Methods of Mathematical Physics - 556 X1
Homework 2 - Solutions

1. *Recall that we define the orthogonal complement as in class: If S is a vector space, and T is a subspace, then we define the* **orthogonal complement** *of T:*

$$T^{\perp} := \{y \in S : \langle x, y \rangle = 0 \text{ for all } x \in T\}.$$

To get warmed up, we'll consider a few concrete examples.

(a) *First let $S = \mathbb{R}^3$ and let T be the (x, y)-plane. What is $T^{\perp}$? What is its dimension?*

(b) *Now let T be any plane containing the origin. What is $T^{\perp}$? What is its dimension?*

(c) *Let T be any line in $\mathbb{R}^3$ through the origin. What is $T^{\perp}$?*

(d) *Again let T be any plane in $\mathbb{R}^3$ containing the origin. What is $(T^{\perp})^{\perp}$?*

Solution.

(a) The (x, y)-plane can be parametrized by vectors of the form $(x, y, 0)$, where $x, y \in \mathbb{R}$. It is easy to see that any vector of the form $(0, 0, z)$ will be orthogonal to all of these vectors. To prove that this is the only possibility, consider the following equation:

$$0 = \langle (x, y, 0), (\alpha, \beta, \gamma) \rangle = \alpha x + \beta y$$

Now, we need this to be zero *for all* choices of x and y. If $\alpha \neq 0$, then notice by choosing $x = 1, y = 0$, we get a nonzero answer, so $\alpha = 0$, same argument for β. Therefore $\alpha = \beta = 0$, and therefore $T^{\perp}$ is the z-axis.

(b) We know from calculus that any plane containing the origin can be parametrized by the equation

$$\alpha x + \beta y + \gamma z = 0,$$

where (α, β, γ) is a normal vector to the plane. However, using the language of inner products, this saying that we are looking (x, y, z) so that

$$\langle (\alpha, \beta, \gamma), (x, y, z) \rangle = 0,$$

which means that (x, y, z) is in the plane if and only if it is orthogonal to (α, β, γ). Therefore the line generated by the vector (α, β, γ) is $T^{\perp}$.

(c) Consider a line through the origin and choose any vector lying along the line, let's call it (α, β, γ). By definition, $T^{\perp}$ is the set of all solutions to

$$\langle (\alpha, \beta, \gamma), (x, y, z) \rangle = 0$$

but this is just the equation for a plane whose normal vector is (α, β, γ).

(d) By part (b), $T^{\perp}$ is the line containing the normal vector to T, and according to (c), the orthogonal complement of a line is the plane whose normal vector is contained in the line, therefore $(T^{\perp})^{\perp} = T$.

2. *Now we prove these things in general. Let S be a n-dimensional vector space, and T a k-dimensional subspace. Prove*

(a) *$T^{\perp}$ is also a subspace of S,*

(b) *$\dim(T^{\perp}) = n - k$,*

(c) *$(T^{\perp})^{\perp} = T$.*

Solution.

(a) We need to show that if $x, y \in T^{\perp}$, then so is $\alpha x + \beta y$ for all α, b. But, by definition, we know that

$$\langle x, z \rangle = \langle y, z \rangle = 0, \text{ for all } z \in T.$$

Then we know, for any $z \in T$,

$$\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle = 0,$$

and thus $\alpha x + \beta y \in T^{\perp}$ as well.

(b) The easiest way to first assume that we have an orthonormal basis for S, some of whose vectors are also an orthonormal basis for T. Specifically, we mean that we have $\{v_1, v_2, \ldots, v_n\}$ as an orthonormal basis for S, and $\{v_1, v_2, \ldots, v_k\}$ is an orthonormal basis for T. (We will show later that we can always construct such an object.)

So we claim that $\{v_{k+1}, \ldots, v_n\}$ is a basis for $T^{\perp}$. Since there are clearly $n-k$ vectors in this list, we are done. To see this, choose any $y \in T^{\perp}$. Since $\{v_1, \ldots, v_n\}$ form a basis, we know that we can write

$$y = \sum_{i=1}^{n} \alpha_i v_i.$$

Since the v form an orthonormal basis, we know that

$$\langle y, v_j \rangle = \alpha_j,$$

but we also know that for all $i = 1, \ldots, k$, y is orthogonal to v_i, which means $\alpha_i = 0$. Therefore we can write

$$y = \sum_{i=k+1}^{n} \alpha_i v_i$$

for any $y \in T^{\perp}$, and we are done.

