readers are referred to the recent books [2] and [3] for a comprehensive account on this subject.

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Suppose that $\mathcal{M}$ is a finite von Neumann algebra equipped with a normal faithful tracial state τ. We say that $\mathcal{M}$ has the Haagerup property if there exists a sequence of normal completely positive maps $\{\Phi_n\}$ on $\mathcal{M}$ such that (0) $\tau \circ \Phi_n \leq \tau$, (1) each Φ_n defines a compact operator $\tilde{\Phi}_n$ on $L_2(\mathcal{M}, \tau)$, and (2) $\| \tilde{\Phi}_n(x) - x \|_\tau \to 0$ for all $x \in \mathcal{M}$. We note that condition (0) in the definition guarantees that each Φ_n defines a bounded linear map $\tilde{\Phi}_n$ on the Hilbert space $L_2(\mathcal{M}, \tau)$ and the definition is actually independent from the choice of the tracial state on $\mathcal{M}$ (see Jolissaint [15]). If Γ is a countable discrete group, it was shown by Choda [4] that Γ has the Haagerup property if and only if its group von Neumann algebra $L(\Gamma)$ has the von Neumann algebra Haagerup property.

The Haagerup property for untial C*-algebras admitting a faithful tracial state has been studied recently by the first author [6]. Applying a similar technique used in Choda [4], he showed that a countable discrete group Γ has the Haagerup property if and only if the reduced group C*-algebra $C^*_\lambda(\Gamma)$ has the C*-algebra Haagerup property. The goal of this paper is to develop a Hilbert module version of Haagerup property for general C*-algebras and study the corresponding Haagerup property for actions of groups on unital C*-algebras.

Let $\mathcal{A} \subseteq \mathcal{B}$ be unital C*-algebras and let $\mathcal{E} : \mathcal{B} \to \mathcal{A}$ be a (not necessarily tracial) faithful conditional expectation. Then $\mathcal{E}$ induces a right Hilbert $\mathcal{A}$-module structure on $\mathcal{B}$ and we can define the Hilbert $\mathcal{A}$-module Haagerup property for $\mathcal{B}$ with respect to $\mathcal{E}$. We are interested in the case when $\mathcal{B} = \Gamma \ltimes_{\alpha,r} \mathcal{A}$ is the reduced C*-algebra crossed product of an action $\alpha : \Gamma \curvearrowright \mathcal{A}$ on a unital C*-algebra $\mathcal{A}$. In this case, we can regard $\mathcal{A}$ as a C*-subalgebra of $\Gamma \ltimes_{\alpha,r} \mathcal{A}$ and there exists a canonical faithful conditional expectation $\mathcal{E}$ from $\Gamma \ltimes_{\alpha,r} \mathcal{A}$ onto $\mathcal{A}$. We recall these definitions in Section 2. Section 3 is devoted to the study of bounded $\mathcal{A}$-bimodule multiplier maps Φ on $\Gamma \ltimes_{\alpha,r} \mathcal{A}$ and the associated multipliers $h : \Gamma \to \mathcal{A}$ which satisfy $\Phi(\lambda_s) = \lambda_s h_s$ for all $s \in \Gamma$. We show in Theorem 3.6 that an action $\alpha : \Gamma \curvearrowright \mathcal{A}$ has the Haagerup property if and only if the C*-algebra crossed product $\Gamma \ltimes_{\alpha,r} \mathcal{A}$ has the Hilbert $\mathcal{A}$-module Haagerup property.

It is important to note that if $h : \Gamma \to \mathcal{A}$ is a bounded multiplier of an action $\alpha : \Gamma \curvearrowright \mathcal{A}$, then the range of h must be contained in the center $\mathcal{Z}(\mathcal{A})$ of $\mathcal{A}$. Since the restriction of α_s to $\mathcal{Z}(\mathcal{A})$ defines a *-automorphism on $\mathcal{Z}(\mathcal{A})$ for each $s \in \Gamma$, this induces an action $\alpha^{\mathcal{Z}(\mathcal{A})}$ on the center $\mathcal{Z}(\mathcal{A})$. Therefore, the action $\alpha : \Gamma \curvearrowright \mathcal{A}$ has the Haagerup property if and only if the induced action $\alpha^{\mathcal{Z}(\mathcal{A})} : \Gamma \curvearrowright \mathcal{Z}(\mathcal{A})$ has the Haagerup property.

We focus on the actions of countable discrete groups on unital commutative C*-algebras $\mathcal{A} = C(X)$ in Section 4. In this case, each action $\alpha : \Gamma \curvearrowright C(X)$ is uniquely associated with an action $\alpha : \Gamma \curvearrowright X$ on the corresponding compact Hausdorff space X. It turns out that the action $\alpha : \Gamma \curvearrowright C(X)$ has the Haagerup property if and only if the left transformation groupoid $\Gamma \ltimes X$ associated with the action $\alpha : \Gamma \curvearrowright X$ has the Haagerup property (see Proposition 4.1 and the proof of [9, Theorem 4.1]). We investigate the trivial action α_∞ of Γ on $C(\{\infty\}) = \mathbb{C}$, and the induced left translation actions α^∞ and $\tilde{\alpha}$ of Γ on $C(\Gamma \cup \{\infty\}) = c(\Gamma)$ and $C(\beta\Gamma) = \ell_\infty(\Gamma)$, respectively. As we have seen from [6] that the trivial action $\alpha_\infty : \Gamma \curvearrowright C(\{\infty\})$ has the Haagerup property if and only if the group Γ has the Haagerup property. We show in Theorem 4.5 that this is equivalent to the action $\alpha^\infty : \Gamma \curvearrowright C(\Gamma \cup \{\infty\})$ having the Haagerup property. On the other hand, we see in Theorem 4.2 that the action $\tilde{\alpha} : \Gamma \curvearrowright C(\beta\Gamma)$ has the Haagerup property if and only if the group Γ is coarsely embeddable into a Hilbert space in the sense of Gromov [8].

In Section 5, we consider $\mathcal{A} = \ell_\infty(\Gamma)$ and identify $\Gamma \ltimes_{\alpha,r} \ell_\infty(\Gamma)$ with the uniform Roe algebra $C_u^*(\Gamma)$, which is the unital C*-subalgebra of $B(\ell_2(\Gamma))$ spanned by all finite sums $\sum_s \lambda_s f_s$ with $f_s \in \ell_\infty(\Gamma)$. We first show in Proposition 5.1 and Proposition 5.2 that there is a one-to-one correspondence between (completely) bounded $\ell_\infty(\Gamma)$-bimodule maps and completely bounded and normal $\ell_\infty(\Gamma)$-bimodule multiplier maps on $C_u^*(\Gamma)$. We then prove a contractive Schur multiplier characterization for coarse embedding property in Theorem 5.3. We end the paper by discussing a uniformly bounded Schur multiplier characterization for exact groups in Section 6.

Finally, we remark that it is natural to ask whether we can develop a von Neumann algebra version of Hilbert module Haagerup property. In this case, we would like to consider a normal von Neumann algebra dynamical system $(\mathcal{M}, \Gamma, \alpha)$ and consider the von Neumann algebra crossed product $\mathcal{M} \rtimes_\alpha^\sigma \Gamma$. If Φ is a normal bounded $\mathcal{M}$-bimodule multiplier map on $\mathcal{M} \rtimes_\alpha^\sigma \Gamma$ and $h : \Gamma \to \mathcal{M}$ is the corresponding $\mathcal{M}$-valued multiplier such that $\Phi(\lambda_s) = \lambda_s h_s$ for all $s \in \Gamma$. We can conclude from Proposition 3.3 that Φ induces a bounded $\mathcal{M}$-module map $\tilde{\Phi}$ on the right (self-dual) Hilbert $\mathcal{M}$-module $M_{\Gamma,1}(\mathcal{M})$, which is the weak* closure of $M_{\Gamma,1} \otimes^{min} \mathcal{M}$. We can see from the proof of Proposition 3.4 that if $h \in C_0(\Gamma, \mathcal{M})$, then the induced map $\tilde{\Phi}$ is compact on $M_{\Gamma,1}(\mathcal{M})$. However, the converse is not necessarily true for general von Neumann algebra. The difficulty is that in this case, we may not be able to approximate each compact map $\tilde{\Phi}$ on $M_{\Gamma,1}(\mathcal{M})$ by finite-rank maps having the form $T = \sum\limits_{s,t \in F} \theta_{\lambda_s a_s, \lambda_t b_t}$. For instance, this may fail for some rank-one map on the right Hilbert $\ell_\infty(\mathbb{Z})$-module $M_{\mathbb{Z},1}(\ell_\infty(\mathbb{Z}))$. Therefore, we can not obtain the corresponding equivalent statements in Proposition 3.4, Proposition 3.5 and Theorem 3.6 in the von Neumann algebra setting, unless the von Neumann algebra is finite dimensional.

2. HILBERT $\mathcal{A}$-MODULE AND C*-ALGEBRA CROSSED PRODUCT

Suppose that $\mathcal{A}$ is a unital C*-subalgebra of a unital C*-algebra $\mathcal{B}$ and suppose that $\mathcal{E}$ is a faithful conditional expectation from $\mathcal{B}$ onto $\mathcal{A}$. Then $\mathcal{E}$ is a unital completely positive $\mathcal{A}$-bimodule map on $\mathcal{B}$. We can define an $\mathcal{A}$-valued inner product

$$\langle x|y\rangle_{\mathcal{E}} = \mathcal{E}(x^*y) \tag{2.1}$$

and a new norm $\|x\|_{\mathcal{E}} = \|\langle x|x\rangle_{\mathcal{E}}\|^{\frac{1}{2}}$ on $\mathcal{B}$. We adopt the physics notation in (2.1) by considering the conjugate linearity in the 1st component and the linearity in the 2nd. This induces a right pre-Hilbert $\mathcal{A}$-module structure on $\mathcal{B}$ (see Lance [20]). We let $\mathcal{H}_{\mathcal{A}}$ denote the Hilbert $\mathcal{A}$-module completion of $\mathcal{B}$.

If Φ is a bounded $\mathcal{A}$-bimodule map on $\mathcal{B}$ such that

$$\mathcal{E}(\Phi(x)^*\Phi(x)) \le k\mathcal{E}(x^*x) \tag{2.2}$$

for some $k > 0$, then Φ determines a bounded right Hilbert $\mathcal{A}$-module map $\tilde{\Phi}$ on $\mathcal{B}$ (and thus on $\mathcal{H}_{\mathcal{A}}$) since

$$\|\tilde{\Phi}(x)\|_{\mathcal{E}} = \|\mathcal{E}(\Phi(x)^*\Phi(x))\|^{\frac{1}{2}} \le k^{\frac{1}{2}}\|\mathcal{E}(x^*x)\|^{\frac{1}{2}} = k^{\frac{1}{2}}\|x\|_{\mathcal{E}}.$$

If Φ is completely positive on $\mathcal{B}$, then we only need to consider the condition

$$\mathcal{E}\circ\Phi \le k\,\mathcal{E} \tag{2.3}$$

for some $k > 0$ since in this case, we can conclude from the Schwarz inequality

$$0 \le \Phi(x)^*\Phi(x) \le \|\Phi\|\Phi(x^*x)$$

that

$$\mathcal{E}(\Phi(x)^*\Phi(x)) \le \|\Phi\|\mathcal{E}(\Phi(x^*x)) \le k\|\Phi\|\mathcal{E}(x^*x) \tag{2.4}$$

for all $x \in \mathcal{B}$.

