FINITE GROUPS OF MOTIONS

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Discussion of orthogonal groups and finite subgroups of $SO(3)$. Based on Artin, *Algebra*, Ch. 5.

1. Orthogonal and special orthogonal groups

The **orthogonal group** $O(n)$ is the set of $A \in GL_n(\mathbb{R})$ such that left multiplication preserves dot product: $Av \cdot Aw = v \cdot w$ for all $v, w \in \mathbb{R}^n$.

Proposition 1.1. $O(n) = \{ A \in GL_n(\mathbb{R}) \mid A^t A = I \}$.

The **special orthogonal group** $SO(n)$ is the subgroup of $O(n)$ elements with determinant 1. It is a subgroup of index 2, so $|O(n) : SO(n)| = 2$.

Example. $O(1) = \{\pm 1\}$.

Example. $SO(2)$ is the group of rotations of the plane, consisting of

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \qquad \theta \in [0, 2\pi),$$

rotation through angle θ.

Example. $O(2)$ is the group of rotations and reflections of the plane, consisting of R_θ as above, and

$$F_\theta = \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ -\sin 2\theta & -\cos 2\theta \end{pmatrix}, \qquad \theta \in [0, \pi),$$

reflection through the line at angle θ.

We have

$$R_\theta R_\eta = R_{\theta+\eta}, \qquad F_\theta F_\eta = R_{2(\theta-\eta)}, \qquad R_\theta F_\eta = F_{\eta+(\theta/2)} = F_\eta R_{-\theta}.$$

Example. $SO(3)$ and $O(3)$. The spectral theorem (aka principal axis theorem) implies that every non-identity element of $SO(3)$ is rotation around some axis, and every element of $O(3) - SO(3)$ is a either a reflection through a plane, or a composition of a reflection through a plane followed by rotation around the axis perpendicular to the plane.

Proposition 1.2. *Every finite subgroup of $SO(2)$ is isomorphic to C_n for some n. Every finite subgroup of $O(2)$ is isomorphic to either C_n or D_{2n}. (C_n is the cyclic group of order n, and D_{2n} is the dihedral group of order $2n$.)*

Note: in each case, there is only one subgroup isomorphic to some C_n, but there is an infinite collection of subgroups isomorphic to D_{2n} (all of which are conjugate to each other).

Note: the groups I am calling C_2 and D_2 are isomorphic, but not conjugate in $O(2)$.

Thus, we have really classified finite subgroups of $O(2)$ up to conjugacy.

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2. Finite subgroups of $SO(3)$

Proposition 2.1. *Every finite subgroup of $SO(3)$ is isomorphic to one of the following.*

1. C_n, *for some* $n \geq 1$.
2. D_{2n}, *for some* $n \geq 2$.
3. *The symmetry group of the tetrahedron (isomorphic to A_4).*
4. *The symmetry group of the octahedron/cube (isomorphic to S_4).*
5. *The symmetry group of the icosahedron/dodecahedron (isomorphic to A_5).*

This is actually a classification of finite subgroups up to conjugacy.

Given $G \leq SO(3)$, and let $n = |G|$. A **pole** of G is a unit vector x in $\mathbb{R}^3$ such that there exists a non-identity $g \in G$ such that $gx = x$. Let X be the set of poles.

1. Each non-identity element of G is a rotation, and thus contributes two poles $\{\pm x\}$ to X. Two elements of G share poles if and only if they are both rotations about the same axis.
2. If $x \in X$, the stabilizer subgroup G_x is the set of all rotations with axis $\{\pm x\}$, together with the identity. Thus G_x is cyclic of order $c_x \geq 2$.
3. The group G acts on X: if x is a pole (fixed by $g \in G$) and $h \in G$, then hx is a pole (fixed by $hgh^{-1} \in G$).
4. Let $O_1, \ldots, O_r$ be the orbits of G acting on X. Let $c_k = |G_x|$ for any $x \in O_k$, so
$$|O_k| = |G : G_x| = |G| \,/\, |G_x| = n/c_k.$$
For convenience, order things so that $2 \leq c_1 \leq c_2 \leq \cdots \leq c_r$.

Proof of finite subgroup classification.

1. Partition $G - \{1\}$ into subsets $G_x - \{1\}$, corresponding to pairs $\pm x$ of poles, giving
$$2(n-1) = \sum_{x \in X} |G_x - \{1\}|,$$
which reduces to
$$2 - \frac{2}{n} = \sum_k \left(1 - \frac{1}{c_k}\right).$$
2. See that $2 - 2/n \in [1, 2)$ and $1 - 1/c_k \in [1/2, 1)$, so $r \in \{2, 3\}$.
3. Show that if $r = 2$, then $c_1 = c_2 = n$ and G is cyclic.
4. With $r = 3$, formula becomes
$$1 + \frac{2}{n} = \frac{1}{c_1} + \frac{1}{c_2} + \frac{1}{c_3},$$
where $2 \leq c_1 \leq c_2 \leq c_3$. Show that $c_1 = 2$.
5. Show that if $c_2 = 2$, then $G \approx D_n$ and $c_3 = n/2$.
6. Show that if $c_1 = 2$ and $c_2 \geq 3$, there are only three possible solutions, which correspond to symmetry groups of the platonic solids.

$(c_1, c_2, c_3) = (2, 3, 3)$,	$n = 12$,	$6 + 4 + 4$ poles,
$(c_1, c_2, c_3) = (2, 3, 4)$,	$n = 24$,	$12 + 8 + 6$ poles,
$(c_1, c_2, c_3) = (2, 3, 5)$,	$n = 60$,	$30 + 20 + 12$ poles.

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