Now, to show we can construct such a basis, simply do the following. Pick any basis for T, and make it orthonormal with Gram-Schmidt, so we now have an orthonormal basis for T, namely $\{v_1, \ldots, v_k\}$. What we now do is extend this basis to form a basis for S, making sure to keep it orthonormal at each step. For example, we start by choosing any $\phi_{k+1} \in S$ which is not in the span of $\{v_1, \ldots, v_k\}$. Using Gram-Schmidt, we can make this orthogonal to all of the $v_1, \ldots, v_k$, and call the new vector v_{k+1}. Now, this is either a basis for S, or it is not. If it is, we are done, but if not, just continue the procedure: if we don't yet have a basis for S, add a vector not in the span of the vectors we already have (and we know it must exist, since we haven't spanned S yet), make it orthogonal to all of the previous vectors, and repeat. We know this must terminate, since we cannot have a list of linearly independent vectors longer than n.

(c) What is $(T^{\perp})^{\perp}$, in words? It is the set of all vectors orthogonal to every vector in $T^{\perp}$, i.e.

$$(T^{\perp})^{\perp} = \{z \in S : \langle z, y \rangle = 0 \text{ for all } y \in T^{\perp}\}.$$

However, notice that if $z \in T$, then $\langle z, y \rangle = 0$ for all y in $T^{\perp}$ (by the definition of $T^{\perp}$) and therefore $z \in (T^{\perp})^{\perp}$. Therefore, $T \subseteq (T^{\perp})^{\perp}$. All we need to show is that $(T^{\perp})^{\perp} \subseteq T$, and we are done. But consider part (b). We know that $\dim(T^{\perp}) = n-k$, and thus $\dim((T^{\perp})^{\perp}) = n-(n-k) = k$. But since T and $(T^{\perp})^{\perp}$ have the same dimension, and one is a subset of the other, they must both be equal. To see this, for example, choose a basis for T. This is a linearly independent list of vectors of length k. Since $(T^{\perp})^{\perp}$ is also k-dimensional, we cannot add another vector from $(T^{\perp})^{\perp}$ to this list and make it still linearly independent, so therefore this list spans $(T^{\perp})^{\perp}$ as well. But then this means that this list is a basis for $(T^{\perp})^{\perp}$ as well, and since the two subspaces have the same basis they must be equal.

Note: This proof does not work in infinite dimensions!

3. We stated the Fredholm Alternative in class as $R(A) = N(A^*)^\perp$. Show that the following three other statements are logically equivalent to this one:

   (a) $N(A) = R(A^*)^\perp$,
   (b) $R(A^*) = N(A)^\perp$,
   (c) $N(A^*) = R(A)^\perp$.

   Solution. Take the orthogonal complement of both sides, and we have

$$R(A)^\perp = (N(A^*)^\perp)^\perp = N(A^*).$$

   This establishes (c). Now go back to the original equation and replace A with A^*, giving

$$R(A^*) = (N((A^*)^*)^\perp = (N(A))^\perp,$$

   giving (b). Now, go back to the original, and do both at once, giving

$$R(A^*)^\perp = (N((A^*)^*))^\perp)^\perp = N(A),$$

   giving (a).

4. *Let us say that we are observing two experiments simultaneously, and we collected the following six pairs of data points:*

$$(5, 3.4), (6, 2.1), (9, 4), (1, 3.5), (7, 11), (5, 6.3),$$

   where the first number corresponds to the results of the first experiment, etc.

   (a) Assume that there is a linear relationship between the first measurement x and the second measurement y, namely that $y = \alpha x + \beta$. Compute the least-squares best approximation for (α, β). Compute the total error made by this approximation.
   (b) Now assume that there is a quadratic relationship between y and x, namely that $y = \alpha x^2 + \beta x + \gamma$. Compute the least-squares best approximation for (α, β, γ). Compute the total error made by this approximation. Is this better than the previous guess? Why should (or shouldn't) it be?

   Solution.

   As a check, we've put the data into Matlab and asked for the best linear and polynomial fits, see Figure 1.

