Methods of Mathematical Physics - 556 X1
Homework 1 - Solutions

1. *(Problem 1.1.2 from Keener.) Show that in any inner product space,*

$$\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right). \tag{1}$$

Interpret this geometrically in $\mathbb{R}^2$.

Solution. Using the rule that $\|x\|^2 = \langle x, x\rangle$, just expand everything in sight.

The left-hand side is

$$\begin{aligned}\langle x+y, x+y\rangle + \langle x-y, x-y\rangle &= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle + \langle x,x\rangle - \langle x,y\rangle - \langle y,x\rangle + \langle y,y\rangle \\ &= 2(\langle x,x\rangle + \langle y,y\rangle).\end{aligned}$$

In $\mathbb{R}^2$, this just says that the sum of the lengths of the diagonals of a parallelogram is the same as the perimeter.

2. *(Problem 1.1.3 from Keener.)*

   (a) Verify that in an inner product space,

$$\text{Re } \langle x, y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right).$$

   (b) Show that in any real inner product space there is at most one inner product which generates the same induced norm.

   (c) In $\mathbb{R}^n$ *with* $n > 1$, *show that*

$$\|x\|_p := \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$$

   can be induced by an inner product if and only if $p = 2$. *(*Hint: *Use both 1.1.2. and 1.1.3. here!)*

Solution.

(a) Expand

$$\begin{aligned}\langle x+y, x+y\rangle - \langle x-y, x-y\rangle &= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle - (\langle x,x\rangle - \langle x,y\rangle - \langle y,x\rangle + \langle y,y\rangle) \\ &= 2(\langle x,y\rangle + \langle y,x\rangle).\end{aligned}$$

Since $\langle y, x\rangle = \overline{\langle x, y\rangle}$, and $z + \overline{z} = 2\text{Re}z$ for any complex number z, then

$$2(\langle x,y\rangle + \langle y,x\rangle) = 4(\text{Re}(\langle x,y\rangle)).$$

(b) Since we can write any inner product in terms of its induced norm, then equality of norms implies equality of inner products. For example, let's say that we have two inner products on our space, $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$. These induce two norms by

$$\|x\|_1 = \sqrt{\langle x,x\rangle_1}, \quad \|x\|_2 = \sqrt{\langle x,x\rangle_2}.$$

Now assume that $\|x\|_1 = \|x\|_2$ for all vectors x. But then

$$\text{Re}\,\langle x,y\rangle_1 = \frac{1}{4}\left(\|x+y\|_1^2 - \|x-y\|_1^2\right) = \frac{1}{4}\left(\|x+y\|_2^2 - \|x-y\|_2^2\right) = \text{Re}\,\langle x,y\rangle_2\,.$$

Since we have a real inner-product space, $\langle x,y\rangle_1$ and $\langle x,y\rangle_2$ are both real, and thus equal.

(c) First notice that if $p=2$, then this is just the standard Euclidean norm on $\mathbb{R}^n$ and it is of course induced by the standard dot product. So all we need to show is that if $p \neq 2$, then this is not induced by an inner product, or, equivalently, if it is induced by an inner product, then p must be 2. So, let us consider the two vectors $e_1 = (1,0,0,0,\ldots,0)$ and $e_2 = (0,1,0,0,\ldots,0)$. Notice that

$$\|e_1\|_p = \|e_2\|_p = 1.$$

But on the other hand,

$$e_1 + e_2 = (1,1,0,0,\ldots,0), \quad e_1 - e_2 = (1,-1,0,0,\ldots,0),$$

and

$$\|e_1 + e_2\|_p = \|e_1 - e_2\|_p = (1+1)^{1/p} = 2^{1/p}.$$

Assuming that $\|\cdot\|_p$ is induced by an inner product, then using (1) this means that

$$2^{2/p} + 2^{2/p} = 2(1+1),$$

or

$$2^{4/p} = 4.$$

This means that $4/p = 2$ and thus $p = 2$.