Let ξ and $\eta \in \mathcal{H}_{\mathcal{A}}$. We can define a *rank-one* map $\theta_{\xi,\eta}$ on $\mathcal{H}_{\mathcal{A}}$ by letting

$$\theta_{\xi,\eta}(x) = \xi\langle\eta|x\rangle_{\mathcal{E}}.$$

A *finite-rank* map on $\mathcal{H}_{\mathcal{A}}$ has the form $T = \sum\limits_{i=1}^{n}\theta_{\xi_i,\eta_i}$ for some $\xi_i, \eta_i \in \mathcal{H}_{\mathcal{A}}$. It is clear that these are bounded $\mathcal{A}$-module maps. We say that a bounded $\mathcal{A}$-module map T on $\mathcal{H}_{\mathcal{A}}$ is *compact* if it is the norm limit of a sequence of finite-rank maps on $\mathcal{H}_{\mathcal{A}}$.

We say that a C*-algebra $\mathcal{B}$ has the *Hilbert $\mathcal{A}$-module Haagerup property* with respect to $\mathcal{E}$ if there exists a sequence of completely positive $\mathcal{A}$-bimodule maps $\{\Phi_n\}$ on $\mathcal{B}$ such that (0) $\mathcal{E}\circ\Phi_n \le \mathcal{E}$, (1) each Φ_n defines a compact $\mathcal{A}$-module map $\tilde{\Phi}_n$ on $\mathcal{H}_{\mathcal{A}}$, and (2) $\|\ \tilde{\Phi}_n(x) - x\ \|_{\mathcal{E}} \to 0$ for all $x \in \mathcal{B}$ (and thus for all $x \in \mathcal{H}_{\mathcal{A}}$). If τ is a faithful state on $\mathcal{B}$, we can obtain a faithful conditional expectation $\mathcal{E}_\tau = \tau 1$ from $\mathcal{B}$ onto the unital

C*-subalgebra $\mathcal{A} = \mathbb{C}1$. Then the C*-algebra $\mathcal{B}$ has the *Haagerup property* with respect to τ if and only if $\mathcal{B}$ has the Hilbert $\mathbb{C}1$-module Haagerup property with respect to $\mathcal{E}_\tau$.

Suppose that Γ is a countable discrete group and we let $\mathcal{A} \subseteq \mathcal{B}(H)$ be a unital C*-algebra on a Hilbert space H. If $\alpha : \Gamma \curvearrowright \mathcal{A}$ is an action of Γ on $\mathcal{A}$, i.e. if we have a group homomorphism α from Γ into the group $Aut(\mathcal{A})$ of *-automorphisms on $\mathcal{A}$, then we can obtain a new representation of $\mathcal{A}$ on $\ell_2(\Gamma) \otimes H$ given by

$$\pi(a)(\delta_s \otimes \xi) = \delta_s \otimes \alpha_{s^{-1}}(a)(\xi)$$

for all $\xi \in H$ and $\delta_s \in \ell_2(\Gamma)$. Let $\lambda : \Gamma \to \mathcal{B}(\ell_2(\Gamma))$ be the *left regular representation* of Γ on $\ell_2(\Gamma)$ defined by

$$(\lambda_s \xi)(t) = \xi(s^{-1}t).$$

We can obtain a unitary representation $\tilde{\lambda}_s = \lambda_s \otimes 1$ of Γ on $\ell_2(\Gamma) \otimes H$. This gives us a *covariant representation* $(\pi, \tilde{\lambda})$ of the C*-dynamical system $(\mathcal{A}, \Gamma, \alpha)$ on $\ell_2(\Gamma) \otimes H$ since it satisfies

$$\pi(\alpha_s(a)) = \tilde{\lambda}_s \pi(a) \tilde{\lambda}_s^* \tag{2.5}$$

for all $a \in \mathcal{A}$ and $s \in \Gamma$. Let $C_c(\Gamma, \mathcal{A})$ denote the space of functions $f : \Gamma \to \mathcal{A}$ with compact (and thus finite) support on Γ. Then we get a unital *-subalgebra

$$\tilde{\lambda} \times \pi(C_c(\Gamma, \mathcal{A})) = \{\sum_{s \in \Gamma} \tilde{\lambda}_s \pi(a_s) : s \in \Gamma \text{ with finitely many non-zero } \ a_s \in \mathcal{A}\}$$

in $\mathcal{B}(\ell_2(\Gamma) \otimes H)$. If there is no confusion, we identify $a \in \mathcal{A}$ with $\pi(a) \in \mathcal{B}(\ell_2(\Gamma) \otimes H)$ and write λ_s for $\tilde{\lambda}_s$. In this case, we simply use $C_c(\Gamma, \mathcal{A})$ for $\tilde{\lambda} \times \pi(C_c(\Gamma, \mathcal{A}))$, and we can write elements in $C_c(\Gamma, \mathcal{A})$ as $x = \sum\limits_{s \in \Gamma} \lambda_s a_s$. The reduced C*-algebra crossed product $\Gamma \rtimes_{\alpha,r} \mathcal{A}$ is defined to be the norm closure of $C_c(\Gamma, \mathcal{A})$ in $\mathcal{B}(\ell_2(\Gamma) \otimes H)$.

It is known (see Brown and Ozawa's book [2, Proposition 4.1.9]) that there exists a faithful conditional expectation $\mathcal{E}$ from $\Gamma \ltimes_{\alpha,r} \mathcal{A}$ onto $\mathcal{A}$, which is defined by

$$\mathcal{E}(\sum_{s \in \Gamma} \lambda_s a_s) = a_e. \tag{2.6}$$

This conditinal expectation $\mathcal{E}$ satisfies the following equivariant condition

$$\alpha_s(\mathcal{E}(x)) = \mathcal{E}(\lambda_s x \lambda_{s^{-1}}) \tag{2.7}$$

for all $s \in \Gamma$ and $x \in \Gamma \ltimes_{\alpha,r} \mathcal{A}$ (see [2, Remark 4.1.10]). In general, $\mathcal{E}$ is not tracial unless $\mathcal{A}$ is commutative and the action is trivial.

As we have seen in (2.1) that we can use $\mathcal{E}$ to define a right pre-Hilbert $\mathcal{A}$-module structure on the crossed product $\Gamma \ltimes_{\alpha,r} \mathcal{A}$ and the $\mathcal{A}$-valued inner product is given by

$$\langle x | y \rangle_{\mathcal{E}} = \mathcal{E}(\sum_{s,t \in \Gamma} a_s^* \lambda_s^* \lambda_t b_t) = \sum_{s \in \Gamma} a_s^* b_s. \tag{2.8}$$

for $x = \sum\limits_{s\in\Gamma} \lambda_s a_s$ and $y = \sum\limits_{t\in\Gamma} \lambda_t b_t$ in $C_c(\Gamma, \mathcal{A})$. With this expression, we can identify the right Hilbert $\mathcal{A}$-module $\mathcal{H}_\mathcal{A}$ with the spatial tensor product $M_{\Gamma,1}(\mathbb{C}) \otimes^{min} \mathcal{A}$, where we let $M_{\Gamma,1}(\mathbb{C})$ denote the column Hilbert space $\ell_2(\Gamma)$. We can also identify $\{\lambda_s\}_{s\in\Gamma}$, which forms an orthonormal basis for $\mathcal{H}_\mathcal{A}$, with the canonical orthonormal basis $\{e_s \otimes 1\}_{s\in\Gamma}$ in $M_{\Gamma,1}(\mathbb{C}) \otimes^{min} \mathcal{A}$.

3. MULTIPLIERS AND HAAGERUP PROPERTY FOR ACTIONS $\alpha : \Gamma \curvearrowright \mathcal{A}$

Let Γ be a countable discrete group and $\alpha : \Gamma \curvearrowright \mathcal{A}$ be an action of Γ on a unital C*-algebra $\mathcal{A}$. A map $h : \Gamma \to \mathcal{A}$ is a *bounded multiplier* with respect to the action α if there exists a bounded $\mathcal{A}$-bimodule map Φ on the crossed product $\Gamma \times_{\alpha,r} \mathcal{A}$ such that $\Phi(\lambda_s) = \lambda_s h_s$ for all $s \in \Gamma$. We call Φ the *bounded $\mathcal{A}$-bimodule multiplier map* associated with h. In this case, $h : \Gamma \to \mathcal{A}$ is a bounded map with

$$\|h_s\| = \|\Phi(\lambda_s)\| \leq \|\Phi\|.$$

We say that a multiplier $h : \Gamma \to \mathcal{A}$ is *completely bounded* (respectively, *completely positive*) if the associated $\mathcal{A}$-bimodule multiplier map Φ is completey bounded (respectively, completely positive) on $\Gamma \times_{\alpha,r} \mathcal{A}$.

It is important to note that if $h : \Gamma \to \mathcal{A}$ is a bounded multiplier with respect to the action $\alpha : \Gamma \curvearrowright \mathcal{A}$, then we have

$$\begin{aligned} \alpha_{s^{-1}}(a)h_s &= \lambda_{s^{-1}} a \lambda_s h_s = \lambda_{s^{-1}} \Phi(a\lambda_s) = \lambda_{s^{-1}} \Phi(\lambda_s \alpha_{s^{-1}}(a)) \\ &= \lambda_{s^{-1}} \Phi(\lambda_s)\alpha_{s^{-1}}(a) = \lambda_{s^{-1}}(\lambda_s h_s)\alpha_{s^{-1}}(a) = h_s \alpha_{s^{-1}}(a) \end{aligned}$$

for all $a \in \mathcal{A}$ and $s \in G$. Since $\alpha_{s^{-1}}$ is a *-automorphism on $\mathcal{A}$, we have $\alpha_{s^{-1}}(\mathcal{A}) = \mathcal{A}$ and thus h_s must be contained in the center $\mathcal{Z}(\mathcal{A})$ of $\mathcal{A}$.

Now if $h : \Gamma \to \mathcal{Z}(\mathcal{A}) \subseteq \mathcal{A}$ is a completely positive multiplier with respect to the action $\alpha : \Gamma \curvearrowright \mathcal{A}$, there exists a completely positive $\mathcal{A}$-bimodule multiplier map Φ on $\Gamma \times_{\alpha,r} \mathcal{A}$ such that $\Phi(\lambda_s) = \lambda_s h_s$ for all $s \in \Gamma$. In this case, we have

$$[\lambda_{s_i^{-1}s_j} h_{s_i^{-1}s_j}] = [\Phi(\lambda_{s_i^{-1}s_j})] \geq 0 \text{ in } M_n(\Gamma \ltimes_{\alpha,r} \mathcal{A})$$

for all $s_1, \cdots, s_n \in \Gamma$. If we let U denote the diagonal matrix with λ_{s_i} at the (i,i)-position, then it is easy to see that

$$[\alpha_{s_j}(h_{s_i^{-1}s_j})] = [\lambda_{s_j} h_{s_i^{-1}s_j} \lambda_{s_j}^*] = U[\lambda_{s_i^{-1}s_j} h_{s_i^{-1}s_j}]U^* \geq 0 \text{ in } M_n(\mathcal{A}). \tag{3.1}$$

Motivated by (3.1), we say that a map $h : \Gamma \to \mathcal{Z}(\mathcal{A}) \subseteq \mathcal{A}$ is *positive definite* with respect to the action $\alpha : \Gamma \curvearrowright \mathcal{A}$ if for any $s_1, \cdots, s_n \in \Gamma$, we have

$$[\alpha_{s_j}(h_{s_i^{-1}s_j})] \geq 0 \text{ in } M_n(\mathcal{A}). \tag{3.2}$$

We have seen from the above discussion that if $h : \Gamma \to \mathcal{A}$ is a completely positive multiplier, then h is a positive definite map satisfying (3.2). We show in Theorem 3.2 that the converse is also true, i.e. every positive definite map $h : \Gamma \to \mathcal{A}$ with respect to the action $\alpha : \Gamma \curvearrowright \mathcal{A}$ must be a completely positive multiplier and the natural map

$$\Phi(\sum_{s\in\Gamma} \lambda_s a_s) = \sum_{s\in\Gamma} \lambda_s h_s a_s \tag{3.3}$$

on $C_c(\Gamma, \mathcal{A})$ extends to a completely positive $\mathcal{A}$-bimodule multiplier map on $\Gamma \times_{\alpha,r} \mathcal{A}$. For this purpose, we first need to recall the following lemma about $\mathcal{Z}(\mathcal{A})$-valued Schur product map. We include a brief proof for the convenience of readers. The readers are referred to Paulsen's book [22] and Pisier's book [23] for scalar valued Schur products.