   (a) In this case, we can simply write out and solve the normal equation directly. We are trying to minimize $\|Ax - b\|$, where

$$A = \begin{pmatrix} 5 & 1 \\ 6 & 1 \\ 9 & 1 \\ 1 & 1 \\ 7 & 1 \\ 5 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 3.4 \\ 2.1 \\ 4 \\ 3.5 \\ 11 \\ 6.3 \end{pmatrix}.$$

   We compute

$$A^*A = \begin{pmatrix} 217 & 33 \\ 33 & 6 \end{pmatrix}, \quad A^*b = \begin{pmatrix} 177.6 \\ 30.3 \end{pmatrix}$$

   Notice that no matter how many data points we have, A^*A will always have the following form: the top left corner of A^*A is the sum of the squares of the data in the first slot, the two off-diagonal elements are the sum of the data, and the bottom right is the number of data points. As for A^*b, the top element will be the sum $\sum x_i y_i$ and the bottom will be $\sum y_i$.
   Solving $A^*A(\alpha, \beta)^T = A^*b$ to three decimals gives

$$\alpha = 0.3085, \quad \beta = 3.354.$$

   So our linear regression suggests $0.3085x + 3.354$ as the best linear fit to the data.

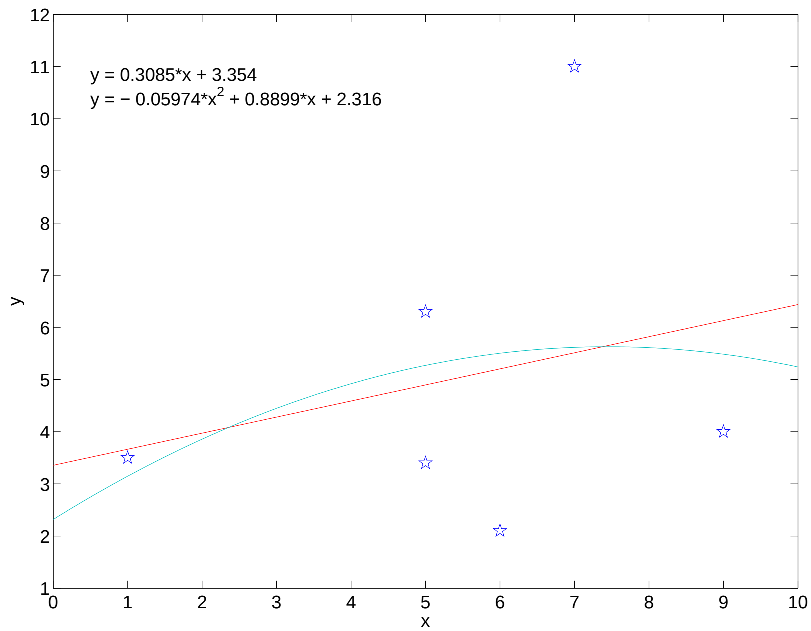

Figure 1: Here we have plotted the data in Matlab and computed the best linear and quadratic fits. Matlab agrees with our computations

(b) We could do this one directly as well, but it will be tedious. Let us compute the Moore-Penrose pseudo-inverse A', if it exists then the least-squares solution will be $A'b$. (A computer will be useful now.)

So we are minimizing $A(\alpha, \beta, \gamma)^T = b$, where

$$A = \begin{pmatrix} 25 & 5 & 1 \\ 36 & 6 & 1 \\ 81 & 9 & 1 \\ 1 & 1 & 1 \\ 49 & 7 & 1 \\ 25 & 5 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 3.4 \\ 2.1 \\ 4 \\ 3.5 \\ 11 \\ 6.3 \end{pmatrix}.$$

We compute

$$A^*A = \begin{pmatrix} 11509 & 1539 & 217 \\ 1539 & 217 & 33 \\ 217 & 33 & 6 \end{pmatrix},$$

and computing

$$(A^*A)^{-1}A^*y = (-0.05974, 0.8899, 2.316)^T.$$

So the best quadratic fit is

$$-0.05974x^2 + 0.8899x + 2.316.$$

5. *(Problem 1.5.2 from Keener.) Let D be an $m \times n$ matrix with entries $d_{ij} = \sigma_i \delta_{ij}$. That is, all of the off-diagonal entries of D are zero. Let D' be the least-squares pseudo-inverse of D. Show that D' is a $n \times m$ matrix whose entries are given by $(D')_{ij} = \sigma_i^{-1}\delta_{ij}$ whenever $\sigma_i \neq 0$, and $(D')_{ij} = 0$ otherwise.*

Solution. By definition, D' is the matrix for which $x = D'b$ always satisfies

(a) $D^*Dx = D^*b$,

(b) $\langle x, w\rangle = 0$ when $w \in N(D)$.

First we characterize $w \in N(D)$. Define e_j to be the vector with a 1 in the jth slot and 0 elsewhere. Then $De_j = 0$ if and only if $D_{jj} = 0$. If, for example, all $D_{jj} \neq 0$, then $N(D) = \{0\}$.