3. (Problem 1.1.5 from Keener.) Show that

$$\langle f, g\rangle = \int_0^1 \left(f(x)\overline{g(x)} + f'(x)\overline{g'(x)}\, dx \right)$$

is an inner product on the space of all continuously differentiable functions defined on $[0,1]$, i.e. on

$$C^1([0,1]) := \{f\colon [0,1] \to \mathbb{C} : f(x), f'(x) \text{ are continuous}\}.$$

Solution. We compute:

$$\begin{aligned}
\langle \alpha f + \beta g, h\rangle &= \int_0^1 (\alpha f(x) + \beta g(x))\overline{h(x)} + (\alpha f(x) + \beta g(x))'\overline{h'(x)}\, dx \\
&= \int_0^1 \alpha f(x)\overline{h(x)} + \beta g(x)\overline{h(x)} + \alpha f'(x)\overline{h'(x)} + \beta g'(x)\overline{h'(x)}\, dx \\
&= \int_0^1 \alpha f(x)\overline{h(x)} + \alpha f'(x)\overline{h'(x)} + \beta g(x)\overline{h(x)} + \beta g'(x)\overline{h'(x)}\, dx \\
&= \alpha \int_0^1 f(x)\overline{h(x)} + f'(x)\overline{h'(x)}\, dx + \beta \int_0^1 g(x)\overline{h(x)} + g'(x)\overline{h'(x)}\, dx \\
&= \alpha \langle f, h\rangle + \beta \langle g, h\rangle.
\end{aligned}$$

It is clear from the definition that

$$\langle g, f\rangle = \overline{\langle f, g\rangle}.$$

Finally

$$\langle f, f\rangle = \int_0^1 |f(x)|^2 + |f'(x)|^2\, dx.$$

The integrand is nonnegative so the integral is. Moreover, if $\langle f, f\rangle = 0$, then

$$\int_0^1 |f(x)|^2 = 0.$$

We want to show that the only continuous function on $[0,1]$ whose integral is zero is the zero function. So we argue by contradiction: Assume f is positive somewhere, i.e. there is an x' such that $f(x') > 0$. Since f is continuous, this means that for any $\epsilon > 0$, there is a $\delta > 0$ such that if $|x - x'| < \delta$, then $|f(x) - f'(x)| < \epsilon$. Choose $\epsilon = f(x')/2$, and this means that for all $|x - x'| < \delta$, we have $f(x) > f(x')/2$. But then this means that

$$\int_{x'-\delta}^{x'+\delta} |f(x)|^2 \, dx > \delta f'(x) > 0,$$

and of course

$$\int_0^1 |f(x)|^2 \, dx > \int_{x'-\delta}^{x'+\delta} |f(x)|^2 \, dx.$$

Therefore if f is positive anywhere, the integral of its square must be positive. Conversely, if $\langle f, f \rangle = 0$, then $f = 0$.

NB. This argument only applies to *continuous* functions; if we don't assume that f is continuous then of course we can have $f > 0$ at points but $\int f^2 = 0$, e.g. choose any finite number of points $\{x_1, x_2, \ldots, x_n\}$, and define

$$f(x) = \begin{cases} 1, & x = x_i \text{ for some } i, \\ 0, & \text{else.} \end{cases}$$

4. *(Problem 1.1.8 from Keener.)* *Verify that the choice* $\gamma = \langle x, y \rangle / \|y\|^2$ *minimizes* $\|x - \gamma y\|^2$. *Show then that* $|\langle x, y \rangle|^2 = \|x\|^2 \|y\|^2$ *if and only if* x *and* y *are linearly dependent.*

Solution. Let us first assume the inner product space is real, and we get

$$\|x - \gamma y\|^2 = \langle x - \gamma y, x - \gamma y \rangle = \langle x, x \rangle - 2\gamma \langle x, y \rangle + \gamma^2 \langle y, y \rangle .$$

