INFINITE DIHEDRAL GROUP

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Discussion of the solution to the last problem of PS6.

Let $R_a, F_a : \mathbb{R} \to \mathbb{R}$ be defined by

$$R_a(x) = x + a, \qquad F_a(x) = -(x - a) + a = -x + 2a.$$

(1) Show that $G = \{ R_n, F_{n/2} \mid n \in \mathbb{Z} \}$ is a group, under composition of functions.
(2) Give a generators and relations description of G. (Hint: Use only two generators.)
(3) There is an evident action of the group G on the set $\mathbb{R}$. For each $x \in \mathbb{R}$, describe the stabilizer group G_x.
(4) Classify all subgroups of G up to conjugacy, and determine which ones are normal. For all normal $N \trianglelefteq G$, describe the quotient group G/N.

The group G is called the **infinite dihedral group**.

In what follows, I write

$$(a + b\mathbb{Z}) \stackrel{\text{def}}{=} \{ a + bk \mid k \in \mathbb{Z} \} \subset \mathbb{R};$$

for $b > 0$, this is an infinite set of evenly spaced points in $\mathbb{R}$.

(1) To see that we get a group, we can compute in general:

$$R_a R_b = R_{a+b}, \qquad F_a F_b = R_{2(a-b)}, \quad R_a F_b = F_{b+a/2} = F_b R_{-a}.$$

If a and b are integers, so is $a + b$; if a is an integer and b is a half integer, $b + a/2$ is a half integer, and if a and b are half integers, then $2(a - b)$ is an integer. This proves G is closed under composition.

It is useful to compute formulas for conjugation:

$$R_a R_b R_{-a} = R_b, \qquad R_a F_b R_{-a} = F_{b+a}, \qquad F_a R_b F_a = R_{-b}, \qquad F_a F_b F_a = F_{2a-b}.$$

Note that R_a is "translation" of the line by a units, while F_b is "reflection" of the line through the point b. Thus G consists of translations by integer units, and reflection through integer and half-integer points. The group G is actually the symmetry group of the subset $\mathbb{Z} \subset \mathbb{R}$, under "rigid motions and reflections". It is *also* the symmetry group of $(\frac{1}{2} + \mathbb{Z})$; you can think of the sets $\mathbb{Z}$ and $(\frac{1}{2} + \mathbb{Z})$ as being "dual" to each other.

(2) Let $R = R_1$ and $F = F_0$. Then $G \approx \langle R, F \mid F^2 = 1, RF = FR^{-1} \rangle$. To see that R and F generate, note that $R^n = R_n$ and $R^n F = F_{n/2}$. So see that these relations suffices, let $H = \langle r, f \mid f^2 = 1, rf = fr^{-1} \rangle$. There is a homomorphism $\phi\colon H \to G$ by $\phi(r) = R$ and $\phi(f) = F$; it is well-defined because $\phi(f^2) = F^2 = 1$ and $\phi(rfrf^{-1}) = RFRF^{-1} = 1$. Since R and F generate G, the map ϕ is surjective. To see that ϕ is an isomorphism, note that any element in H can be written in the form $r^i f^j$ with $i \in \mathbb{Z}$ and $j \in \{0, 1\}$; this is because the relation $rf = fr^{-1}$ allows you to move all copies of r to the left. Since $\phi(r^i f^j) = R^i F^j$ is 1 only if $i = j = 0$ (assuming $j \in \{0, 1\}$), we see ϕ is injective.

(3) It is clear that if $R_a(x) = x + a = x$, then $a = 0$. If $F_a(x) = -x + 2a = x$, then $x = a$. Thus,
   (a) $G_x = 1$ if $x \in \mathbb{R} - \{\frac{1}{2}\mathbb{Z}\}$,
   (b) $G_x = \{1, F_x\}$ if $x \in \frac{1}{2}\mathbb{Z}$.

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(4) Here is a list of conjugacy classes of subgroups of G. The most interesting subgroup is T_1, the subgroup of *all* translations. We can organize the subgroups $H \le G$ according to what their intersection with T_1 looks like, together with the fact that $H/T_1 \cap H \le G/T_1 \approx \mathbb{Z}/2$. If $H \le T_1$, then we are in cases (a) or (b) below. If $H \not\le T_1$, and $H \cap T_1 = 1$, then we are in case (c) below. If $H \not\le T_1$ and $H \cap T_1 = T_m \ne 1$, then we are in case (d) below.

(a) 1, the trivial subgroup.

(b) $T_m = \{ R_{km} \mid k \in \mathbb{N} \}$ for each $m \ge 1$ (isomorphic to $\mathbb{Z}$). None of these are conjugate to each other, all are normal. These subgroups correspond to translations by multiples of the integer m. This is the group of translation symmetries of the subset $m\mathbb{Z} \subset \mathbb{R}$ (or more generally, of any subset $(a + m\mathbb{Z})$ with $a \in \mathbb{R}$).

(c) $U_m = \{1, F_{m/2}\}$ for each $m \in \mathbb{Z}$ (isomorphic to $\mathbb{Z}/2$). These come in two conjugacy clases, corresponding to even m and odd m, because $R_n F_{m/2} R_{-n} = F_{m/2+n}$, and $F_{n/2} F_{m/2} F_{n/2} = F_{n-m/2}$. These conjugacy classes correspond to: reflection through an integer, and reflection through a half-integer. None of these subgroups are normal.

(d) $V_{m,i} = \{ R_{km}, F_{(km+i)/2} \mid k \in \mathbb{Z} \}$, where $m \ge 1$ and $i \in \{0, 1, \ldots, m-1\}$. (Note that $V_{m,i} = V_{m,j}$ if $i \equiv j \mod m$.) The group $V_{m,i}$ is the subgroup of G which are symmetries of $(i/2 + m\mathbb{Z})$, and also symmetries of the "dual" set $(i/2 + m/2 + m\mathbb{Z})$. The groups $V_{m,i}$ are all isomorphic to G (use $r = R_m$ and $f = F_{i/2}$). We can compute

$$R_n V_{m,i} R_{-n} = V_{m,i+2n}, \qquad F_{n/2} V_{m,i} F_{n/2} = V_{m,2n-i}.$$

So, the conjugacy classes are organized as follows.

(i) If m is even, then there are two conjugacy classes of groups $V_{m,i}$, corresponding to whether $i \in \{0, 1, \ldots, m-1\}$ is even or odd.

(ii) If m is odd, then there is a single conjugacy class of groups $V_{m,i}$. (Because $V_{m,i} = V_{m,m+i} = R_{(m-1)/2} V_{m,i+1} R_{-(m-1)/2}$, since $(m-1)/2$ is an integer if m is odd.)

Thus, the subgroups $G = V_{1,0}$ and $V_{2,0}$ and $V_{2,1}$ are normal; the other $V_{m,i}$s are not normal.

For the normal subgroups, we have the following.

(a) $1 \trianglelefteq G$, and $G/1 \approx G$.

(b) $T_m \trianglelefteq G$, and $G/T_m \approx D_{2m}$.

(c) $V_{2,i} \trianglelefteq G$ for $i = 0, 1$, and $G/V_{2,i} \approx \mathbb{Z}/2$.

(d) $V_{1,0} = G \trianglelefteq G$, and $G/G \approx 1$.

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