Lemma 3.1. *Suppose that $b = [b_{ij}]$ is a positive matrix in $M_n(\mathcal{Z}(\mathcal{A}))$. Then the $\mathcal{A}$-valued Schur product defines a completely positive map*

$$T_b : [a_{ij}] \to [a_{ij} b_{ij}]$$

from $M_n(\mathcal{A})$ into $M_n(\mathcal{A})$.

Proof. Since $b \geq 0$ in $M_n(\mathcal{Z}(\mathcal{A}))$, there exists $c \in M_n(\mathcal{Z}(\mathcal{A}))$ such that $b = c^* c$. Suppose $\mathcal{A} \subseteq \mathcal{B}(H)$ and $a = [a_{ij}] \geq 0$ in $M_n(\mathcal{A}) \subseteq \mathcal{B}(H^n)$. For any $v_1, \cdots, v_n \in H$, we have

$$\textstyle\sum_{i,j=1}^n \langle v_i | a_{ij} b_{ij} v_j\rangle = \sum_{i,j=1}^n \langle v_i | a_{ij} (\sum_{k=1}^n c_{ki}^* c_{kj}) v_j\rangle = \sum_{k=1}^n \sum_{i,j=1}^n \langle c_{ki} v_i | a_{ij} c_{kj} v_j\rangle \geq 0.$$

This shows that $T_b(a) = [b_{ij} a_{ij}] \geq 0$ in $M_n(\mathcal{A})$. Therefore, T_b is a positive map from $M_n(\mathcal{A})$ into $M_n(\mathcal{A})$. The complete positivity of T_b follows from a standard matricial argument. □

Theorem 3.2. *Let $h : \Gamma \to \mathcal{Z}(\mathcal{A}) \subseteq \mathcal{A}$ be a positive definite map with respect to the action $\alpha : \Gamma \curvearrowright \mathcal{A}$. Then h is a completely positive multiplier with respect to α and the natural map Φ on $C_c(\Gamma, \mathcal{A})$ given in (3.3) extends to a completely positive $\mathcal{A}$-bimodule multiplier map on $\Gamma \times_{\alpha,r} \mathcal{A}$.*

Proof. Suppose that $h : \Gamma \to \mathcal{Z}(\mathcal{A})$ is a positive definite map with respect to the action $\alpha : \Gamma \curvearrowright \mathcal{A}$. We want to show that the map $\Phi(\sum_{s\in\Gamma} \lambda_a a_s) = \sum_{s\in\Gamma} \lambda_a h_s a_s$ can be extended to a completely positive map on $\Gamma \ltimes_{\alpha,r} \mathcal{A}$.

According to [2, Corollary 4.16] (with a slight index modification since we consider the left action of Γ on $\mathcal{A}$), an element $\sum\limits_{s\in\Gamma} \lambda_s a_s \in C_c(\Gamma, \mathcal{A})$ is positive if and only if for any $s_1, \cdots, s_n \in \Gamma$, we have $[\alpha_{s_j}(a_{s_i^{-1}s_j})] \geq 0$ in $M_n(\mathcal{A})$. Since the map $h : \Gamma \to \mathcal{Z}(\mathcal{A})$ is positive definite, we have $[\alpha_{s_j}(h_{s_i^{-1}s_j})] \geq 0$ in $M_n(\mathcal{Z}(\mathcal{A}))$. This shows that

$$[\alpha_{s_j}(h_{s_i^{-1}s_j} a_{s_i^{-1}s_j})] = [\alpha_{s_j}(h_{s_i^{-1}s_j}) \cdot \alpha_{s_j}(a_{s_i^{-1}s_j})] \geq 0 \text{ in } M_n(\mathcal{A})$$

by Lemma 3.1. Applying [2, Corollary 4.16] again, we see that the map Φ given in (3.3) is positive on $C_c(\Gamma, \mathcal{A})$. Passing to the matricial level, we see that the map Φ is completely positive on $C_c(\Gamma, \mathcal{A})$. Now assume $\| \sum\limits_{s\in\Gamma} \lambda_s a_s \| \leq 1$ in $C_c(\Gamma, \mathcal{A}) \subseteq \Gamma \ltimes_{\alpha,r} \mathcal{A}$. Then we get

$$\begin{bmatrix} 1 & \sum_{s\in\Gamma} \lambda_s a_s \\ (\sum_{s\in\Gamma} \lambda_s a_s)^* & 1 \end{bmatrix} \geq 0$$

in $C_c(\Gamma, M_2(\mathcal{A})) \subseteq M_2(\Gamma \ltimes_{\alpha,r} \mathcal{A}) = \Gamma \ltimes_{\alpha,r} M_2(\mathcal{A})$, and thus

$$\Phi_2\left(\begin{bmatrix} 1 & \sum_{s\in\Gamma} \lambda_s a_s \\ (\sum_{s\in\Gamma} \lambda_s a_s)^* & 1 \end{bmatrix}\right) = \begin{bmatrix} h_e & \sum_{s\in\Gamma} \lambda_s h_s a_s \\ (\sum_{s\in\Gamma} \lambda_s h_s a_s)^* & h_e \end{bmatrix} \geq 0.$$

From this, we can conclude that

$$\| \Phi(\sum_{s\in\Gamma} \lambda_s a_s) \| = \| \sum_{s\in\Gamma} \lambda_s h_s a_s \| \leq \| h_e \| .$$

Therefore, Φ is bounded on $C_c(\Gamma, \mathcal{A})$ and thus can be extended to a completely positive $\mathcal{A}$-bimodule map on $\Gamma \ltimes_{\alpha,r} \mathcal{A}$. □

We note that if the action $\alpha : \Gamma \curvearrowright \mathcal{A}$ acts trivially on all h_s, i.e. $\alpha_t(h_s) = h_s$ for all $s, t \in \Gamma$, then h is positive definite with respect to the action α if and only if h is a positive definite map in the usual sense, i.e.

$$[h_{s_i^{-1}s_j}] = [\alpha_{s_j}(h_{s_i^{-1}s_j})] \geq 0 \ \text{ in } M_n(\mathcal{A}).$$

This is always the case when $\mathcal{A} = \mathbb{C}$ since any action α acts trivially on $\mathbb{C}$. Therefore, Theorem 3.2 generalizes the well-known result that every positive definite function $h : \Gamma \to \mathbb{C}$ is uniquely associated with a completely positive map Φ on the C*-algerba $\Gamma \ltimes_{\alpha,r} \mathbb{C} = C^*_\lambda(\Gamma)$ such that $\Phi(\lambda_s) = \lambda_s h_s$. In the rest of this paper, we assume that all multipliers $h : \Gamma \to \mathcal{Z}(\mathcal{A}) \subseteq \mathcal{A}$ under the consideration are related to some action $\alpha : \Gamma \curvearrowright \mathcal{A}$. To simplify our notation, we sometimes do not mention the action α if there is no confusion.

Now we are ready to discuss the connection between (completely) bounded $\mathcal{A}$-bimodule multiplier maps Φ on $\Gamma \ltimes_{\alpha,r} \mathcal{A}$ and the corresponding induced Hilbert $\mathcal{A}$-module maps $\tilde{\Phi}$ on $\mathcal{H}_\mathcal{A}$.

Proposition 3.3. *Let Φ be a bounded $\mathcal{A}$-bimodule multiplier map on $\Gamma \ltimes_{\alpha,r} \mathcal{A}$. Then for any $x \in C_c(\Gamma, \mathcal{A})$, we have*

$$\mathcal{E}(\Phi(x)^*\Phi(x)) \leq \|\Phi\|^2 \mathcal{E}(x^*x).$$

If, in addition, Φ is positive on $\Gamma \ltimes_{\alpha,r} \mathcal{A}$, we get

$$\mathcal{E} \circ \Phi \leq \|\Phi\| \, \mathcal{E}.$$

Therefore, Φ *induces a bounded* $\mathcal{A}$*-module map* $\tilde{\Phi}$ *on* $\mathcal{H}_{\mathcal{A}}$ *with* $\|\tilde{\Phi}\|_{\mathcal{E}} \leq \|\Phi\|$.

Proof. Since Φ is a bounded $\mathcal{A}$-bimodule multiplier map, we have $\Phi(\lambda_s) = \lambda_s h_s$ with $h_s \in \mathcal{Z}(\mathcal{A})$ and $\|h_s\| \leq \|\Phi\|$ for all $s \in \Gamma$. Given any $x = \sum\limits_{s\in\Gamma} \lambda_s a_s \in C_c(\Gamma, \mathcal{A})$, we have

$$\begin{aligned}\mathcal{E}(\Phi(x)^*\Phi(x)) &= \mathcal{E}(\sum_{s,t\in\Gamma} a_s^* h_s^* \lambda_{s^{-1}} \lambda_t h_t a_t) = \sum_{s,t\in\Gamma} a_s^* h_s^* \mathcal{E}(\lambda_{s^{-1}}\lambda_t) h_t a_t \\ &= \sum_{s\in\Gamma} a_s^* h_s^* h_s a_s \leq \|\Phi\|^2 \sum_{s\in\Gamma} a_s^* a_s = \|\Phi\|^2\, \mathcal{E}(x^*x).\end{aligned}$$

Then Φ defines a bounded Hilbert $\mathcal{A}$-module map $\tilde{\Phi}$ on $\mathcal{H}_{\mathcal{A}}$ with $\|\tilde{\Phi}\|_{\mathcal{E}} \leq \|\Phi\|$. If Φ is completely positive, we have

$$\begin{aligned}\mathcal{E}\circ\Phi(x^*x) &= \sum_{s,t\in\Gamma} \mathcal{E}(a_s^*\Phi(\lambda_{s^{-1}t})a_t) = \sum_{s,t\in\Gamma} \mathcal{E}(a_s^* \lambda_{s^{-1}t} h_{s^{-1}t} a_t) \\ &= \sum_{s\in\Gamma} a_s^* h_e a_s \leq \|\Phi\|(\sum_{s\in\Gamma} a_s^* a_s) \leq \|\Phi\|\mathcal{E}(x^*x).\end{aligned}$$

□

Let $h : \Gamma \to \mathcal{A}$ be a bounded multiplier. We say that h is *vanishing at infinity*, i.e. $h \in C_0(\Gamma, \mathcal{A})$, if for arbitrary $\varepsilon > 0$, there exists a finite set $F \subseteq \Gamma$ such that $\|h_s\| < \varepsilon$ for all $s \notin F$.

Proposition 3.4. *Let* Φ *be a bounded* $\mathcal{A}$*-bimodule multiplier map on* $\Gamma \ltimes_{\alpha,r} \mathcal{A}$*. Then the corresponding bounded multiplier* $h : \Gamma \to \mathcal{A}$ *is vanishing at infinity if and only if the induced map* $\tilde{\Phi}$ *is compact on* $\mathcal{H}_{\mathcal{A}}$.