Notice that this is a quadratic polynomial with real coefficients, and as long as $y \neq 0$, the coefficient on the quadratic term is positive. This means that this is a concave-up function which is going to have exact one critical point, and this critical point is a global minimum. To compute the critical point, we differentiate with respect to γ and set equal to zero, giving

$$-2 \langle x, y \rangle + 2\gamma \langle y, y \rangle = 0,$$

or

$$\gamma = \frac{\langle x, y \rangle}{\langle y, y \rangle}.$$

Now, let's assume that we have a complex vector space. We then have

$$\|x - \gamma y\|^2 = \langle x - \gamma y, x - \gamma y \rangle = \langle x, x \rangle - \overline{\gamma} \langle x, y \rangle - \gamma \langle y, x \rangle + |\gamma|^2 \langle y, y \rangle .$$

We can rewrite this as

$$\langle x, x \rangle - 2\mathrm{Re}(\overline{\gamma} \langle x, y \rangle) + |\gamma|^2 \langle y, y \rangle .$$

This is a complex expression and we're trying to minimize with respect to a complex parameter. One of the most efficient things to do in such a scenario is to write everything out in terms of real and imaginary parts, which will give two real parameters. If we write

$$\gamma = \alpha + i\beta, \quad \langle x, y \rangle = \zeta + i\omega,$$

then we get

$$F(\alpha, \beta) := \langle x, x \rangle - 2(\alpha\zeta + \beta\omega) + (\alpha^2 + \beta^2) \langle y, y \rangle .$$

Think of this as a function of the two real variables α, β which we want to minimize over. To find a minimum, we need to first find the critical points of F, and we have

$$\frac{\partial F}{\partial \alpha} = -2\zeta + 2\alpha \langle y, y \rangle, \quad \frac{\partial F}{\partial \beta} = -2\omega + 2\beta \langle y, y \rangle. \tag{2}$$

To see if this critical point is a local min, local max, or saddle, we compute the Hessian of F, namely:

$$H = \begin{pmatrix} \frac{\partial^2 F}{\partial x \partial x} & \frac{\partial^2 F}{\partial x \partial y} \\ \frac{\partial^2 F}{\partial y \partial x} & \frac{\partial^2 F}{\partial y \partial y} \end{pmatrix} = \begin{pmatrix} 2\langle y, y \rangle & 0 \\ 0 & 2\langle y, y \rangle \end{pmatrix}.$$

Again, as long as $y \neq 0$ the determinant and entries of H are positive and thus this critical point is a local minimum. Solving (2) gives

$$\alpha = \frac{\zeta}{\langle y, y \rangle}, \quad \beta = \frac{\omega}{\langle y, y \rangle},$$

or

$$\text{Re}\gamma = \frac{\text{Re}\langle x, y \rangle}{\langle y, y \rangle}, \quad \text{Im}\gamma = \frac{\text{Im}\langle x, y \rangle}{\langle y, y \rangle}.$$

Therefore

$$\gamma = \frac{\langle x, y \rangle}{\langle y, y \rangle}.$$

5. *(Problem 1.1.9 from Keener.) For any $w(x) > 0$, we can define the inner product*

$$\langle f, g \rangle = \int_a^b f(x) g(x) w(x)\, dx$$

for real continuous functions defined on $[a, b]$ and taking real values. ($w(x)$ is a "weight" function — think of it like a weighted average — we have considered the case where $w \equiv 1$ before.) Start with the basis $\{1, x, x^2, x^3, x^4\}$ for P_4. Use the Gram-Schmidt algorithm to generate an orthogonal set of polynomials when we choose:

(a) $a = -1, b = 1, w(x) \equiv 1$ *(Legendre polynomials),*

(b) $a = -1, b = 1, w(x) = (1 - x^2)^{-1/2}$ *(Chebyshev polynomials),*

(c) $a = 0, b = \infty, w(x) = e^{-x}$ *(Laguerre polynomials),*

(d) $a = -\infty, b = \infty, w(x) = e^{-x^2}$ *(Hermite polynomials).*

Hint: *You might find a computer algebra system (e.g. Maple, Mathematica) useful here, but you can do it by hand with some work as well.*

Solution. The Gram-Schmidt algorithm says that if we want to turn the linearly independent set $\{y_0, y_1, \ldots, y_n\}$ of vectors into an orthogonal set, we compute

$$v_j = y_j - \sum_{i=0}^{j-1} \frac{\langle y_j, v_i \rangle}{\langle v_i, v_i \rangle} v_i.$$

(We index the list from 0 instead of 1, just for convenience below.) We see that in all of these cases, every number which appears in the algorithm will be the inner product of two polynomials, and since inner products are linear in each slot, if we know the inner product of all possible monomials, then this is enough information to compute everything. So we compute all the monomial integrals to begin.

