Methods of Mathematical Physics - 556 X1
Final Exam — Take-home
Due December 17, 2008 — 5pm

Take-home exam rules: you can use Keener or any other textbook, any paper, computer program, etc. you like, but you may not discuss the solutions with your classmates or others. In short, you are allowed to consult a source if and only if it is not human. Of course, all assertions need to be proved.

1. In each of the following problems, you are given a vector space V and a function with domain $V \times V$ and range $\mathbb{R}$. You are to determine whether or not this function defines an inner product.

   (a) $V = L^2[0,1], \langle f,g \rangle = \int_0^1 f(x)g(x)\,dx$.

   (b) $V = L^2[0,1], \langle f,g \rangle = \int_0^1 f(x)g(x)\omega(x)\,dx$ where $\omega(x)$ is some fixed function defined on $[0,1]$. What needs to be true about ω to make this an inner product?

   (c) $V = \mathbb{R}^n$, let A be the $n \times n$ matrix with entry A_{ij} in the ith row and jth column. Then define

   $$\langle x,y \rangle = \sum_{i,j=1}^{n} A_{ij} x_i y_j.$$

   (Notice that this is just the dot product of x and Ay.) What has to be true about A to make this an inner product?

   (d) $V = \mathbb{R}^2$, same function as last problem where we choose

   $$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

   What has to be true about the numbers $\alpha, \beta, \gamma, \delta$ to make this an inner product? (What I'm asking here is, what do the conditions you obtained in (c) translate to in terms of the entries of A?)

2. We have considered both self-adjoint and anti-self-adjoint operators, i.e. operators so that $L^* = L$ and $L^* = -L$, respectively.

   (a) Can we generalize this past ± 1, namely, for which $\lambda \in \mathbb{C}$ do there exist operators L such that $L^* = \lambda L$?

   (b) As we know, if we assume that L satisfies $L^* = \lambda L$ when $\lambda = \pm 1$, then we can say something about the eigenvalues of L, namely that if $\lambda = 1$, then the eigenvalues of L are real, and if $\lambda = -1$, then the eigenvalues of L are imaginary. Can we say something similar for other λ?

3. As we discussed in class, the set of rational numbers, $\mathbb{Q}$, is not complete, meaning that not every Cauchy sequence converges. Give examples of:

   (a) a Cauchy sequence which does not converge,

   (b) a Cauchy sequence which does converge.

4. For each of the following cases, you are given a normed vector space V, and a linear operator L on V (or a set of linear operators on V). Either prove that the linear operator is bounded (and give the best estimate of its norm that you can!) or prove that it is unbounded:

   (a) Let L be any linear operator on $\mathbb{R}^n$ where we use the Euclidean norm:

   $$\|x\|^2 = \sum_{i=1}^{n} x_i^2.$$

(b) Let L be any linear operator on $\mathbb{R}^n$ where we use the p-norm:

$$\|x\|^p = \sum_{i=1}^{n} x_i^p.$$

(c) Let V be all functions in $L^2[0,1]$ which have a derivative (i.e. all $f \in L^2$ such that $f' \in L^2$ as well), and define

$$Lf(x) = x^2 \frac{df}{dx}.$$

(d) Let $V = L^2[0,1]$ and $Lf(x) = x^2 f(x)$.

(e) Let $V = L^2[0,1]$ and

$$Lf(x) = \int_0^1 (x-y)^4 f(y)\, dy.$$

5. Write down the Green's function and construct the solution in terms of the Green's function of

$$u'' = f(x), \quad u(0) = 1, u(1) = 1.$$

Hint: You might find it more convenient to change variables first.

6. Recall the heat equation on a finite interval, i.e.

$$u_t = u_{xx}, \quad u(0) = u(1) = 0,$$

and how we showed in class that all solutions decay to zero exponentially quickly. Can you make the same argument for the following PDE? Why or why not? Describe the pattern and try to develop a general method for equations of this type.

(a) $u_t = u_x, \quad u(0) = u(1) = 0,$

(b) $iu_t = u_{xx}, \quad u(0) = u(1) = 0,$

(c) $u_t = u_{xxxx}, \quad u(0) = u(1) = 0,$

(d) $u_t = -u_{xx} - u_{xxxx}, \quad u(0) = u(1) = 0.$

DONE.