

Partial Differential Equations – Math 442 C13/C14

Fall 2009

Homework 6 — due November 18

1. Let $D = \{(x, y) : x^2 + y^2 < 4\}$, and solve

$$\begin{aligned}\Delta u &= 0, \text{ in } D, \\ u &= 3 - 2 \cos \theta, \quad r = 2.\end{aligned}$$

2. Let D be the square $[0, 1]^2$. Solve $\Delta u = 0$ subject to the boundary conditions

$$u(0, y) = 0, \quad u_x(1, y) = 0, \quad u(x, 0) = x^2 - 2x, \quad u_y(x, 1) = 0.$$

3. In class, we derived difference approximations for the second derivative in x of the form

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u_{k+1}^n - 2u_k^n + u_{k-1}^n}{(\Delta x)^2},$$

where we have defined $u_k^n = u(k\Delta x, n\Delta t)$. Use a similar methodology to get a difference approximation for the *fourth* derivative in terms of $u_{k+2}^n, u_{k+1}^n, u_k^n, u_{k-1}^n, u_{k-2}^n$. What is the size of the error in your approximation?

4. Consider the heat equation $u_t = u_{xx}$ defined on $x \in [0, 5], t > 0$ with initial condition $u(x, 0) = x(5 - x)$ and boundary conditions $u(0, t) = u(5, t) = 0$.
- (a) Use the discretization scheme we defined in class (forward difference in time, second centered difference in space) with $\Delta x = 1, \Delta t = 1/4$. Compute two time steps forward (i.e. compute the solution at $t = 1/2$).
 - (b) Do the same, except now choose $\Delta t = 1/8$. Compute forward four steps, again computing until time $t = 1/2$.
 - (c) Now set $\Delta x = 1/2$, and return Δt to $1/4$. Compute two steps forward.
 - (d) Compare all of the answers obtained above; explain your observations.
5. (**Strauss 8.2.11.**) Write down a discretization scheme for $u_t = au_{xx} + bu$ where we use forward difference in time, and centered second difference in space. Define $\rho = \Delta t/(\Delta x)^2$ and find the condition on ρ for this scheme to be stable.