As always, note that if we want an *orthonormal* set of vectors, once we have an orthogonal set we just scale each vector by its length. So for this problem we will just make the set orthogonal and skip the scaling.

(a) Here our inner product is

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\,dx.$$

Computing the integrand for all monomials, we have

$$\int_{-1}^{1} x^p\,dx = \frac{1^{p+1} - (-1)^{p+1}}{p+1} = \begin{cases} 0, & p \text{ odd}, \\ \frac{2}{p+1}, & p \text{ even} \end{cases}$$

Since we're always going to choose $y_k = x^k$, we have

$$\langle y_j, y_k\rangle = \begin{cases} 0, & j+k \text{ odd}, \\ \frac{2}{j+k+1}, & j+k \text{ even} \end{cases}$$

So we have

$$\begin{aligned}
v_0 &= 1, \\
v_1 &= y_1 - \frac{\langle y_1, v_0\rangle}{\langle v_0, v_0\rangle} v_0 = x - 0 = x, \\
v_2 &= y_2 - \frac{\langle y_2, v_0\rangle}{\langle v_0, v_0\rangle} v_0 - \frac{\langle y_2, v_1\rangle}{\langle v_1, v_1\rangle} v_1 = x^2 - \frac{2/3}{2} 1 - 0 = x^2 - 1/3, \\
v_3 &= y_3 - \frac{\langle y_3, v_0\rangle}{\langle v_0, v_0\rangle} v_0 - \frac{\langle y_3, v_1\rangle}{\langle v_1, v_1\rangle} v_1 - \frac{\langle y_3, v_2\rangle}{\langle v_2, v_2\rangle} v_2 = x^3 - 0 - \frac{2/5}{2/3} x - 0 = x^3 - 3x/5.
\end{aligned}$$

Thus our orthogonal set is $\{1, x, x^2 - 1/3, x^3 - 3x/5\}$.

(b) Here our inner product is

$$\langle f, g\rangle = \int_{-1}^{1} \frac{f(x)g(x)}{\sqrt{1-x^2}}\,dx.$$

We note that we will only need monomials up to degree 5. Computing

$$\int_{-1}^{1} \frac{x^p}{\sqrt{1-x^2}}\,dx$$

for $p = 0, \ldots, 5$ gives $\{\pi, 0, \pi/2, 0, 3\pi/8, 0\}$. (As before, the odd powers give zero since the weight function is itself even.) Again, we replace p with $j+k$ to compute $\langle y_j, y_k\rangle$. So we have

$$\begin{aligned}
v_0 &= 1, \\
v_1 &= y_1 - \frac{\langle y_1, v_0\rangle}{\langle v_0, v_0\rangle} v_0 = x - 0 = x, \\
v_2 &= y_2 - \frac{\langle y_2, v_0\rangle}{\langle v_0, v_0\rangle} v_0 - \frac{\langle y_2, v_1\rangle}{\langle v_1, v_1\rangle} v_1 = x^2 - \frac{\pi/2}{\pi} 1 - 0 = x^2 - 1/2, \\
v_3 &= y_3 - \frac{\langle y_3, v_0\rangle}{\langle v_0, v_0\rangle} v_0 - \frac{\langle y_3, v_1\rangle}{\langle v_1, v_1\rangle} v_1 - \frac{\langle y_3, v_2\rangle}{\langle v_2, v_2\rangle} v_2 = x^3 - 0 - \frac{3\pi/8}{\pi/2} x - 0 = x^3 - 3x/4.
\end{aligned}$$

Thus our orthogonal set is $\{1, x, x^2 - 1/2, x^3 - 3x/4\}$.