Proof. Suppose that $h \in C_0(\Gamma, \mathcal{A})$. Then for any $k \in \mathbb{N}$, there exists a finite subset $F_k \subseteq \Gamma$ such that $\|h_t\| < \frac{1}{k}$ for all $t \notin F_k$. Consider the finite-rank map

$$T_k = \sum_{t\in F_k} \theta_{\lambda_t h_t, \lambda_t}$$

on $\mathcal{H}_{\mathcal{A}}$. For $x = \sum\limits_{s\in\Gamma} \lambda_s a_s \in C_c(\Gamma, \mathcal{A})$, we have

$$T_k(x) = \sum_{t\in F_k} \theta_{\lambda_t h_t, \lambda_t}(x) = \sum_{t\in F_k} \lambda_t h_t \langle \lambda_t | \sum_{s\in\Gamma} \lambda_s a_s\rangle_{\mathcal{E}} = \sum_{t\in F_k} \lambda_t h_t a_t.$$

Since $\tilde{\Phi}(x) = \sum\limits_{t\in\Gamma} \lambda_t h_t a_t$, we get

$$\| (\tilde{\Phi} - T_k)(x) \|_{\mathcal{E}}^2 = \| \sum_{t\notin F_k} \lambda_t h_t a_t \|_{\mathcal{E}}^2 = \| \sum_{t\notin F_k} a_t^* h_t^* h_t a_t \| \leq \frac{1}{k^2} \| x \|_{\mathcal{E}}^2 .$$

This shows that $\tilde{\Phi}$ is the norm limit of the finite-rank maps T_k and thus is compact on $\mathcal{H}_{\mathcal{A}}$.

On the other hand, let us suppose that $\tilde{\Phi}$ is compact on $\mathcal{H}_{\mathcal{A}}$. For any $\varepsilon > 0$, there exists a finite-rank map $T = \sum\limits_{i=1}^{n} \theta_{\xi_i, \eta_i}$ on $\mathcal{H}_{\mathcal{A}}$ such that $\|\tilde{\Phi} - T\|_{\mathcal{E}} < \varepsilon$. Now with an appropriate modification, we can replace ξ_i, η_i in $\mathcal{H}_{\mathcal{A}}$ by $\tilde{\xi}_i, \tilde{\eta}_i$ in $C_c(\Gamma, \mathcal{A})$. Therefore, without loss of generality, we can assume that T has the form $T = \sum\limits_{s,t\in F} \theta_{\lambda_s a_s, \lambda_t b_t}$ for some finite subset $F \subseteq \Gamma$. For any $r \notin F$, we have

$$T(\lambda_r) = \sum_{s,t\in F} \theta_{\lambda_s a_s, \lambda_t b_t}(\lambda_r) = \sum_{s,t\in F} \lambda_s a_s b_t^* \langle \lambda_t | \lambda_r \rangle_{\mathcal{E}} = 0.$$

Since $\tilde{\Phi}(\lambda_r) = \lambda_r h_r$, we get

$$\|h_r\| = \|h_r^* h_r\|^{\frac{1}{2}} = \|\mathcal{E}(\Phi(\lambda_r)^* \Phi(\lambda_r))\|^{\frac{1}{2}} = \|\tilde{\Phi}(\lambda_r)\|_{\mathcal{E}} = \|\tilde{\Phi}(\lambda_r) - T(\lambda_r)\|_{\mathcal{E}} \leq \varepsilon$$

for any $r \notin F$. This shows that $h \in C_0(\Gamma, \mathcal{A})$. □

Let $h_n : \Gamma \to \mathcal{A}$ be a sequence of bounded multipliers. We say that $\{h_n\}$ converges to the constant function 1 pointwisely on Γ if we have $\|h_{n,s} - 1\| \to 0$ for each $s \in \Gamma$.

Proposition 3.5. *Let $\{\Phi_n\}$ be a sequence of uniformly bounded $\mathcal{A}$-bimodule multiplier maps on $\Gamma \ltimes_{\alpha,r} \mathcal{A}$. Then the corresponding bounded multipliers $\{h_n\}$ converge to the constant function 1 pointwisely on Γ if and only if the induced maps $\{\tilde{\Phi}_n\}$ converge to the identity map id in the point-norm topology on $\mathcal{H}_{\mathcal{A}}$.*

Proof. Let us first suppose that $h_n \to 1$ pointwisely on Γ. Since there exists a positive number $k > 0$ such that $\|\tilde{\Phi}_n\|_{\mathcal{E}} \leq \|\Phi_n\| \leq k$ for all $n \in \mathbb{N}$, it sufficies to show that $\|\tilde{\Phi}_n(x) - x\|_{\mathcal{E}} \to 0$ for all $x = \sum\limits_{s\in\Gamma} \lambda_s a_s \in C_c(\Gamma, \mathcal{A})$. For each $x \in C_c(\Gamma, \mathcal{A})$, there exists a finite subset $F \subseteq \Gamma$ such that we can write $x = \sum\limits_{s\in F} \lambda_s a_s$. Let us fix such an element x and fix this finite subset $F \subseteq \Gamma$.

For every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\|h_{n,s} - 1\| < \varepsilon$ for all $s \in F$ and $n \geq n_0$. It follows that

$$\begin{aligned} \|\tilde{\Phi}_n(x) - x\|_{\mathcal{E}}^2 &= \|\sum_{s\in F} \lambda_s (h_{n,s} - 1) a_s\|_{\mathcal{E}}^2 = \|\sum_{s\in F} a_s^* (h_{n,s}^* - 1)(h_{n,s} - 1) a_s\| \\ &\leq \epsilon^2 \|\sum_{s\in F} a_s^* a_s\| = \varepsilon^2 \|x\|_{\mathcal{E}}^2. \end{aligned}$$

This shows that $\|\tilde{\Phi}_n(x) - x\|_{\mathcal{E}} \to 0$.

On the other hand, let us suppose that $\|\tilde{\Phi}_n(x) - x\|_{\mathcal{E}} \to 0$ for all $x \in \mathcal{H}_{\mathcal{A}}$. Then for each $s \in \Gamma$, we have

$$\|h_{n,s} - 1\|^2 = \|(h_{n,s} - 1)^* (h_{n,s} - 1)\| = \|\tilde{\Phi}_n(\lambda_s) - \lambda_s\|_{\mathcal{E}}^2 \to 0.$$

This shows that $h_n \to 1$ pointwisely on Γ. □

Let Γ be a countable discrete group and α be an action of Γ on a unital C*-algebra $\mathcal{A}$. We say that the action $\alpha : \Gamma \curvearrowright \mathcal{A}$ has the *Haagerup property* if there exists a sequence of positive definite multipliers $\{h_n\}$ in $C_0(\Gamma, \mathcal{A})$ such that $h_n \to 1$ pointwisely on Γ.

Theorem 3.6. *Let Γ be a countable discrete group and α be an action of Γ on a unital C^*-algebra $\mathcal{A}$. Then the following are equivalent:*

1. *the action $\alpha : \Gamma \curvearrowright \mathcal{A}$ has the Haagerup property,*
2. *the reduced C^*-algebra crossed product $\Gamma \ltimes_{\alpha,r} \mathcal{A}$ has the Hilbert $\mathcal{A}$-module Haagerup property.*

Proof. Suppose that the action α has the Haagerup property. Then there exists a sequence of positive definite multipliers $h_n : \Gamma \to \mathcal{Z}(\mathcal{A}) \subseteq \mathcal{A}$ such that $h_n \in C_0(\Gamma, \mathcal{A})$ and $\|h_{n,s} - 1\| \to 0$ for each $s \in \Gamma$. It follows from Theorem 3.2 that the corresponding multiplier maps

$$\Phi_n(\sum_{s\in\Gamma} \lambda_s a_s) = \sum_{s\in\Gamma} \lambda_s h_{n,s} a_s$$

extend to completely positive $\mathcal{A}$-bimodule multiplier maps on $\Gamma \ltimes_{\alpha,r} \mathcal{A}$ with norm

$$\|\Phi_n\| = \|\Phi_n(1)\| = \|h_{n,e}\|.$$

Since $\Phi_n(1) = h_{n,e}$ converges to 1 in $\mathcal{A}$, the sequence $\{\|\Phi_n(1)\|\}$ is bounded. Then $\{\Phi_n\}$ is a sequence of uniformly bounded and completely positive $\mathcal{A}$-bimodule multiplier maps on $\Gamma \ltimes_{\alpha,r} \mathcal{A}$. It follows from Proposition 3.4 and Proposition 3.5 that the induced maps $\tilde{\Phi}_n$ are compact and converge to the identity map id in the point-norm topology on $\mathcal{H}_{\mathcal{A}}$. This shows that $\Gamma \ltimes_{\alpha,r} \mathcal{A}$ has the Hilbert $\mathcal{A}$-module Haagerup property.

On the other hand, suppose that $\Gamma \ltimes_{\alpha,r} \mathcal{A}$ has the Hilbert $\mathcal{A}$-module Haagerup property. Then there exists a sequence of completely positive $\mathcal{A}$-bimodule maps Φ_n on $\Gamma \ltimes_{\alpha,r} \mathcal{A}$ such that (0) $\mathcal{E} \circ \Phi_n \leq \mathcal{E}$, (1) each induced map $\tilde{\Phi}_n$ is compact on $\mathcal{H}_{\mathcal{A}}$ and (2) the sequence of maps $\{\tilde{\Phi}_n\}$ converges to the indentity map id on $\mathcal{H}_{\mathcal{A}}$ in the point-norm topology. In general, these maps Φ_n need not be multiplier maps on $\Gamma \ltimes_{\alpha,r} \mathcal{A}$. We need to use Haagerup's trick (see [13]) to obtain a sequence of maps

$$h_n : s \in \Gamma \to h_{n,s} = \mathcal{E}(\lambda_s^* \Phi_n(\lambda_s)) = \langle \lambda_s | \tilde{\Phi}_n(\lambda_s) \rangle_{\mathcal{E}} \in \mathcal{A}.$$

Since $\mathcal{E}$ and Φ_n are $\mathcal{A}$-bimodule maps, we can apply (2.5) to get

$$\begin{aligned} h_{n,s} a &= \mathcal{E}(\lambda_s^* \Phi_n(\lambda_s a)) = \mathcal{E}(\lambda_s^* \Phi_n(\alpha_s(a) \lambda_s)) \\ &= \mathcal{E}(\lambda_s^* \alpha_s(a) \Phi_n(\lambda_s)) = a \mathcal{E}(\lambda_s^* \Phi_n(\lambda_s)) = a h_{n,s} \end{aligned}$$

for all $a \in \mathcal{A}$ and $s \in \Gamma$. This shows that $h_n : \Gamma \to \mathcal{Z}(\mathcal{A})$ for all $n \in \mathbb{N}$. Next we show that each $h_n : \Gamma \to \mathcal{Z}(\mathcal{A})$ is a positive definite map and thus is a positive definite multiplier by Theorem 3.2. Indeed, for any $s_1, \cdots, s_m \in \Gamma$, it follows from (2.7) that we have

$$\begin{aligned}\Big[\alpha_{s_j}(h_{n,s_i^{-1}s_j})\Big] &= \Big[\alpha_{s_j}(\mathcal{E}(\lambda^*_{s_i^{-1}s_j}\Phi_n(\lambda_{s_i^{-1}s_j})))\Big]\\ &= \Big[\mathcal{E}(\lambda_{s_j}\lambda_{s_j^{-1}s_i}\Phi_n(\lambda_{s_i^{-1}s_j})\lambda_{s_j^{-1}})\Big]\\ &= \Big[\mathcal{E}(\lambda_{s_i}\Phi_n(\lambda_{s_i^{-1}s_j})\lambda_{s_j^{-1}})\Big] \geq 0 \text{ in } M_m(\mathcal{A}).\end{aligned}$$

Then conditions (0), (1), and (2) for $\{\Phi_n\}$ imply that $\{h_n\}$ is a sequence of positive definite multipliers in $C_0(\Gamma, \mathcal{A})$ such that $\|h_{n,s} - 1\| \to 0$ for each $s \in \Gamma$. Therefore, the action $\alpha : \Gamma \curvearrowright \mathcal{A}$ has the Haagerup property. □

We note that if $\mathcal{A} = \mathbb{C}$, then the natural conditional expectation $\mathcal{E}$ from $C^*_\lambda(\Gamma) = \Gamma \ltimes_{\alpha,r} \mathbb{C}$ onto $\mathbb{C}$ is just the canonical tracial state τ on $C^*_\lambda(\Gamma)$. Therefore, the trivial action $\alpha : \Gamma \curvearrowright \mathbb{C}$ has the Haagerup property, i.e. the group Γ has the Haagerup property, if and only if the group C*-algebra $C^*_\lambda(\Gamma)$ has the Hilbert $\mathbb{C}$-module Haagerup property, i.e. $C^*_\lambda(\Gamma)$ has the Haagerup property. So Theorem 3.6 is a natural Hilbert module generalization of [6, Theorem 2.6].

