Partial Differential Equations – Math 442 C13/C14
Fall 2009
Homework 6 Solutions

---

1. Let $D = \{(x,y) : x^2 + y^2 < 4\}$, and solve

$$\Delta u = 0, \text{ in } D,$$
$$u = 3 - 2\cos\theta, \quad r = 2.$$

Solution: Using the formula generated in class, we have that

$$u(r,\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} r^n (A_n \cos(n\theta) + B_n \sin(n\theta)),$$

for some unknown A_n, B_n. Plugging in the boundary condition $r = 2$ gives

$$u(r,2) = \frac{A_0}{2} + \sum_{n=1}^{\infty} 2^n (A_n \cos(n\theta) + B_n \sin(n\theta)) = 3 - 2\cos\theta,$$

from which we can see that $A_0 = 6, A_1 = -1$, and all the rest of the coefficients are zero, and thus the solution is

$$u(r,\theta) = 3 + 2r\cos(\theta).$$

---

2. Let D be the square $[0,1]^2$. Solve $\Delta u = 0$ subject to the boundary conditions

$$u(0,y) = 0, \quad u_x(1,y) = 0, \quad u(x,0) = x^2 - 2x, \quad u_y(x,1) = 0.$$

Solution: First make the Ansatz

$$u(x,y) = A(x)B(y),$$

separating as usual gives

$$A''(x) + \lambda A(x) = 0, \quad A(0) = A'(1) = 0,$$
$$B''(y) - \lambda B(y) = 0, \quad B'(1) = 0.$$

Solving the first system gives

$$A_n(x) = \sin(\omega_n x), \quad \omega_n = (n + 1/2)\pi, n = 1,2,3,\ldots, \lambda_n = \omega_n^2.$$

From this we obt $^{?}33^{T}E < 8B >: D2^{?}321331^{WR}bR^{K}K - q < 91 >< 9C >< 9D > :^{G} < 95 > m < 9F > < 9D >^{N} *; < 9D > < o <$

---

3. In class, we derived difference approximations for the second derivative in x of the form

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u_{k+1}^n - 2u_k^n + u_{k-1}^n}{(\Delta x)^2},$$

where we have defined $u_k^n = u(k\Delta x, n\Delta t)$. Use a similar methodology to get a difference approximation for the *fourth* derivative in terms of $u_{k+2}^n, u_{k+1}^n, u_k^n, u_{k-1}^n, u_{k-2}^n$. What is the size of the error in your approximation?

Solution: We write

$$\begin{aligned}
u_{k-2}^n &= u_k^n - 2(\Delta x)\frac{\partial u}{\partial x} + \frac{4(\Delta x)^2}{2}\frac{\partial^2 u}{\partial x^2} - \frac{8(\Delta x)^3}{6}\frac{\partial^3 u}{\partial x^3} + \frac{16(\Delta x)^4}{24}\frac{\partial^4 u}{\partial x^4} + O((\Delta x)^5) \\
u_{k-1}^n &= u_k^n - (\Delta x)\frac{\partial u}{\partial x} + \frac{(\Delta x)^2}{2}\frac{\partial^2 u}{\partial x^2} - \frac{(\Delta x)^3}{6}\frac{\partial^3 u}{\partial x^3} + \frac{(\Delta x)^4}{24}\frac{\partial^4 u}{\partial x^4} + O((\Delta x)^5) \\
u_k^n &= u_k^n \\
u_{k+1}^n &= u_k^n + (\Delta x)\frac{\partial u}{\partial x} + \frac{(\Delta x)^2}{2}\frac{\partial^2 u}{\partial x^2} + \frac{(\Delta x)^3}{6}\frac{\partial^3 u}{\partial x^3} + \frac{(\Delta x)^4}{24}\frac{\partial^4 u}{\partial x^4} + O((\Delta x)^5) \\
u_{k+2}^n &= u_k^n + 2(\Delta x)\frac{\partial u}{\partial x} + \frac{4(\Delta x)^2}{2}\frac{\partial^2 u}{\partial x^2} + \frac{8(\Delta x)^3}{6}\frac{\partial^3 u}{\partial x^3} + \frac{16(\Delta x)^4}{24}\frac{\partial^4 u}{\partial x^4} + O((\Delta x)^5)
\end{aligned}$$

So we want to find $\alpha_{-2}, \alpha_{-1}, \alpha_0, \alpha_1, \alpha_2$ so that if we consider the linear combination

$$\sum_{i=-2}^{2} \alpha_i u_{k+i}^n,$$

then all terms before the fourth order cancel. This gives

$$\begin{aligned}
\alpha_{-2} + \alpha_{-1} + \alpha_0 + \alpha_1 + \alpha_2 &= 0, \\
-2\alpha_{-2} - \alpha_{-1} + \alpha_1 + 2\alpha_2 &= 0, \\
4\alpha_{-2} + \alpha_{-1} + \alpha_1 + 4\alpha_2 &= 0, \\
-8\alpha_{-2} - \alpha_{-1} + \alpha_1 + 8\alpha_2 &= 0.
\end{aligned}$$