(c) Here our inner product is

$$\langle f, g\rangle = \int_{0}^{\infty} f(x)g(x)e^{-x}\,dx.$$

Computing the integrand for all monomials, we have

$$\int_0^\infty x^p e^{-x}\,dx = p!$$

If you have seen the "gamma function" before, you know this. If not, to check that this is so, prove it by induction. Clearly the formula is true for $p = 0$, since

$$\int_0^\infty e^{-x} = -(e^{-x})|_{x=0}^{x=\infty} = 1.$$

Now assume the formula is true for p. Then we have

$$\int_0^\infty x^{p+1}e^{-x}\,dx = x^{p+1}(-e^{-x})|_{x=0}^{x=\infty} - \int_0^\infty (p+1)x^p(-e^{-x})\,dx = 0 + (p+1)p! = (p+1)!$$

This gives

$$\langle y_j, y_k\rangle = (j+k)!$$

So we have

$$\begin{aligned}
v_0 &= 1,\\
v_1 &= y_1 - \frac{\langle y_1, v_0\rangle}{\langle v_0, v_0\rangle}v_0 = x - \frac{1!}{0!}1 = x - 1,\\
v_2 &= y_2 - \frac{\langle y_2, v_0\rangle}{\langle v_0, v_0\rangle}v_0 - \frac{\langle y_2, v_1\rangle}{\langle v_1, v_1\rangle}v_1 = x^2 - \frac{2!}{0!}1 - \frac{3! - 2!}{2! - 2*1! + 0!}(x-1) = x^2 - 2 - 4(x-1) = x^2 - 4x + 2,\\
v_3 &= y_3 - \frac{\langle y_3, v_0\rangle}{\langle v_0, v_0\rangle}v_0 - \frac{\langle y_3, v_1\rangle}{\langle v_1, v_1\rangle}v_1 - \frac{\langle y_3, v_2\rangle}{\langle v_2, v_2\rangle}v_2\\
&= x^3 - \frac{3!}{0!}1 - \frac{4! - 3!}{2! - 2*1! + 0!}(x-1) - \frac{5! - 4*4! + 2*3!}{4! - 8*3! + 18*2! - 16*1! + 4*0!}(x^2 - 4x + 2)\\
&= x^3 - 6 - 18(x-1) - \frac{36}{4}(x^2 - 4x + 2) = x^2 - 9x^2 + 18x - 6.
\end{aligned}$$

Thus our orthogonal set is $\{1, x-1, x^2 - 4x + 2, x^3 - 9x^2 + 18x - 6\}$.

(d) Here our inner product is

$$\langle f, g\rangle = \int_{-\infty}^\infty f(x)g(x)e^{-x^2}\,dx.$$

Computing

$$\int_{-\infty}^\infty x^p e^{-x^2}\,dx$$

for $p = 0, \ldots, 5$ gives $\{\sqrt{\pi}, 0, \sqrt{\pi}/2, 0, 3\sqrt{\pi}/4, 0\}$. Then:

$$\begin{aligned}
v_0 &= 1,\\
v_1 &= y_1 - \frac{\langle y_1, v_0\rangle}{\langle v_0, v_0\rangle}v_0 = x - 0 = x,\\
v_2 &= y_2 - \frac{\langle y_2, v_0\rangle}{\langle v_0, v_0\rangle}v_0 - \frac{\langle y_2, v_1\rangle}{\langle v_1, v_1\rangle}v_1 = x^2 - \frac{\sqrt{\pi}/2}{\sqrt{\pi}}1 - 0 = x^2 - 1/2,\\
v_3 &= y_3 - \frac{\langle y_3, v_0\rangle}{\langle v_0, v_0\rangle}v_0 - \frac{\langle y_3, v_1\rangle}{\langle v_1, v_1\rangle}v_1 - \frac{\langle y_3, v_2\rangle}{\langle v_2, v_2\rangle}v_2 = x^3 - 0 - \frac{3\sqrt{\pi}/4}{\sqrt{\pi}/2}x - 0 = x^3 - 3x/2.
\end{aligned}$$

Thus our orthogonal set is $\{1, x, x^2 - 1/2, x^3 - 3x/2\}$.