It is known that the Haggerup property is a strong negation of property (T). We can consider the corresponding $\mathcal{A}$-valued property (T). We say that an action $\alpha : \Gamma \curvearrowright \mathcal{A}$ has the *property (T)* if whenever $h_n : \Gamma \to \mathcal{A}$ is a sequence of completely positive multipliers converging to the constant function 1 pointwisely on Γ, then $h_n \to 1$ uniformly on Γ.

Proposition 3.7. *If an action α of a countable discrete group Γ on a unital C*-algebra $\mathcal{A}$ has both Haagerup property and property (T), then Γ must be a finite group.*

Proof. If the action $\alpha : \Gamma \curvearrowright \mathcal{A}$ has the Haagerup property, there exists a sequence of completely positive multipliers $h_n : \Gamma \to \mathcal{A}$ such that $h_n \to 1$ pointwisely on Γ. Then property (T) implies that $h_n \to 1$ uniformly on Γ. Therefore, the induced compact maps $\tilde{\Phi}_n$ converge to the identity map id in the operator norm on $\mathcal{H}_\mathcal{A}$ and thus id is compact on $\mathcal{H}_\mathcal{A}$. Since $\mathcal{H}_\mathcal{A} = M_{\Gamma,1} \otimes^{min} \mathcal{A}$, we can conclude that

$$\mathcal{K}(\mathcal{H}_\mathcal{A}) \cong \mathcal{K}(\ell_2(\Gamma)) \otimes^{min} \mathcal{A}$$

(see Lance [20]). Then the compactness of id implies that Γ has to be a finite group. □

4. Commutative Case

In this section, we consider the actions $\alpha : \Gamma \curvearrowright C(X)$ of countable discrete groups Γ on unital commutative C*-algebras $\mathcal{A} = C(X)$. It is well-known that each action $\alpha : \Gamma \curvearrowright C(X)$ uniquely corresponds to a left action

$$\alpha : (s, x) \in \Gamma \times X \to s \cdot_\alpha x \in X \tag{4.1}$$

on the compact Hausdorff space X such that

$$\alpha_s(f)(x) = f(s^{-1} \cdot_\alpha x). \tag{4.2}$$

We can also obtain a *left transformation groupoid*

$$G = \Gamma \ltimes X = \{(s, x) : s \in \Gamma, x \in X\},$$

with the unit space $G^{(0)} = \{(e, x) : x \in X\}$ and the source and range maps $\mathfrak{s}, \mathfrak{r} : G \to G^{(0)}$ given by

$$\mathfrak{s}(s, x) = (e, x) \text{ and } \mathfrak{r}(s, x) = (e, s \cdot x).$$

The multiplication on the groupoid G is given by $\alpha \cdot \beta = (st, y)$ for elements $\alpha = (s, x)$ and $\beta = (t, y)$ satisfying $\mathfrak{s}(\alpha) = \mathfrak{r}(\beta)$, i.e. $x = t \cdot y$. It is known that we can isometrically *-isomorphically identify the reduced C*-algebra crossed product $\Gamma \ltimes_{\alpha,r} C(X)$ with the reduced groupoid C*-algebra $C_r^*(G)$.

Each bounded map $h : \Gamma \to C(X)$ uniquely corresponds to a bounded function $\tilde{h} : G \to \mathbb{C}$ on the groupoid G such that

$$\tilde{h}(s, x) = h_s(x). \tag{4.3}$$

It is easy to see that $h \in C_c(\Gamma, C(X))$ (respectively, $h \in C_0(\Gamma, C(X))$) if and only if $\tilde{h} \in C_c(G)$ (respectively, $\tilde{h} \in C_0(G)$). Moreover, the map h is a positive definite multiplier if and only if $\tilde{h}$ is a positive definite function on the groupoid G since for any $\alpha_i = (s_i^{-1}, x) \in G$, we have $\alpha_i^{-1} = (s_i, s_i^{-1} \cdot x)$ and thus

$$\left[\tilde{h}(\alpha_i \cdot \alpha_j^{-1})\right] = \left[\tilde{h}(s_i^{-1} s_j, s_j^{-1} \cdot x)\right] = [\alpha_{s_j}(h_{s_i^{-1} s_j})(x)] \geq 0 \text{ in } M_n(\mathbb{C}).$$

We refer the readers to Tu [24] for the definition of positive definite functions on groupoids. We note that the index in our definition of positive definite functons on the left translation groupoid G is slightly different from that given in [2] and [9], where they considered the right transformation groupoid. Summarize the above disucssion, we can obtain the following proposition.

Proposition 4.1. *An action $\alpha : \Gamma \curvearrowright C(X)$ has the Haagerup property if and only if the corresponding left transformation groupoid $\Gamma \ltimes X$ has the* Haagerup property, *i.e. there exists a sequence of positive definite approximate unit $\tilde{h}_n \in C_0(\Gamma \ltimes X)$.*

In this case, we also say that the action α on the compact Γ-space X has the Haagerup property.

Let $\{\infty\}$ be a single-point topological space. As we have remarked at the end of Section 3 that if α_∞ is the trivial action of Γ on $C(\{\infty\}) = \mathbb{C}$, then $\alpha_\infty : \Gamma \curvearrowright C(\{\infty\})$ has the Haagerup property if and only if the group Γ has the Haagerup property.

Now let us consider the induced left translation action $\tilde{\alpha}$ on the Stone-Cěch compactification $\beta\Gamma$. It is known that $\beta\Gamma$ is a compact Hausdorff space which contains Γ as an open dense subspace and has the universal property that any (continuous) map f from Γ into a compact Hausdorff space X has a unique (continuous) extension $\tilde{f} : \beta\Gamma \to X$. In this case each left translation

$$\alpha_s : t \in \Gamma \to st \in \Gamma \subseteq \beta\Gamma$$

extends uniquely to a left translation $\tilde{\alpha}_s : \beta\Gamma \to \beta\Gamma$. This gives us an action $\tilde{\alpha}$ on $\beta\Gamma$. For this action, we can obtain the following result, which is a consequence of [9, Theorem 2.3 and Theorem 4.2] and Theorem 3.6 of this paper. We outline the proof for the convenience of the reader.

Theorem 4.2. *Let Γ be a countable discrete group. Then there exists a locally finite metric d on Γ such that the following are equivalent:*

(1) *Γ is coarsely embeddable into a Hilbert space;*
(2) *the action $\tilde{\alpha} : \Gamma \curvearrowright C(\beta\Gamma)$ has the Haagerup property;*
(3) *$\Gamma \ltimes_{\tilde{\alpha},r} C(\beta\Gamma)$ has the Hilbert $C(\beta\Gamma)$-module Haagerup property.*

Proof. Let us first recall from Brown and Ozawa [2, Proposition 5.5.2] that if Γ is a countable discrete group, then there exists an increasing sequence

$$\{e\} = E_0 \subseteq E_1 \subseteq E_2 \subseteq \cdots E_n \subseteq \cdots$$

of finite symmetric subsets E_n of Γ such that $E_n E_m \subseteq E_{n+m}$ and $\Gamma = \cup_{n=0}^{\infty} E_n$. We can define a right invariant metric on Γ by letting

$$d(s,t) = min\{n : st^{-1} \in E_n\}. \tag{4.4}$$

According to this definition, we have $d(s,t) \leq n$ if and only if $st^{-1} \in E_n$. Therefore, $s \in \Gamma$ is contained in the closed ball $\bar{B}(t,n)$ if and only if $st^{-1} \in E_n$ or equivalently, $s \in E_n t$. From this we can see that the metric space (Γ, d) is discrete and locally finite, i.e. every closed ball is a finite subset of Γ.

It is known from [9, Theorem 2.3 and Theorem 4.2] that a locally finite metric space (Γ, d) is coarsely embeddable into a Hilbert space, i.e. there exist a function $f : \Gamma \to \mathcal{H}$ and two non-decreasing positive functions ρ_- and ρ_+ on $[0, \infty)$ such that $\lim\limits_{r\to\infty} \rho_-(r) = \infty$ and

$$\rho_-(d(s,t)) \leq \|f(s) - f(t)\| \leq \rho_+(d(s,t))$$

for all $s, t \in \Gamma$, if and only if the left transformation groupoid $\Gamma \ltimes \beta\Gamma$ has the Haagerup property. By Proposition 4.1, this is equivalent to the action $\tilde{\alpha} : \Gamma \curvearrowright C(\beta\Gamma)$ having the Haagerup property . This proves (1) $\Leftrightarrow$ (2).

The equivalence (2) $\Leftrightarrow$ (3) follows from Theorem 3.6. $\square$

In the following, we discuss the connections of these two special actions with actions on other compact spaces.

Theorem 4.3. *Suppose that the action $\alpha : \Gamma \curvearrowright C(X)$ has the Haagerup property and suppose that $\beta : \Gamma \curvearrowright C(Y)$ is another action. If we have a continuous map $\tau : Y \to X$ which is equivariant in the sense that*

$$\alpha_s \circ \tau = \tau \circ \beta_s$$

for all $s \in \Gamma$, then the action $\beta : \Gamma \curvearrowright C(Y)$ must have the Haagerup property.

Proof. Suppose that $\{h_n\}$ is a sequence of positive definite multipliers contained in $C_0(\Gamma, C(X))$ such that $\|h_{n,s} - 1\|_{C(X)} \to 0$ for all $s \in \Gamma$. It is easy to show that

$$k_n : s \in \Gamma \to k_{n,s} = h_{n,s} \circ \tau \in C(Y)$$

is a sequence of bounded maps in $C_0(\Gamma, C(Y)))$ such that $\|k_{n,s} - 1\|_{C(Y)} \to 0$. We only need to show that k_n are all positive definite with respect to the action β. This follows from the fact that for any $s_1, \cdots, s_m \in \Gamma$ and $y \in Y$, we have

$$\begin{aligned} [\beta_{s_j}(k_{n,s_i^{-1}s_j})(y)] &= [k_{n,s_i^{-1}s_j}(s_j^{-1} \cdot_\beta y)] = [h_{n,s_i^{-1}s_j}(\tau(s_j^{-1} \cdot_\beta y))] \\ &= [h_{n,s_i^{-1}s_j}(s_j^{-1} \cdot_\alpha \tau(y))] = [\alpha_{s_j}(h_{n,s_i^{-1}s_j})(\tau(y))] \geq 0 \text{ in } M_m(\mathbb{C}). \end{aligned}$$

This shows that the action β has the Haagerup property. $\square$

As a consequence of Theorem 4.3, we can obtain the following proposition.

Proposition 4.4. *Let $\beta : \Gamma \curvearrowright C(X)$ be an arbitrary action of Γ on a unital commutative C^*-algebra $C(X)$.*

1. *If Γ has the Haagerup property, then so is the action $\beta : \Gamma \curvearrowright C(X)$.*
2. *If $\beta : \Gamma \curvearrowright C(X)$ has the Haagerup property, then so is the action $\tilde{\alpha} : \Gamma \curvearrowright C(\beta\Gamma)$. In this case, the group Γ is coarsely embeddable into a Hilbert space.*

Proof. To prove (1), we can consider the constant map $f : X \to \{\infty\}$. It is clear that f is a continuous map such that $f \circ \beta = \alpha_\infty \circ f$. Therefore, we can obtain (1) by Theorem 4.3.

