FREE GROUPS

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Construction of the free group, different than the one given in D&F VI.3.

1. Monoids

A **monoid** is a set M equipped with a function $(x, y) \mapsto x \cdot y \colon M \times M \to M$ such that

(1) $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ for all $x, y, z \in M$, and
(2) there exists an element $1 \in M$ such that $1 \cdot x = x = x \cdot 1$ for all $x \in M$.

A **homomorphism** of monoids is a function $\phi \colon M \to N$ such that

(1) $\phi(x \cdot y) = \phi(x) \cdot \phi(y)$ for all $x, y \in M$, and
(2) $\phi(1) = 1$.

If M and N are groups, then a homomorphism of monoids $\phi \colon M \to N$ is the same thing as a homomorphism of groups. (If N is a group, then condition (2) is automatically satisfied by any function which satisfies condition (1).)

2. Free monoids

Let S be a set. Let

$$F_{\text{mon}}S = \{\, (a_1, \dots, a_r) \mid a_i \in S, r \geq 0 \,\},$$

be the set of all finite lists of elements of S. Note that there is an element $() \in F_{\text{mon}}S$, the empty list; we write 1 for this element.

This is a monoid, by concatenation of lists:

$$(a_1, \dots, a_r) \cdot (b_1, \dots, b_s) = (a_1, \dots, a_r, b_1, \dots, b_s).$$

It is associative. The element $1 = ()$ is a unit. The monoid $F_{\text{mon}}S$ is called the **free monoid** on S. Write $i \colon S \to F_{\text{mon}}S$ for the inclusion $a \mapsto (a)$; sometimes we will omit the notation for i, and think of S as a subset of $F_{\text{mon}}S$.

Examples. $F_{\text{mon}}\{x\} \approx \mathbb{N}$. $F_{\text{mon}}\{x, y\} = \{1, x, y, x^2, xy, yx, y^2, \dots\}$.

Proposition 2.1. *Let M be a monoid. Given a set function $\phi \colon S \to M$, there exists a unique monoid homomorphism $\Phi \colon F_{\text{mon}}S \to M$ such that $\Phi((a)) = \phi(a)$.*

$$\begin{tikzcd}
S \arrow[r, "i"] \arrow[dr, "\phi"'] & F_{\text{mon}}S \arrow[d, dotted, "\Phi"] \\
& M
\end{tikzcd}$$

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3. Equivalence relation generated by a relation

Let X be a set, and $T \subseteq X \times X$. The **equivalence relation generated by** T is

$$R = \bigcap R_\alpha,$$

where the intersection runs over all equivalence relations R_α on X such that $T \subseteq R_\alpha$.

Proposition 3.1. *Given $T \subseteq X \times X$, let $R \subseteq X \times X$ be the set of $(a,b) \in X \times X$ such that either*

1. *$a = b$, or*
2. *there exists a sequence $a = x_0, x_1, \ldots, x_r = b$ of elements of X, such that for all $i = 1, \ldots, r$ either $(x_{i-1}, x_i) \in T$ or $(x_i, x_{i-1}) \in T$.*

Then R is the equivalence relation generated by T.

4. Construction of the free group

Given a set S, let S' be a duplicate copy of S. To each element $s \in S$, there is a corresponding element $s' \in S'$. The function $s \mapsto s'$ defines a bijection $S \to S'$. It will also be convenient to write $t \mapsto t'$ for the inverse $S' \to S$. Thus $s'' = s$ for all $s \in S \amalg S'$.

Let $X = F_{\mathrm{mon}}(S \amalg S')$. Let $T \subset X \times X$ denote the set of pairs of the form

$$(W_1W_2, W_1aa'W_2), \qquad (W_1W_2, W_1a'aW_2)$$

where $W_1, W_2 \in X$, $a \in S$. Let R be the equivalence relation generated by T.

Proposition 4.1. *If $u, v, w \in X$ such that $u \sim_R v$, then $uw \sim_R vw$ and $wu \sim_R wv$.*

Let $FS = X/R$, the set of equivalence classes of R; write $[W]$ for the equivalence class containing $W \in X$. The monoid structure on X thus descends to a monoid structure on FS, by the rule $[U][V] = [UV]$, so that the quotient map $\pi\colon X \to FS$ is a homomorphism of monoids.

Proposition 4.2. *Every element of FS has an inverse; that is, FS is a group.*

Let $j\colon S \to FS$ denote the map

$$j(s) = [i(s)].$$

Proposition 4.3. *Let G be a group. Given a set function $\phi\colon S \to G$, there exists a unique group homomorphism $\Phi\colon FS \to M$ such that $\Phi(i(a)) = \phi(a)$.*

$$\begin{tikzcd}
S \arrow[r, "j"] \arrow[dr, "\phi"'] & FS \arrow[d, dotted, "\Phi"] \\
& G
\end{tikzcd}$$

5. Reduced words

A **reduced word** in $X = F_{\mathrm{mon}}(S \amalg S')$ is a word $(x_1, \ldots, x_r)$ such that there is no $i = 1, \ldots, r$ with $x_i = s$ and $x_{i+1} = s'$, for some $s \in S \amalg S'$.

Proposition 5.1. *Each equivalence class of R in $X = F_{\mathrm{mon}}(S \amalg S')$ contains exactly one reduced word.*

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