Notice that (a,b,d) are much easier than (c). If we choose an even weight function and a symmetric domain of integration, this makes all odd monomials orthogonal to all even monomials and simplifies the calculation considerably.

6. *(Problem 1.2.2 from Keener.)*

   (a) *Prove that two symmetric matrices are equivalent if and only if they have the same eigenvalues (with the same multiplicities).*
   (b) *Show that if A and B are equivalent, then* $\det A = \det B$.
   (c) *Is the converse true?*

   Solution.

   (a) First of all, we will prove that if A and B are equivalent, then they have the same eigenvalues. If A and B are equivalent, then there is an invertible T such that

   $$A = T^{-1}BT.$$

   So, let's say that $Ax = \lambda x$. Then $BTx = TAx = T(\lambda x) = \lambda Tx$. Thus if x is an eigenvector of A with eigenvalue λ, then Tx is an eigenvector of B with the same eigenvalue.
   Now let us assume that A and B have the same eigenvalues. Since they are both symmetric, they are both diagonalizable, thus we can write

   $$A = T_1^{-1}\Lambda_1 T_1, \quad B = T_2^{-1}\Lambda_2 T_2$$

   for some diagonal $\Lambda_{1,2}$. But since A and B have the same eigenvalues, we can choose $\Lambda_1 = \Lambda_2$, and thus we have

   $$A = T_2 T_1^{-1} B T_1 T_2^{-1} = (T_1 T_2^{-1})^{-1} B (T_1 T_2^{-1}),$$

   and A and B are equivalent.

   (b) We use the fact that the determinant of a product of matrices is the product of the determinants. Then if $B = T^{-1}AT$, we have

   $$\det(B) = \det(T^{-1}AT) = \det(T^{-1})\det(A)\det(T) = \frac{1}{\det(T)}\det(A)\det(T) = \det(A).$$

   (c) No. Take any two matrices with different eigenvalues but the same determinant. Then they are not equivalent (see the proof in part (a)).

7. *(Problem 1.2.4 from Keener.)* *Show that the eigenvalues of an anti-self-adjoint matrix ($A^* = -A$) are imaginary.*

   Solution. Let's say $Ax = \lambda x$, with $x \neq 0$, then

   $$\langle Ax, x\rangle = \langle \lambda x, x\rangle = \lambda \langle x, x\rangle.$$

   On the other hand,

   $$\langle Ax, x\rangle = \langle x, A^* x\rangle = -\langle x, Ax\rangle = -\langle x, \lambda x\rangle = -\overline{\lambda}\langle x, x\rangle.$$

   Since $\langle x, x\rangle > 0$, we have $\lambda = -\overline{\lambda}$, which means λ is imaginary.

8. *(Problem 1.2.5 from Keener.)* *Find a basis for the range and null space of the following matrices:*

   (a)

   $$A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \\ 2 & 5 \end{pmatrix}$$

(b)

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

Solution.

(a) The range of a matrix is spanned by its columns. Since the columns are linearly independent, they form a basis for their span, and thus a basis for the range is $\{(1,1,2),(2,3,5)\}$. For the null space, notice that the first two rows are linearly independent, and thus the only solution to $x+2y=0, x+3y=0$ is $x=y=0$, and thus the null space is $\{0\}$.

(b) We can check that the columns are linearly independent, so again the columns give a basis for the range. Moreover, notice that the only solution to $Ax=0$ is $x=0$, so the null space is $\{0\}$.

9. *(Problem 1.2.6 from Keener.)* *Find an invertible matrix T and a diagonal matrix Λ so that $A = T\Lambda T^{-1}$ for each of the following matrices A:*

(a)

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1/4 & 1/4 & 1/2 \\ 0 & 0 & 1 \end{pmatrix},$$

(b)

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

(c)

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & 1 & 3 \end{pmatrix},$$

(d)

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix},$$

(e)

$$A = \begin{pmatrix} 1/2 & 1/2 & \sqrt{3}/6 \\ 1/2 & 1/2 & \sqrt{3}/6 \\ \sqrt{3}/6 & \sqrt{3}/6 & 5/6 \end{pmatrix}.$$

Solution. In each case, we compute the eigenvalues and eigenvectors of the matrix. We choose Λ to be the diagonal matrix whose entries are the eigenvalues of A, and T to be the matrix whose columns are the associated eigenvectors.