Now, let us compute D^*D. By definition,

$$(D^*D)_{ij} = \sum_{k=1}^{m} D^*_{ik} D_{kj} = \sum_{k=1}^{m} \delta_{ik} D_{ki} \delta_{kj} D_{kj}.$$

If $i, j \leq n$, this last term becomes $\delta_{ij}\sigma_i^2$, and if either $i > n, j > n$, then this term is zero. D^*D is always an $n \times n$ matrix, and if we choose $m > n$, then D^*D is an $n \times n$ matrix whose diagonal entries are exactly σ_i^2. If we choose $m < n$, however, D^*D will be have diagonal entries equal to σ_i^2 for $i = 1, \dots, m$, but after this will simply have zeros.

As a concrete example, imagine that we choose

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix},$$

then

$$D^*D = \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix},$$

whereas if we had chosen

$$D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \\ 0 & 0 \end{pmatrix},$$

then

$$D^*D = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In either case, D^*D is diagonal and the entries are squares of the diagonal entries of D, but if the sizes are not right we have have zero entries in D^*D even if all the diagonal entries of D are nonzero. (Of course, if D is square then everything is simpler.)

So let us assume that $m \geq n$ and the diagonal entries of D are nonzero. Then D^*D is an $n \times n$ diagonal matrix with nonzero entries $(D^*D)_{ii} = D_{ii}^2$, so that D^*D is invertible and $D' = (D^*D)^{-1}D^*$, which we compute

$$D'_{ij} = \sum_{k=1}^{n} (D^*D)^{-1}_{ik} (D^*)_{kj} = \delta_{ik}\sigma_i^{-2}\delta_{kj}\sigma_j = \delta_{ij}\sigma_i^{-1},$$

and we are done.

But what if D^*D is not invertible? This can happen one of two ways: we could have $m < n$, or some of the diagonal entries of D are zero. So we compute: Let $b \in \mathbb{R}^m$ be the vector

$$b = (b_1, b_2, \dots, b_m),$$

then

$$(D^*b)_j = \sum_{k=1}^{m} (D^*)_{ik} b_k = \sum_{k=1}^{m} \delta_{ik} D_{ki} b_k = D_{ii} b_i,$$

so that

$$D^*b = (\sigma_1 b_1, \sigma_2 b_2, \dots, \sigma_m b_m).$$

On the other hand, we know that the diagonal entries of D^*D are σ_i^2, so

$$D^*Dx = (\sigma_1^2 x_1, \sigma_2^2 x_2, \ldots, s_n^2 x_n),$$

with the standard caveat that if $n > m$ then $\sigma_k = 0$ for $i = m+1, \ldots, n$. If we set these equal by part (a) of the definition of pseudo-inverse, we see that computing term by term we get that

$$\sigma_j^2 x_j = \sigma_j b_j$$

for $j = 1, \ldots, \min(m, n)$. From here we see the map which sends b to x must be diagonal, since x_j only depends on b_j. If $\sigma_j \neq 0$, then clearly we need to choose $x_j = \sigma_j^{-1} b_j$. Therefore if $\sigma_j \neq 0$, then $D'_{jj} = \sigma_j^{-1}$. However, if $\sigma_j = 0$, it seems as those we're free to choose x_j, since any x_j will work in the above equation.

However, if we now impose condition (b), we know that if $\sigma_j = 0$, then $De_j = 0$, and therefore $\langle x, e_j \rangle = 0$, which means the jth component of x must be zero. But this means that $D'_{jj} = 0$, and we are done.

6. *Let us consider as a vector space $\mathcal{P}_2(\mathbb{R})$, that is, all polynomials with real coefficients with degree less than or equal to 2. We define the inner product*

$$\langle f, g \rangle = \int_0^1 f(x)g(x)\,dx.$$

Now, consider the function L given by $Lf(x) = f'(x)$. Note that L is a map from $\mathcal{P}_2(\mathbb{R})$ into itself.