Of course this system is underdetermined, but we find that

$$\alpha_{-1} = -4\alpha_{-2}, \alpha_0 = 6\alpha_{-2}, \alpha_1 = \alpha_{-1}, \alpha_2 = \alpha_{-2}.$$

Moreover, if we actually take that linear combination, we find that we obtain

$$\alpha_{-2}\left(\frac{2}{3} - \frac{1}{6} - \frac{1}{6} + \frac{2}{3}\right)(\Delta x)^4 \frac{\partial^4 u}{\partial x^4} + O((\Delta x)^5)$$

or

$$\alpha_{-2}(\Delta x)^4 \frac{\partial^4 u}{\partial x^4} + O((\Delta x)^5),$$

and therefore we choose $\alpha_{-2} = 1/(\Delta x)^4$. This makes the error $O(\Delta x)$.

---

4. Consider the heat equation $u_t = u_{xx}$ defined on $x \in [0,5], t > 0$ with initial condition $u(x,0) = x(5-x)$ and boundary conditions $u(0,t) = u(5,t) = 0$.

   (a) Use the discretization scheme we defined in class (forward difference in time, second centered difference in space) with $\Delta x = 1, \Delta t = 1/4$. Compute two time steps forward (i.e. compute the solution at $t = 1/2$).

   (b) Do the same, except now choose $\Delta t = 1/8$. Compute forward four steps, again computing until time $t = 1/2$.

   (c) Now set $\Delta x = 1/2$, and return Δt to $1/4$. Compute two steps forward.

   (d) Compare all of the answers obtained above; explain your observations.

Solution: In all cases, our discretization will be of the form

$$u_k^{n+1} = (1-2\rho)u_k^n + \rho(u_{k+1}^n + u_{k-1}^n),$$

where $\rho = \Delta t/(\Delta x)^2$.

(a) For this case we have $\rho = 1/4$, so our scheme is

$$u_k^{n+1} = \frac{1}{2}u_k^n + \frac{1}{4}(u_{k+1}^n + u_{k-1}^n).$$

Our first row is given by $0, 4, 6, 6, 4, 0$. The next row up will be given by

$$0, 3.5, 5.5, 5.5, 3.5, 0$$

(notice the left and right edges are zero because of BC!), and finally the third row is

$$0, 3.125, 5, 5, 3.125, 0.$$

The rest of the cases are similar, except more work. In the other two cases we have $\rho = 1/8$ and $\rho = 1$, respectively. You should observe that the first two cases work reasonably, but the third case shows signs of instability.

---

5. **(Strauss 8.2.11.)** Write down a discretization scheme for $u_t = au_{xx} + bu$ where we use forward difference in time, and centered second difference in space. Define $\rho = \Delta t/(\Delta x)^2$ and find the condition on ρ for this scheme to be stable. **Solution:** Using the difference schemes, we have

$$\frac{u_k^{n+1} - u_k^n}{\Delta t} = a\frac{u_{k+1}^n - 2u_k^n + u_{k-1}^n}{(\Delta x)^2} + bu_k^n,$$

and solving gives

$$u_k^{n+1} = (1 - 2a\rho + b\Delta t)u_k^n + a\rho(u_{k+1}^n + u_{k-1}^n).$$

As for stability, we do the standard argument, take $u_k^n = X_k T_n$, and we obtain

$$(1 - 2a\rho + b\Delta t) + a\rho\left(\frac{X_{k+1}}{X_k} + \frac{X_{k-1}}{X_k}\right) = \xi$$

and $T_{n+1} = \xi T_n$, so for stability we need $|\xi| < 1$. We make the Ansatz

$$X_k = (e^{i\alpha(\Delta x)})^k,$$

and furthermore notice that in the limit of small grids the Δt term disappears, and we have

$$\begin{aligned}\xi &= (1-2a\rho) + a\rho(e^{i\alpha(\Delta x)} + e^{-i\alpha(\Delta x)}) \\ &= (1-2a\rho) + 2a\rho\cos(\alpha(\Delta x)) \\ &= 1 - 2a\rho(1 - \cos(\alpha(\Delta x))).\end{aligned}$$

Since the $1 - \cos(\cdot)$ term is always positive, we cannot have $\xi > 1$, but we could have $\xi < -1$. Taking the worst-case scenario into account, i.e. $1 - \cos(\alpha(\Delta x)) < 2$, we need to specify that

$$1 - 4a\rho < -1,$$

or

$$\rho < \frac{1}{2a}.$$