To prove (2), let us fix a point $x_0 \in X$ and consider the continuous map

$$f : s \in \Gamma \to s \cdot_\beta x_0 \in X.$$

We can obtain a unique continuous extension $\tilde{f} : \beta\Gamma \to X$ by the universal property of $\beta\Gamma$. Given any $t \in \Gamma$, we have

$$\tilde{f} \circ \tilde{\alpha}_s(t) = \tilde{f}(st) = f(st) = (st) \cdot_\beta x_0 = s \cdot_\beta (t \cdot_\beta x_0) = \beta_s(t \cdot_\beta x_0) = \beta_s \circ \tilde{f}(t).$$

This shows that $\tilde{f} \circ \tilde{\alpha}_s$ and $\beta_s \circ \tilde{f}$ are equal on Γ. Since Γ is dense on $\beta\Gamma$ and the two maps are continuous on $\beta\Gamma$, they must be equal. This shows that $\tilde{f}$ is an equivariant continuous map from $\beta\Gamma$ into X. Therefore, we can conclude from Theorem 4.3 that the action $\tilde{\alpha} : \Gamma \curvearrowright C(\beta\Gamma)$ has the Haagerup property. It follows from Theorem 4.2 that the group Γ is coarsely embeddable into a Hilbert space. □

Now we wonder what we can say about the Haagerup property related to some other compact topological spaces. One possible candidate is the one-point compactification $\Gamma \cup \{\infty\}$ of Γ. It is clear that the left translation action $\alpha_s(t) = st$ on Γ extends to an action $\alpha^\infty$to $\Gamma \cup \{\infty\}$, which satisfies $\alpha^\infty_s(\infty) = \infty$ for all $s \in \Gamma$. In this case, $C(\Gamma \cup \{\infty\}) = c(\Gamma)$ is the space of all convergent sequences on Γ. As a consequence of Theorem 4.3, we can obtain the following result.

Theorem 4.5. *A countable discrete group* Γ *has the Haagerup property if and only if the induced action* $\alpha^\infty : \Gamma \curvearrowright C(\Gamma \cup \{\infty\})$ *has the Haagerup property.*

Proof. It is known from Proposition 4.4 that if Γ has the Haagerup property, then the induced action $\alpha^\infty : \Gamma \curvearrowright C(\Gamma \cup \{\infty\})$ has the Haagerup property.

On the other hand, let us consider the (continuous) map $g : \infty \in \{\infty\} \to \infty \in \Gamma \cup \{\infty\}$. It is easy to see that g satiafies the equivariant condition $g \circ \alpha_\infty = \alpha^\infty \circ g$ with respect to the trivial action α_∞ on the one-point set $\{\infty\}$ and the induced action α^∞ on $\Gamma \cup \{\infty\}$. Therefore, if the induced action $\alpha^\infty : \Gamma \curvearrowright C(\Gamma \cup \{\infty\})$ has the Haagerup property, then Γ has the Haagerup property. □

Next let us consider the Stone-Cěch boundary $Y = \beta\Gamma \setminus \Gamma$, which is a compact space and the left translation action $\tilde{\alpha}$ on $\beta\Gamma$ restricts to an action on Y. The inclusion map from Y into $\beta\Gamma$ is clearly equivariant. We can obtain the following result.

Corollary 4.6. *The left translation action* $\tilde{\alpha}$ *on* $\beta\Gamma$ *has the Haagerup property if and only if the induced left translation* $\tilde{\alpha}$ *on* $Y = \beta\Gamma \setminus \Gamma$ *has the Haagerup property.*

We wonder whether there is any action α of Γ on some compact topological space X such that the Haagerup property of this action $\alpha : \Gamma \curvearrowright C(X)$ is different from the Haagerup property and coarse embedding property of Γ.

5. COMPLETELY BOUNDED $\ell_\infty(\Gamma)$-BIMODULE MAPS ON $C^*_u(\Gamma)$

It is known that there is an isometric *-isomorphism $\ell_\infty(\Gamma) = C(\beta\Gamma)$, where we can identify $\beta\Gamma$ with the maximal ideal space (or the pure state space) of $\ell_\infty(\Gamma)$. If we let $\alpha : \Gamma \curvearrowright \ell_\infty(\Gamma)$ denote the left translation action on $\ell_\infty(\Gamma)$ given by

$$\alpha_s(f)(t) = (\lambda_s f \lambda^*_s)(t) = f(s^{-1}t),$$

we can identify the induced left translation action $\tilde{\alpha} : \Gamma \curvearrowright C(\beta\Gamma)$ with this action $\alpha : \Gamma \curvearrowright \ell_\infty(\Gamma)$, and we can identify the C*-algebra crossed product $\Gamma \ltimes_{\tilde{\alpha},r} C(\beta\Gamma)$ with the *uniform Roe algebra*

$$C_u^*(\Gamma) = span\{\sum_{s\in\Gamma} \lambda_s f_s : f_s \in \ell_\infty(\Gamma)\}^{-\|\cdot\|} \subseteq \mathcal{B}(\ell_2(\Gamma)).$$

It is easy to see that $C_u^*(G)$ is an $\ell_\infty(\Gamma)$-subbimodule of $\mathcal{B}(\ell_2(\Gamma))$ such that

$$\mathcal{K}(\ell_2(\Gamma)) \cup C_\lambda^*(\Gamma) \subseteq C_u^*(\Gamma) \subseteq \mathcal{B}(\ell_2(\Gamma)).$$

There is a faithful conditional expectation $\mathcal{E}$ from $C_u^*(\Gamma)$ onto $\ell_\infty(\Gamma)$, which is just the restriction of the canonical normal faithful conditional expectation from $\mathcal{B}(\ell_2(\Gamma))$ onto the diagonal subalgebra $\ell_\infty(\Gamma)$.

Let $\mathcal{CB}_{\ell_\infty(\Gamma)}(\mathcal{B}(\ell_2(\Gamma)))$ (respectively, $\mathcal{CB}^\sigma_{\ell_\infty(\Gamma)}(\mathcal{B}(\ell_2(\Gamma))))$ denote the space of completely bounded (respectively, normal completely bounded) $\ell_\infty(\Gamma)$-bimodule maps on $\mathcal{B}(\ell_2(\Gamma))$. Since $\ell_\infty(\Gamma)' = \ell_\infty(\Gamma)$ is an atomic and finite von Neumann algebra on $\ell_2(\Gamma)$, it is known from Hofmeier and Wittstock [14, Lemma 3.5] that every completely bounded $\ell_\infty(\Gamma)$-bimodule map on $\mathcal{B}(\ell_2(\Gamma))$ is automatically normal. Therefore, we have

$$\mathcal{CB}_{\ell_\infty(\Gamma)}(\mathcal{B}(\ell_2(\Gamma))) = \mathcal{CB}^\sigma_{\ell_\infty(\Gamma)}(\mathcal{B}(\ell_2(\Gamma))). \tag{5.1}$$

We also note that since the von Neumann algebra $\ell_\infty(\Gamma)$ is standardly represented on $\ell_2(\Gamma)$, every normal state is a vector state. Therefore, every bounded $\ell_\infty(\Gamma)$-bimodule map Ψ on $\mathcal{B}(\ell_2(\Gamma))$ is automatically completely bounded with $\|\Psi\|_{cb} = \|\Psi\|$ (cf. Effros and Kishimoto [7, Theorem 2.5]).

Proposition 5.1. *Let $\Psi \in \mathcal{CB}_{\ell_\infty(\Gamma)}(\mathcal{B}(\ell_2(\Gamma)))$. For every $t \in \Gamma$, we have*

$$\Psi(\lambda_t) = \lambda_t h_t$$

for some $h_t \in \ell_\infty(\Gamma)$ with $\|h_t\| \le \|\Psi\|_{cb}$. Therefore, $\Phi = \Psi_{|C_u^*(\Gamma)}$ defines a completely bounded $\ell_\infty(\Gamma)$-bimodule multiplier map on $C_u^*(\Gamma)$.

Proof. Let $\Psi \in \mathcal{CB}_{\ell_\infty(\Gamma)}(\mathcal{B}(\ell_2(\Gamma))) = \mathcal{CB}^\sigma_{\ell_\infty(\Gamma)}(\mathcal{B}(\ell_2(\Gamma)))$ and let $M_{\mathbb{N},1}(\ell_\infty(\Gamma))$ denote the space of bounded column vectors $[\xi_n]$ with $\xi_n \in \ell_\infty(\Gamma)$ and

$$\|[\xi_n]\| = \|\sum_{n=1}^\infty \xi_n^*\xi_n\|^{\frac{1}{2}} < \infty.$$

It is known from Haagerup [11] that there exist two bounded column vectors $\xi = [\xi_n]$ and $\eta = [\eta_n]$ in $M_{\mathbb{N},1}(\ell_\infty(\Gamma))$ such that

$$\|\Psi\|_{cb} = \|[\xi_n]\|\|[\eta_n]\| = \|\sum_{n=1}^\infty \xi_n^*\xi_n\|^{\frac{1}{2}}\|\sum_{n=1}^\infty \eta_n^*\eta_n\|^{\frac{1}{2}}$$

and

$$\Psi(x) = \sum_{n=1}^{\infty} \xi_n^* x \eta_n$$

for all $x \in \mathcal{B}(\ell_2(\Gamma))$. In particular, for every $t \in \Gamma$, we have

$$\lambda_t^* \Psi(\lambda_t) = \sum_{n=1}^{\infty} (\lambda_{t^{-1}} \xi_n^* \lambda_t) \eta_n = \sum_{n=1}^{\infty} \alpha_{t^{-1}}(\xi_n^*) \eta_n$$

Since $\alpha_{t^{-1}}$ is a normal *-automorphism on $\ell_\infty(\Gamma)$, $[\alpha_{t^{-1}}(\xi_n)]$ is again a bounded column vector in $M_{\mathbb{N},1}(\ell_\infty(\Gamma))$ such that

$$\|[\alpha_{t^{-1}}(\xi_n)]\| = \|\sum_{n=1}^{\infty} \alpha_{t^{-1}}(\xi_n^* \xi_n)\|^{\frac{1}{2}} = \|\sum_{n=1}^{\infty} \xi_n^* \xi_n\|^{\frac{1}{2}} = \|[\xi_n]\|.$$

Therefore, $h_t = \lambda_t^* \Psi(\lambda_t) = \sum_{n=1}^{\infty} \alpha_{t^{-1}}(\xi_n^*) \eta_n$ is a well-defined element in $\ell_\infty(G)$ such that

$$\Psi(\lambda_t) = \lambda_t h_t.$$

It is clear that

$$\|h_t\| = \|\lambda_t^* \Psi(\lambda_t)\| \leq \|\Psi\| = \|\Psi\|_{cb}.$$

□

Let $\mathcal{CB}_{\ell_\infty(\Gamma)}(C_u^*(\Gamma))$ denote the space of completely bounded $\ell_\infty(\Gamma)$-bimodule maps on $C_u^*(\Gamma)$. We can obtain the following isometric identification.

Proposition 5.2. *The restriction map*

$$\Psi \in \mathcal{CB}_{\ell_\infty(\Gamma)}(\mathcal{B}(\ell_2(\Gamma))) \to \Phi = \Psi|_{C_u^*(\Gamma)} \in \mathcal{CB}_{\ell_\infty(\Gamma)}(C_u^*(\Gamma))$$

defines an isometric isomorphism from $\mathcal{CB}_{\ell_\infty(\Gamma)}(\mathcal{B}(\ell_2(\Gamma)))$ *onto* $\mathcal{CB}_{\ell_\infty(\Gamma)}(C_u^*(\Gamma))$.