(a) *Prove that L is a linear map.*

(b) *Choose the "monomial basis" $\{1, x, x^2\}$, and write the (3×3) matrix representing L with respect to this basis. Recall, we say that the matrix A with entries a_{ij} represents an operator with respect to the basis $\{\phi_1, \ldots, \phi_n\}$ if $L\phi_i = \sum_j a_{ij}\phi_j$ for all i.*

(c) *Describe what the matrix for derivative would look like in $\mathcal{P}_n(\mathbb{R})$.*

(d) *Back to $\mathcal{P}_2(\mathbb{R})$. Orthogonalize the monomial basis using the Gram-Schmidt process. Write the matrix for L with respect to this basis.*

(e) *Compute L^*.*

Hint: *The best way to do this is to pick a basis (let's say the standard monomial basis $\{1, x, x^2\}$. Then if we know that L^*1, L^*x, L^*x^2 are, we are in business. But we know that*

$$\langle L^*x, y \rangle = \langle x, Ly \rangle .$$

Compute the nine numbers $C_{ij} := \left\langle x^i, Lx^j \right\rangle$, $i, j = 0, 1, 2$. Now, to compute L^*1, write it in the form $L^*1 = \alpha + \beta x + \gamma x^2$, and we just need to find the coefficients α, β, γ. But then we know $\left\langle L^*1, x^j \right\rangle = C_{0j}$ for all j. Writing this out gives us three equations in three unknowns, and we can solve this. Do the same for L^*x, L^*x^2.

Solution.

(a) $L(\alpha f + \beta g) = (\alpha f + \beta g)' = \alpha f' + \beta g' = \alpha Lf + \beta Lg.$

(b) We will use the vector $(\alpha, \beta, \gamma) \in \mathbb{R}^3$ to denote the polynomial $\alpha + \beta x + \gamma x^2$. So we have

$$L(1,0,0) = (0,0,0),$$
$$L(0,1,0) = (1,0,0),$$
$$L(0,0,1) = (0,2,0).$$

Thus the matrix for L in this basis will be

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

(c) We can generalize this example. In $\mathcal{P}_n$, we will use as a basis $\{1, x, x^2, \ldots, x^n\}$. But the effect of L is to shift one power down and multiply by the power, so that each column will have exactly one non-zero entry, directly above the diagonal, and this entry will be n, i.e. the matrix looks like

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & \ldots & 0 & 0 \\ & & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & 0 & 0 & \ldots & n-1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 & n \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \end{pmatrix}$$

(d) We compute

$$\left\langle x^i, Lx^j \right\rangle = \int_0^1 x^i j x^{j-1}\, dx = \frac{j}{i+j}.$$

So

$$\langle 1, L1 \rangle = 0, \langle 1, Lx \rangle = \left\langle 1, Lx^2 \right\rangle = 1.$$

If we write $L^*1 = a + bx + cx^2$, then we have

$$\begin{aligned} 0 &= \langle L^*1, 1 \rangle = a + \frac{b}{2} + \frac{c}{3}, \\ 1 &= \langle L^*1, x \rangle = \frac{a}{2} + \frac{b}{3} + \frac{c}{4}, \\ 1 &= \left\langle L^*1, x^2 \right\rangle = \frac{a}{3} + \frac{b}{4} + \frac{c}{5}, \end{aligned}$$

and solving for this gives $a = -6, b = 12, c = 0$, so

$$L^*1 = 12x - 6.$$

From this already we see that the first column of L^* in this basis will be $(-6, 12, 0)^T$, which is not the first row of L, so this will not be the conjugate transpose.

We now compute

$$\langle x, L1 \rangle = 0, \langle x, Lx \rangle = \frac{1}{2}, \left\langle x, Lx^2 \right\rangle = \frac{2}{3}.$$

Again writing $L^*x = a + bx + cx^2$ gives

$$\begin{aligned} 0 &= \langle L^*x, 1 \rangle = a + \frac{b}{2} + \frac{c}{3}, \\ \frac{1}{2} &= \langle L^*x, x \rangle = \frac{a}{2} + \frac{b}{3} + \frac{c}{4}, \\ \frac{2}{3} &= \left\langle L^*x, x^2 \right\rangle = \frac{a}{3} + \frac{b}{4} + \frac{c}{5}, \end{aligned}$$

which gives $a = 2, b = -24, c = 30$, or

$$L^*x = 2 - 24x + 30x^2.$$

Finally, we have

$$\left\langle x^2, L1 \right\rangle = 0, \left\langle x^2, Lx \right\rangle = \frac{1}{3}, \left\langle x^2, Lx^2 \right\rangle = \frac{2}{4}.$$