(a)

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/4 \end{pmatrix}, \quad T = \begin{pmatrix} -2 & 3 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

(b)

$$\Lambda = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad T = \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}.$$

(c)

$$\Lambda = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 0 & -2 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{pmatrix}.$$

(d) *no solution*

(e)

$$\Lambda = \begin{pmatrix} 4/3 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} \sqrt{3}/2 & -1/\sqrt{3} & -1 \\ \sqrt{3}/2 & -1/\sqrt{3} & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

10. *(Problem 1.2.9 from Keener.) The two sets of vectors $\{\phi_i\}_{i=1}^n$ and $\{\psi_i\}_{i=1}^n$ in an inner product space S are said to be* **biorthogonal** *if $\langle \phi_i, \psi_j \rangle = \delta_{ij}$. Assume that $\{\phi_i\}_{i=1}^n$ and $\{\psi_i\}_{i=1}^n$ are biorthogonal.*

    (a) Show that $\{\phi_i\}_{i=1}^n$ and $\{\psi_i\}_{i=1}^n$ each form linearly independent sets.

    (b) Show that if S is n-dimensional, then any vector $x \in S$ can be written as

    $$x = \sum_{i=1}^n \alpha_i \phi_i \tag{3}$$

    where $\alpha_i = \langle x, \psi_i \rangle$.

    (c) Express (3) in matrix form, i.e. show that

    $$x = \sum_{i=1}^n P_i x$$

    where P_i are projection matrices with the properties that $P_i^2 = P_i$ and $P_i P_j = 0$ whenever $i \neq j$. Express the matrix P_i in terms of the vectors ϕ_i, ψ_i.

Solution.

(a) Consider the equation

$$\sum_{i=1}^n \alpha_i \phi_i = 0.$$

Take the inner product with ψ_j, and we get

$$0 = \left\langle \sum_{i=1}^n \alpha_i \phi_i, \psi_j \right\rangle = \sum_{i=1}^n \alpha_i \langle \phi_i, \psi_j \rangle = \sum_{i=1}^n \alpha_i \delta_{ij} = \alpha_j.$$

We can do the same for all j, and thus all $\alpha_j = 0$.

(b) If S is n-dimensional, then the ϕ_i form a basis. This means that we can write every $x \in S$ uniquely as

$$x = \sum_{i=1}^n \alpha_i \phi_i.$$

To compute α_i, we have

$$\langle x, \psi_j \rangle = \left\langle \sum_{i=1}^n \alpha_i \phi_i, \psi_j \right\rangle = \sum_{i=1}^n \alpha_i \delta_{ij} = \alpha_j.$$

(c) Let's put the matrix P_j in the standard basis. If the matrix P_j has the property that $P_j x = \langle \psi_j, x \rangle \phi_j$, by (ii) we are done. Take the matrix whose columns are multiples of ϕ_j, then $P_j x$ will be a multiple of ϕ_j. More specifically, the kth column of P_j is $(\psi_j)_k \phi_j$, that is, in the kth column we multiply ϕ_j by the kth entry of the vector ψ_j.

Even more concretely, if we have

$$\phi_j = (\phi_{j,1}, \phi_{j,2}, \ldots, \phi_{j,n}), \quad \psi_j = (\psi_{j,1}, \psi_{j,2}, \ldots, \psi_{j,n}),$$

then we set the kth row, and lth column of the matrix P_j to be $\phi_{j,k}\psi_{j,l}$. We then have

$$(P_j x)_k = \sum_l (P_j)_{kl} x_l = \sum_l \phi_{j,k}\psi_{j,l} x_l = \langle \psi_j, x \rangle \phi_{j,k}$$

and thus $P_j x = \langle \psi_j, x \rangle \psi_j$.