Methods of Mathematical Physics - 556 X1
Homework 3
Due October 10, 2008

1. (Problem 2.1.1 from Keener.) Verify that ℓ^2 is an inner product space. Specifically, show that if $x, y \in \ell^2$, then
$$\langle x, y \rangle = \sum_{k=0}^{\infty} x_k y_k$$
is defined and satisfies the properties of an inner product. (Here we're assuming that our sequences are real, so no need for the complex conjugate.)

   Hint: Think about how we proved Bessel's Inequality in class.

2. (Problem 2.1.3 from Keener.) Show that the sequence $x_n = \sum_{k=1}^{n} \frac{1}{k!}$ is a Cauchy sequence. Since the reals are complete, this means it converges. To which number does this sequence converge?

3. Show that the sequence $x_n = \sum_{k=1}^{n} \frac{1}{k}$ is not a Cauchy sequence.

4. Consider the Hilbert space L^2. Prove that the list of vectors
$$\{\cos(nx)\}_{n=0}^{\infty} \cup \{\sin(nx)\}_{n=1}^{\infty}$$
is an infinite orthogonal list in L^2 with respect to the inner product
$$\langle f, g \rangle = \int_0^{2\pi} f(x)g(x)\,dx.$$

   Hint: You will need some trigonometric identities to solve this problem, e.g. you will need to compute integrals like
$$\int_0^{2\pi} \cos(mx)\cos(nx)\,dx.$$
Recall that we can use Euler's formula to get, for example,
$$\begin{aligned}\cos((m+n)x) &= \mathrm{Re}(e^{i(m+n)x}) = \mathrm{Re}(e^{imx}e^{inx})\\ &= \mathrm{Re}((\cos(mx) + i\sin(mx))(\cos(nx) + i\sin(nx)) = \cos(mx)\cos(nx) - \sin(mx)\sin(nx).\end{aligned}$$
If you recombine these formulas in a clever way, you can do all of the integrals.

5. (Problem 2.2.1 from Keener.) Find the best quadratic polynomial fit to the function $f(x) = |x|$, where we choose as inner product
$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\omega(x)\,dx,$$
for each of the weights $\omega(x) = 1, \sqrt{1-x^2}, (1-x^2)^{-1/2}$.

   Hint: You might find it convenient to compute some orthogonal polynomials for each weight and then compute the answer in terms of these polynomials — and we already did most of the work here on the last homework!

6. (Problem 2.2.9 from Keener.) Suppose that $\{\phi_n(x)\}_{n=0}^{\infty}$ is a set of orthonormal polynomials, where we choose the inner product
$$\langle f, g \rangle = \int_a^b f(x)g(x)\omega(x)\,dx,$$
($\omega(x) > 0$) and assume that $\phi_n(x)$ is a polynomial of degree n with leading coefficient k_n (specifically, we mean that
$$\phi_n(x) = k_n x^n + (\text{terms of power } n-1 \text{ or less}).$$
Then show:

(a) If f is a polynomial of degree less than n, then $\langle \phi_n, f \rangle = 0$.

(b) Show that every polynomial of degree n can be written in the form

$$\sum_{i=0}^{n} \alpha_i \phi_i$$

for some numbers α_i.

(c) the polynomials satisfy a recurrence relation of the form

$$\phi_{n+1}(x) = (A_n x + B_n)\phi_n(x) - C_n \phi_{n-1}(x),$$

for every n, where $A_n = k_{n+1}/k_n$. Compute B_n, C_n in terms of A_n, A_{n-1}, ϕ_n.

Hint: What do we know about $\phi_{n+1}(x) - A_n x \phi_n(x)$? Use part (b), take the inner product with ϕ_j, what do you get? Also, notice that for this inner product, $\langle xf, g \rangle = \langle f, xg \rangle$.

7. (Problem 2.2.10 from Keener.) Consider the inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\omega(x)\, dx$$

($\omega(x) > 0$).

(a) Show that

$$P_n(x) := \frac{1}{\omega(x)} \frac{d^n}{dx^n} \left(\omega(x)(1 - x^2)^n \right)$$

is orthogonal to every polynomial of degree less than n.

(b) Show then that the functions

$$\phi_n(x) := \frac{P_n(x)}{\|P_n(x)\|^2} = \frac{\frac{1}{\omega(x)} \frac{d^n}{dx^n} \left(\omega(x)(1 - x^2)^2 \right)}{\int_{-1}^{1} \frac{1}{\omega(x)} \left(\frac{d^n}{dx^n} \left(\omega(x)(1 - x^2)^2 \right) \right)^2 dx}$$

form an orthonormal set.

Hint: for part (a), we proved this in class when $\omega \equiv 1$. Adapt that argument to this case.

8. (Problem 2.2.14 from Keener.) Suppose that $f(t)$ and $g(t)$ are 2π-periodic functions with Fourier series representations

$$f(t) = \sum_{k=-\infty}^{\infty} f_k e^{ikt}, \quad g(t) = \sum_{k=-\infty}^{\infty} g_k e^{ikt}.$$

Now define

$$h(t) = \int_{0}^{2\pi} f(t - x)g(x)\, dx.$$

Compute the Fourier series for h.