Again writing $L^*x = a + bx + cx^2$ gives

$$\begin{aligned} 0 &= \langle L^*x, 1\rangle = a + \frac{b}{2} + \frac{c}{3}, \\ \frac{1}{3} &= \langle L^*x, x\rangle = \frac{a}{2} + \frac{b}{3} + \frac{c}{4}, \\ \frac{2}{4} &= \left\langle L^*x, x^2\right\rangle = \frac{a}{3} + \frac{b}{4} + \frac{c}{5}, \end{aligned}$$

which gives $a = 3, b = -26, c = 30$, or

$$L^*x^2 = 3 - 26x + 30x^2.$$

Since L^* is linear, we have

$$\begin{aligned} L^*(a + bx + cx^2) &= a(12x - 6) + b(2 - 24x + 30x^2) + c(-3/2 - 7x + 15x^2) \\ &= (-6a + 2b + 3c) + (12a - 24b - 26c)x + (30b + 30c)x^2. \end{aligned}$$

Or we could write the matrix for L^* in this basis as

$$\begin{pmatrix} -6 & 2 & 3 \\ 12 & -24 & -26 \\ 0 & 30 & 30 \end{pmatrix}$$

That was a lot of arithmetic, to check that it is true, we need to check that

$$\left\langle L^*(a + bx + cx^2), \alpha + \beta x + \gamma x^2\right\rangle = \left\langle a + bx + cx^2, L(\alpha + \beta x + \gamma x^2)\right\rangle$$

for all $a, b, c, \alpha, \beta, \gamma$, but direct computation shows both sides are equal to

$$a\beta + a\gamma + \frac{b\beta + c\gamma}{2} + \frac{c\beta + 2c\gamma}{3}.$$

7. **Bonus.** *Now that we know L^*, compute the matrix with respect to the basis $\{1, x, x^2\}$. You will note that this is* not *the conjugate transpose of the matrix we calculated above. Explain why it isn't, specifically, how the proof we showed in class does not apply here.*

   Solution. We have computed the matrix above, and it is certainly not the conjugate transpose of the matrix for L. The reason the proof we use in class does not apply here is that the basis we've chosen is not orthonormal with respect to this inner product. If it were, the matrix for L^* would be the transpose of that for L.

8. **Extra special bonus.** *We saw above that the matrix for L^* is not the conjugate transpose of the matrix for L. Can we fix this by keeping the monomial basis and changing the inner product?*

   (a) It is possible to define an inner product on $\mathcal{P}_2(\mathbb{R})$ for which the matrix of L^* equals the conjugate transpose of the matrix of L? What properties must such an inner product have?

   (b) Is it possible to find one which is defined by an inner product of the form

   $$\langle f, g\rangle = \int_L^R f(x)g(x)\,dx$$

   for some real L, R? Why or why not?

(c) Is it possible to find one defined by an inner product of the form

$$\langle f, g \rangle = \int_0^1 f(x)g(x)\omega(x)\,dx$$

where $\omega(x)$ is a positive continuous function on $[0,1]$? If so, determine what properties $\omega(x)$ must have, and give an example of such an $\omega(x)$.

Solution.

(a) We need to find an inner product and a basis such that the basis is orthonormal, and then we can do this. Certainly, there exist such inner products (pick any inner product you like, and make a basis orthonormal). However, we can also work the other way, and choose an inner product such that the monomials are orthonormal, and the way we do that is just to require that $\langle x^i, x^j \rangle = \delta_{ij}$. For example, this would then give

$$\langle a + bx + cx^2, \alpha + \beta x + \gamma x^2 \rangle = a\alpha + b\beta + c\gamma.$$

(b) No, it's not. We can check that if we want $\langle 1, x \rangle = \langle 1, x^2 \rangle = \langle x, x^3 \rangle = 0$, then we need to find L, R such that

$$\int_L^R x\,dx = \int_L^R x^2\,dx \int_L^R x^3\,dx = 0,$$

but this gives

$$R^2 - L^2 = R^3 - L^3 = R^4 - L^4 = 0.$$

The first equation gives $R = \pm L$, and of course we do not choose $R = L$, since then the inner product is always zero, so we need $R = -L$. But then the second equation then becomes $2R^3 = 0$, or $R = 0$. So there is no solution.

(c) However, we might be able to get around this problem by assigning a weight to the integral. However, there is no way to make the monomials into an orthogonal basis. Notice that

$$\langle x^2, x^2 \rangle = \int_0^1 x^4\omega(x)\,dx = \langle 1, x^4 \rangle.$$

We know $\langle x^2, x^2 \rangle > 0$, but if we have an orthonormal basis then we must have $\langle 1, x^4 \rangle = 0$, which is a contradiction.