Methods of Mathematical Physics - 556 X1
Homework 2
Due September 26, 2008

1. Recall that we define the orthogonal complement as in class: If S is a vector space, and T is a subspace, then we define the **orthogonal complement** of T:

$$T^{\perp} := \{y \in S : \langle x, y \rangle = 0 \text{ for all } x \in T\}.$$

   To get warmed up, we'll consider a few concrete examples.

   (a) First let $S = \mathbb{R}^3$ and let T be the (x, y)-plane. What is $T^{\perp}$? What is its dimension?

   (b) Now let T be any plane containing the origin. What is $T^{\perp}$? What is its dimension?

   (c) Let T be any line in $\mathbb{R}^3$ through the origin. What is $T^{\perp}$?

   (d) Again let T be any plane in $\mathbb{R}^3$ containing the origin. What is $(T^{\perp})^{\perp}$?

2. Now we prove these things in general. Let S be a n-dimensional vector space, and T a k-dimensional subspace. Prove

   (a) $T^{\perp}$ is also a subspace of S,

   (b) $\dim(T^{\perp}) = n - k$,

   (c) $(T^{\perp})^{\perp} = T$.

3. We stated the Fredholm Alternative in class as $R(A) = N(A^*)^{\perp}$. Show that the following three other statements are logically equivalent to this one:

   (a) $N(A) = R(A^*)^{\perp}$,

   (b) $R(A^*) = N(A)^{\perp}$,

   (c) $N(A^*) = R(A)^{\perp}$.

4. Let us say that we are observing two experiments simultaneously, and we collected the following six pairs of data points:

$$(5, 3.4), (6, 2.1), (9, 4), (1, 3.5), (7, 11), (5, 6.3),$$

   where the first number corresponds to the results of the first experiment, etc.

   (a) Assume that there is a linear relationship between the first measurement x and the second measurement y, namely that $y = \alpha x + \beta$. Compute the least-squares best approximation for (α, β). Compute the total error made by this approximation.

   (b) Now assume that there is a quadratic relationship between y and x, namely that $y = \alpha x^2 + \beta x + \gamma$. Compute the least-squares best approximation for (α, β, γ). Compute the total error made by this approximation. Is this better than the previous guess? Why should (or shouldn't) it be?

5. (Problem 1.5.2 from Keener.) Let D be an $m \times n$ matrix with entries $d_{ij} = \sigma_i \delta_{ij}$. That is, all of the off-diagonal entries of D are zero. Let D' be the least-squares pseudo-inverse of D. Show that D' is a $n \times m$ matrix whose entries are given by $(D')_{ij} = \sigma_i^{-1} \delta_{ij}$ whenever $\sigma_i \neq 0$, and $(D')_{ij} = 0$ otherwise.

6. Let us consider as a vector space $\mathcal{P}_2(\mathbb{R})$, that is, all polynomials with real coefficients with degree less than or equal to 2. We define the inner product

$$\langle f, g \rangle = \int_0^1 f(x) g(x)\, dx.$$

   Now, consider the function L given by $Lf(x) = f'(x)$. Note that L is a map from $\mathcal{P}_2(\mathbb{R})$ into itself.

(a) Prove that L is a linear map.

(b) Choose the "monomial basis" $\{1, x, x^2\}$, and write the (3×3) matrix representing L with respect to this basis. Recall, we say that the matrix A with entries a_{ij} represents an operator with respect to the basis $\{\phi_1, \dots, \phi_n\}$ if $L\phi_i = \sum_j a_{ij}\phi_j$ for all i.

(c) Describe what the matrix for derivative would look like in $\mathcal{P}_n(\mathbb{R})$.

(d) Back to $\mathcal{P}_2(\mathbb{R})$. Orthogonalize the monomial basis using the Gram-Schmidt process. Write the matrix for L with respect to this basis.

(e) Compute L^*.

Hint: The best way to do this is to pick a basis (let's say the standard monomial basis $\{1, x, x^2\}$. Then if we know that L^*1, L^*x, L^*x^2 are, we are in business. But we know that

$$\langle L^*x, y \rangle = \langle x, Ly \rangle .$$

Compute the nine numbers $C_{ij} := \left\langle x^i, Lx^j \right\rangle$, $i, j = 0, 1, 2$. Now, to compute L^*1, write it in the form $L^*1 = \alpha + \beta x + \gamma x^2$, and we just need to find the coefficients α, β, γ. But then we know $\left\langle L^*1, x^j \right\rangle = C_{0j}$ for all j. Writing this out gives us three equations in three unknowns, and we can solve this. Do the same for L^*x, L^*x^2.

7. **Bonus.** Now that we know L^*, compute the matrix with respect to the basis $\{1, x, x^2\}$. You will note that this is *not* the conjugate transpose of the matrix we calculated above. Explain why it isn't, specifically, how the proof we showed in class does not apply here.

8. **Extra special bonus.** We saw above that the matrix for L^* is not the conjugate transpose of the matrix for L. Can we fix this?

(a) It is possible to define an inner product on $\mathcal{P}_2(\mathbb{R})$ for which the matrix of L^* equals the conjugate transpose of the matrix of L? What properties must such an inner product have?

(b) Is it possible to find one which is defined by an inner product of the form

$$\langle f, g \rangle = \int_L^R f(x)g(x)\,dx$$

for some real L, R? Why or why not?

(c) Is it possible to find one defined by an inner product of the form

$$\langle f, g \rangle = \int_0^1 f(x)g(x)\omega(x)\,dx$$

where $\omega(x)$ is a positive continuous function on $[0, 1]$? If so, determine what properties $\omega(x)$ must have, and give an example of such an $\omega(x)$.