Therefore, every completely bounded $\ell_\infty(\Gamma)$*-bimodule map on* $C_u^*(\Gamma)$ *is automatically a completely bounded* $\ell_\infty(\Gamma)$*-bimodule multiplier map on* $C_u^*(\Gamma)$.

Proof. If $\Psi \in \mathcal{CB}_{\ell_\infty(\Gamma)}(\mathcal{B}(\ell_2(G)))$ is a completely bounded $\ell_\infty(\Gamma)$-bimodule map on $\mathcal{B}(\ell_2(\Gamma)$, then we have $\Phi = \Psi|_{C_u^*(\Gamma)} \in \mathcal{CB}_{\ell_\infty(\Gamma)}(C_u^*(\Gamma))$ by Proposition 5.1. On the other hand, suppose that Φ is a completely bounded $\ell_\infty(\Gamma)$-bimodule map on $C_u^*(\Gamma)$. We can regard Φ as a completely bounded $\ell_\infty(\Gamma)$-bimodule map from $C_u^*(\Gamma)$ into the injective von Neumann algebra $\mathcal{B}(\ell_2(\Gamma))$. Then it is known from Wittstock [25, Theorem 3.1] that there exists a completely bounded $\ell_\infty(\Gamma)$-bimodule extension Ψ from $\mathcal{B}(\ell_2(\Gamma))$ into $\mathcal{B}(\ell_2(\Gamma))$ such that $\|\Psi\|_{cb} = \|\Phi\|_{cb}$. Since the map Ψ must be normal on $\mathcal{B}(\ell_2(\Gamma))$ (see (5.1)) and $C_u^*(\Gamma)$ is σ-weakly dense in $\mathcal{B}(\ell_2(\Gamma))$, Ψ has to be the unique extension of Φ. This shows that the restriction map

$$\Psi \in \mathcal{CB}_{\ell_\infty(\Gamma)}(\mathcal{B}(\ell_2(\Gamma))) \to \Phi = \Psi|_{C_u^*(\Gamma)} \in \mathcal{CB}_{\ell_\infty(\Gamma)}(C_u^*(\Gamma))$$

is an isometric isomorphism from $\mathcal{CB}_{\ell_\infty(\Gamma)}(\mathcal{B}(\ell_2(\Gamma)))$ onto $\mathcal{CB}_{\ell_\infty(\Gamma)}(C^*_u(\Gamma))$. Therefore, we can conclude from Proposition 5.1 that every Φ in $\mathcal{CB}_{\ell_\infty(\Gamma)}(C^*_u(\Gamma))$ is a completely bounded $\ell_\infty(\Gamma)$-bimodule multiplier map on $C^*_u(\Gamma)$. □

Let us fix the canonical othonormal basis $\{\delta_s\}_{s\in\Gamma}$ for $\ell_2(\Gamma)$. Then every element $x\in\mathcal{B}(\ell_2(\Gamma))$ can be expressed as an infinite matrix $x=[x_{st}]$. A function $\phi:\Gamma\times\Gamma\to\mathbb{C}$ is called a *Schur multiplier* if the Schur product defines a bounded (and thus a completely bounded and normal) $\ell_\infty(\Gamma)$-bimodule map

$$M_\phi : x\in\mathcal{B}(\ell_2(\Gamma))\to[\phi(s,t)x_{st}]\in\mathcal{B}(\ell_2(\Gamma)).$$

It is known from Proposition 5.2 that the restriction of M_ϕ to $C^*_u(\Gamma)$ is a completely bounded $\ell_\infty(\Gamma)$-bimodule multiplier map on $C^*_u(\Gamma)$ and the corresponding multiplier map $h:\Gamma\to\ell_\infty(\Gamma)$ is given by

$$h_s(t)=\phi(st,t)\ \text{ or equivalently,}\quad \phi(s,t)=h_{st^{-1}}(t). \tag{5.2}$$

It is easy to see that ϕ is a positive definite Schur multiplier if and only if h is a positive definte multiplier with respect to the action $\alpha:\Gamma\curvearrowright\ell_\infty(\Gamma)$. In this case, M_ϕ is completely positive. We have $\phi(s,s)=1$ for all $s\in\Gamma$ if and only if M_ϕ is unital.

Now let us consider the metric d on Γ discussed in the proof of Theorem 4.2. We let

$$\triangle_R=\{(s,t)\in\Gamma\times\Gamma : d(s,t)\le R\}$$

denote the *strip* bounded by some positive number $R<\infty$, and let

$$\triangle=\{(s,s)\in\Gamma\times\Gamma : s\in\Gamma\}=\triangle_0$$

denote the diagonal of $\Gamma\times\Gamma$. We say that a Schur multiplier ϕ is contained in $C_0(\Gamma\times\Gamma,\triangle)$ if for every $\varepsilon>0$, there exists a positive $R>0$ such that $|\phi(s,t)|<\varepsilon$ for all $(s,t)\notin\triangle_R$. It is easy to verify that a multiplier $h:\Gamma\to\ell_\infty(\Gamma)$ is contained in $C_0(\Gamma,\ell_\infty(\Gamma))$ if and only if the corresponding Schur multiplier ϕ is contained in $C_0(\Gamma\times\Gamma,\triangle)$. It is also easy to see that a sequence of Schur multipliers $\{\phi_n\}$ converges to the constant function 1 uniformly on strips $\triangle_R$ for all $R>0$ if and only if the corresponding sequence of multipliers $\{h_n\}$ converges to the the constant function 1 pointwisely on Γ.

Therefore, as it was shown in [9, Theorem 2.3], a countable discrete group Γ with the locally finite metric d is coarsely embeddable into a Hilbert space if and only if there exists a sequence of positive definite Schur multipliers $\{\phi_n\}$ in $C_0(\Gamma\times\Gamma,\triangle)$ such that $\phi_n\to1$ uniformly on strips. In the following, we show that the positive definite condition in [9, Theorem 2.3] can be replaced by contractive Schur multipliers.

Theorem 5.3. *Let Γ be a countable discrete group with locally finite metric d. Then the following are equivalent:*

(1) Γ *is coarsely embeddable into a Hilbert space;*

(2) *there exists a sequence of contractive Schur multipliers* $\{\phi_n\}$ *in* $C_0(\Gamma\times\Gamma,\triangle)$ *such that* $\phi_n\to 1$ *uniformly on strips* $\triangle_R$ *for all* $R>0$*;*

(3) *there exists a sequence of completely contractive multipliers* $\{h_n\}$ *of the action* $\tilde\alpha$ *in* $C_0(\Gamma,\ell_\infty(\Gamma))$ *such that* $h_n\to 1$ *pointwisely on* Γ*.*

Proof. We only need to prove (2) $\Rightarrow$ (1). Our proof is largely motivated from the proof of [5, Proposition 2.1]. By assumption, for any $R>0$ and $0<\epsilon<\frac{1}{10}$, there exist two contractive maps $\xi,\eta:\Gamma\to H$ such that the contractive Schur multiplier $\phi(s,t)=\langle\xi(s)|\eta(t)\rangle$ satisfies

(i) $\sup\{|1-\phi(s,t)|: d(s,t)\le R, s,t\in\Gamma\}\le\epsilon$;

(ii) $\lim\limits_{S\to\infty}\sup\{|\phi(s,t)|: d(s,t)\ge S, s,t\in\Gamma\}=0$.

Condition (i) implies that if $d(s,t)\le R$, we have

$$\begin{aligned}\| \xi(s)-\eta(t)\|^2 &= \|\xi(s)\|^2+\|\eta(t)\|^2-2Re\langle\xi(s)|\eta(t)\rangle\\ &\le 2(1-Re\phi(s,t))\le 2|1-\phi(s,t)|<2\varepsilon.\end{aligned}$$

This shows that

$$\sup\{\|\xi(s)-\eta(t)\|: d(s,t)\le R, s,t\in\Gamma\}\le\sqrt{2\epsilon}.$$

Condition (i) also implies that for any $s\in\Gamma$,

$$1-\epsilon \le Re\phi(s,s)=Re\langle\xi(s)|\eta(s)\rangle\le\|\xi(s)\|\cdot\|\eta(s)\|\le\tfrac{1}{2}(\|\xi(s)\|^2+\|\eta(s)\|^2).$$

Therefore, we get

$$2(1-\varepsilon)\le\|\xi(s)\|^2+\|\eta(s)\|^2\le 2.$$

Condition (ii) implies that there exists a sufficiently large $S>0$ such that $|\phi(s,t)|<\varepsilon$ for all $d(s,t)\ge S$. Since $d(s,t)=d(t,s)$, we also have $|\phi(t,s)|<\varepsilon$. Then for $d(s,t)\ge S$, we get

$$\begin{aligned}\|\xi(s)-\eta(t)\|^2+\|\xi(t)-\eta(s)\|^2 &= \|\xi(s)\|^2+\|\eta(t)\|^2+\|\xi(t)\|^2+\|\eta(s)\|^2\\ &- 2Re\phi(s,t)-2Re\phi(t,s)\ge 4-8\varepsilon>3\frac{2}{10}.\end{aligned}$$

Since

$$\|\xi(t)-\eta(s)\|^2=\|\xi(t)\|^2+\|\eta(s)\|^2-2Re\phi(t,s)\le 2+2\varepsilon<2\frac{2}{10},$$

we can conclude that

$$\|\xi(s)-\eta(t)\|^2>3\frac{2}{10}-\|\xi(t)-\eta(s)\|^2>1.$$

Applying a natural induction precedure, we can choose two sequences of contractive maps $\xi_n,\eta_n:\Gamma\to H_n$ and an increasing sequence of positive real numbers $S_0=0<$

$S_1 < S_2 < \cdots$, which tends to infinity, such that for every $n \geq 1$ and every $(s,t) \in \Gamma \times \Gamma$ we have

(a) $\| \xi_n(s) - \eta_n(t) \| \leq \frac{1}{n^2}$, if $d(s,t) \leq \sqrt{n}$;

(b) $\| \xi_n(s) - \eta_n(t) \| \geq 1$, if $d(s,t) \geq S_n$.

Choose a base point $s_0 \in \Gamma$ and define two maps f and g from Γ into the infinite Hilbert space direct sum $\bigoplus\limits_{n=1}^{\infty} H_n$ by letting

$$f(s) = (\xi_1(s) - \eta_1(s_0)) \oplus (\xi_2(s) - \eta_2(s_0)) \oplus \cdots$$

and

$$g(t) = (\eta_1(t) - \xi_1(s_0)) \oplus (\eta_2(t) - \xi_2(s_0)) \oplus \cdots .$$

It follows from (a) that f and g are well defined maps such that

$$\begin{aligned} \| f(s) - g(t) \|^2 &= \sum_{k=1}^{\infty} \| \xi_k(s) - \eta_k(t) + \xi_k(s_0) - \eta_k(s_0) \|^2 \\ &\leq 2 \sum_{k=1}^{\infty} \| \xi_k(s) - \eta_k(t) \|^2 + 2 \sum_{k=1}^{\infty} \| \xi_k(s_0) - \eta_k(s_0) \|^2 . \end{aligned}$$

If n is such that $\sqrt{n-1} \leq d(s,t) < \sqrt{n}$, then for all $k \geq n$ we have $\| \xi_k(s) - \eta_k(t) \| < \frac{1}{k^2}$. It follows that

$$\begin{aligned} \sum_{k=1}^{\infty} \| \xi_k(s) - \eta_k(t) \|^2 &= \sum_{1 \leq k \leq n-1} \| \xi_k(s) - \eta_k(t) \|^2 + \sum_{k \geq n} \| \xi_k(s) - \eta_k(t) \|^2 \\ &\leq 4(n-1) + \sum_{k \geq n} \frac{1}{k^4} \leq 4d(s,t)^2 + c, \end{aligned}$$

where we let $c = \sum\limits_{k=1}^{\infty} \frac{1}{k^4} < \infty$. Since $d(s_0, s_0) = 0$, we obtain

$$\sum_{k=1}^{\infty} \| \xi_k(s_0) - \eta_k(s_0) \|^2 \leq \sum_{k=1}^{\infty} \frac{1}{k^4} = c$$

and thus

$$\| f(s) - g(t) \|^2 \leq 8d(s,t)^2 + 4c.$$

From this, we can obtain $\| f(t) - g(t) \| \leq 2\sqrt{c}$ and thus

$$\begin{aligned} \| f(s) - f(t) \| &\leq \| f(s) - g(t) \| + \| g(t) - f(t) \| \\ &\leq 2\sqrt{2d(s,t) + c} + 2\sqrt{c} = \rho_+(d(s,t)). \end{aligned}$$

On the other hand, if n is such that $S_n \leq d(s,t) < S_{n+1}$ we have

$$\begin{aligned} \| f(s) - g(t) \|^2 &= \sum_{k=1}^{\infty} \| \xi_k(s) - \eta_k(t) + \xi_k(s_0) - \eta_k(s_0) \|^2 \\ &\geq \sum_{1 \leq k \leq n} \| \xi_k(s) - \eta_k(t) + \xi_k(s_0) - \eta_k(s_0) \|^2 \\ &\geq \sum_{1 \leq k \leq n} | \, \| \xi_k(s) - \eta_k(t) \| - \| \xi_k(s_0) - \eta_k(s_0) \| \, |^2 \\ &\geq \sum_{1 \leq k \leq n} \| \xi_k(s) - \eta_k(t) \|^2 - 2 \sum_{1 \leq k \leq n} \| \xi_k(s) - \eta_k(t) \| \cdot \| \xi_k(s_0) - \eta_k(s_0) \| . \end{aligned}$$

For $1 \leq k \leq n$, we have $S_k \leq S_n \leq d(s,t)$. It follows from (b) that

$$\sum_{1\leq k\leq n} \| \xi_k(s) - \eta_k(t) \|^2 \geq n-1 = \rho'_-(d(s,t))^2,$$

where we let $\rho'_- = \sum\limits_{n=1}^{\infty} \sqrt{n-1}\chi_{[S_n,S_{n+1})}$. Since ξ and η are contractive maps, we also get

$$\sum_{1\leq k\leq n} \| \xi_k(s) - \eta_k(t) \| \cdot \| \xi_k(s_0) - \eta_k(s_0) \| \leq \sum_{1\leq k\leq n} 2\cdot \| \xi_k(s_0) - \eta_k(s_0) \| \leq 2 \cdot \sum_{1\leq k\leq n} \tfrac{1}{k^2} \leq 8.$$

So

$$\| f(s) - g(t) \|^2 \quad \geq \quad \rho'_-(d(s,t))^2 - 16 \geq (\rho'_-(d(s,t)) - 4)^2$$

and thus we get

$$\begin{aligned} \| f(s) - f(t) \| &\geq \| f(s) - g(t) \| - \| g(t) - f(t) \| \\ &\geq \rho'_-(d(s,t)) - 4 - 2\sqrt{c} = \rho_-(d(s,t)). \end{aligned}$$

This shows that Γ is coarsely embeddable into a Hilbert space. □

The following result is an immediate consequence of Theorem 5.3.

Corollary 5.4. *A countable discrete group* Γ *is coarsely embeddable into a Hilbert space if and only if there exists a sequence of completely contractive* $\ell_\infty(\Gamma)$*-bimodule maps* $\{\Phi_n\}$ *on* $C^*_u(\Gamma)$ *such that (1) each induced map* $\tilde{\Phi}_n$ *is compact on* $\mathcal{H}_{\ell_\infty(\Gamma)}$ *and (2)* $\{\tilde{\Phi}_n\}$ *converges to the identity map id in the point-norm topology on* $\mathcal{H}_{\ell_\infty(\Gamma)}$.

6. A SCHUR MULTIPLIER CHARACTERIZATION FOR EXACT GROUPS

According to Kirchberg [17, 18, 19], we say that a locally compact group Γ is exact if the reduced group C*-algebra $C^*_\lambda(\Gamma)$ is exact, i.e. we have the short exact sequence

$$0 \to \mathcal{K}(\ell_2) \otimes^{min} C^*_\lambda(\Gamma) \to \mathcal{B}(\ell_2) \otimes^{min} C^*_\lambda(\Gamma) \to \mathcal{Q}(\ell_2) \otimes^{min} C^*_\lambda(\Gamma) \to 0$$

of C*-algebras. It was shown by Ozawa [21] that a countable discrete group Γ is exact if and only if there is a sequence of positive definite Schur multipliers $\phi_n : \Gamma \times \Gamma \to \mathbb{C}$ such that (1) each ϕ_n is supported on a strip $\triangle_S$ for some $S > 0$ and (2) $\phi_n(s,t) \to 1$ uniformly on strips $\triangle_R$ with $R > 0$. This is equivalent to saying that there exists a sequence of positive definite multipliers $h_n : \Gamma \to \ell_\infty(\Gamma)$ in $C_c(\Gamma, \ell_\infty(\Gamma))$ such that $h_n \to 1$ pointwisely on Γ. From this result, it is easy to see that every exact discrete group is coarsely embeddable into a Hilbert space.

In general, we can say that an action $\alpha : \Gamma \curvearrowright C(X)$ is *amenable* if there exists a sequence of positive definite multpliers $h_n : \Gamma \to C(X)$ in $C_c(\Gamma, C(X))$ such that $h_n \to 1$ pointwisely on Γ. This definition is equivalent to that given by Anantharaman-Delaroche [1] (see proof in [2, Theorem 4.4.3]). Therefore, we can say that a discrete group Γ is amenable if and only if the trivial action $\alpha : \Gamma \curvearrowright \mathbb{C}$ is amenable, and a discrete group Γ is exact if and only if the left translation action $\alpha : \Gamma \curvearrowright \ell_\infty(\Gamma)$ is amenable. Moreover, we can obtain the following analogue of Theorem 4.3 and Proposition 4.4.

Theorem 6.1. *Suppose that the action $\alpha : \Gamma \curvearrowright C(X)$ is amenable and suppose that $\beta : \Gamma \curvearrowright C(Y)$ is another action. If we have a continuous map $\tau : Y \to X$ which is equivariant in the sense that*

$$\alpha_s \circ \tau = \tau \circ \beta_s$$

for all $s \in \Gamma$, then the action $\beta : \Gamma \curvearrowright C(Y)$ must be amenable.

Proposition 6.2. *Let $\beta : \Gamma \curvearrowright C(X)$ be an arbitrary action of Γ on $C(X)$.*

1. *If Γ is amenable, then so is the action $\beta : \Gamma \curvearrowright C(X)$.*
2. *If $\beta : \Gamma \curvearrowright C(X)$ is amenable, then so is the action $\tilde{\alpha} : \Gamma \curvearrowright \ell_\infty(\Gamma) = C(\beta\Gamma)$.*

In the following, we show that the positive definite Schur multipliers in Ozawa's result [21] can be replaced by uniformly bounded Schur multipliers. For this purpose, we need to recall from Haagerup [12] that a bounded linear map $\Phi : \mathcal{A} \to \mathcal{B}$ between two C*-algebras is called *decomposable* if there exist two completely positive maps $\Psi_i : \mathcal{A} \to \mathcal{B}$ $(i = 1, 2)$ such that the induced map

$$\tilde{\Phi} : a \in \mathcal{A} \to \begin{bmatrix} \Psi_1(a) & \Phi(a) \\ \Phi(a)^* & \Psi_2(a) \end{bmatrix} \in M_2(\mathcal{B}) \tag{6.1}$$

is completely positive. The *decomposable norm* of Φ is defined by

$$\|\Phi\|_{dec} = \inf\{\max\{\|\Psi_1\|, \|\Psi_2\|\}\},$$

where the infimum is taken over all possible Ψ_1 and Ψ_2 in (6.1). In general, we have $\|\Phi\|_{cb} \le \|\Phi\|_{dec}$ and if $\mathcal{B}$ is an injective C*-algebra, we have $\|\Phi\|_{cb} = \|\Phi\|_{dec}$.

Theorem 6.3. *Let Γ be a countable discrete group. Then Γ is exact if and only if there exists a sequence of uniformly bounded Schur multipliers $\{\phi_n\}$ such that each ϕ_n is supported on a strip $\triangle_{S_n}$ for some $S_n > 0$ and $\phi_n(s,t) \to 1$ uniformly on strips $\triangle_R$ for all $R > 0$.*

Proof. We only need to prove the sufficiency. Suppose that $\{\phi_n\}$ is a sequence of Schur multipliers satisfying the hypothesis. Since each ϕ_n is supported on a strip $\triangle_{S_n}$, there exists a finite subset $F_n \subseteq \Gamma$ such that $\phi_n(s,t) = 0$ whenever $st^{-1} \notin F_n$. It follows from (5.2) that $h_{n,s} = 0$ for each $s \notin F_n$. Let Φ_n to be the restriction of M_{ϕ_n} to $C^*_\lambda(\Gamma)$. For any $x = \sum\limits_{s\in\Gamma} \lambda_s a_s \in \lambda(\mathbb{C}[\Gamma])$ with $a_s \in \mathbb{C}$, we have

$$\Phi_n(x) = \sum_{s\in F_n} \lambda_s h_{n,s} a_s.$$

Therefore, each Φ_n is a finite-rank map from $C^*_\lambda(\Gamma)$ into $\mathcal{B}(\ell_2(\Gamma))$ with

$$\|\Phi_n\|_{dec} = \|\Phi_n\|_{cb} \leq \|\phi_n\| < k$$

for some positive $k > 0$. It is known from [16] (also see Pisier [23, Theorem 12.7]) that for each $n \in \mathbb{N}$ there exist completely bounded maps

$$v_n : C^*_\lambda(\Gamma) \to M_{k(n)} \text{ and } w_n : M_{k(n)} \to \mathcal{B}(\ell_2(\Gamma))$$

such that $\Phi_n = w_n \circ v_n$ and $\|v_n\|_{cb}\|w_n\|_{cb} < k$.

Since $\phi_n(s,t) \to 1$ uniformly on each strip $\triangle_R$, the corresponding multipliers $h_n : \Gamma \to \ell_\infty(\Gamma)$ converge to the constant function 1 pointwisely on Γ. Since these Schur multipliers ϕ_n are uniformly boundedness, the corresponding completely bounded maps $\Phi_n = M_{\phi_n}|_{C^*_\lambda(\Gamma)}$ are uniformly bounded and converge to the inclusion map $\iota : C^*_\lambda(\Gamma) \hookrightarrow \mathcal{B}(\ell_2(\Gamma))$ in the point-norm topology. This shows that the C*-algebra $C^*_\lambda(\Gamma)$ is k-exact and thus is exact. $\square$

As a consequence of Theorem 6.3, we can obtain the following $\ell_\infty(\Gamma)$-module characterization for exact groups.

Theorem 6.4. *Let Γ be a countable discrete group. Then Γ is exact if and only if there exists a sequence of completely bounded $\ell_\infty(\Gamma)$-bimodule maps $\{\Phi_n\}$ on $C^*_u(\Gamma)$ with $\|\Phi_n\|_{cb} < k$ for some positive number $k > 0$ such that*

(i) *each Φ_n induces a finite-rank map of the form $\tilde{\Phi}_n = \sum_{s,t\in F_n} \theta_{\lambda_s f_s, \lambda_t g_t}$ on $\mathcal{H}_{\ell_\infty(\Gamma)}$;*

(ii) *$\|\tilde{\Phi}_n(x) - x\|_{\mathcal{E}} \to 0$ for all $x \in \mathcal{H}_{\ell_\infty(\Gamma)}$.*

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