Partial Differential Equations – Math 442 C13/C14
Fall 2009
Homework 5 Solutions

1. **(Strauss 5.2.2.)** Show that $\cos(x) + \cos(\alpha x)$ is periodic if α is a rational number and compute its period. What happens if α is not rational?

   Solution: Let us first notice that if f, g are both periodic with period p, then so is their sum:

   $$(f+g)(x+p) = f(x+p) + g(x+p) = f(x) + g(x) = (f+g)(x).$$

   So we simply have to show that $\cos(x), \cos(\alpha x)$ share a period. Note that the periods of $\cos(x)$ are $2n\pi$ with $n \in \mathbb{Z}$, and the periods of $\cos(\alpha x)$ are $2m\pi/\alpha$, where $m \in \mathbb{Z}$. The question is then, do these two sets of numbers share an element, i.e. is there an n and an m so that

   $$2n\pi = \frac{2m\pi}{\alpha}?$$

   If $\alpha = p/q$, then we have

   $$2n\pi = \frac{2mq\pi}{p}, \text{ or } n = \frac{mq}{p}.$$

   This has many solutions, choose, for example, $m = p, n = q$.

   If α is irrational, then this does not work; in fact, solving the first equation gives

   $$\alpha = \frac{m}{n}$$

   which is only possible if α is rational.

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2. Define $f(x) = x^3$ on the interval $[0, 1]$. Compute its Fourier sine series and its Fourier cosine series.

   Solution: The Fourier sine series coefficients are given by

   $$B_n = \frac{2}{L} \int_0^L f(x) \sin(n\pi x/L)\, dx,$$

   and in this case $L = 1$, so we need to compute

   $$B_n = 2 \int_0^1 x^3 \sin(n\pi x)\, dx.$$

   Integrating by parts several times gives

   $$B_n = \frac{(-1)^n 2(6 - n^2\pi^2)}{n^3\pi^3},$$

   and then the F.S.S. is $\sum_{n=1}^{\infty} B_n \sin(n\pi x)$.

   The Fourier cosine series coefficients are given by

   $$A_n = 2 \int_0^1 x^3 \cos(n\pi x)\, dx.$$

   Integrating by parts several times gives

   $$A_n = \frac{6 + (-1)^n(3n^2\pi^2 - 6)}{n^4\pi^4},$$

and then the F.C.S. is

$$\frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(n\pi x).$$

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3. Consider the function $f(x) = x$ on the interval $[-\pi, \pi]$. Compute the full Fourier series for $f(x)$. Use Parseval's Identity to compute

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Solution: Since f is odd, we know all of the cosine terms will be zero, so we compute

$$B_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin(nx)\, dx = \frac{2(-1)^{n+1}}{n}.$$

Parseval's Identity tells us that

$$\sum_{n=1}^{\infty} B_n^2 \left\langle \sin(nx), \sin(nx) \right\rangle = \left\langle f, f \right\rangle,$$

or

$$\sum_{n=1}^{\infty} \frac{4}{n^2}\pi = \int_{-\pi}^{\pi} x^2\, dx = \frac{2\pi^3}{3}.$$

Dividing gives

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

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4. Solve the heat equation given by

$$\begin{aligned} u_t &= k u_{xx}, \quad x \in [0, L], t > 0, \\ u(x, 0) &= x, \\ u(0, t) &= u(L, t) = 0. \end{aligned}$$

Solution: We make the Ansatz

$$u(x, t) = A(x)B(t),$$

and plugging in this gives us

$$A(x)B'(t) = kA''(x)B(t),$$

or

$$\frac{A''(x)}{A(x)} = \frac{B'(t)}{kB(t)} = -\lambda.$$

Thus we first solve

$$A''(x) + \lambda A(x) = 0, \quad A(0) = A(L) = 0,$$

and we know the solutions of this are

$$A_n(x) = \sin(n\pi x/L), \lambda_n = \frac{n^2\pi^2}{L^2}.$$

Then we have

$$B_n = C_n e^{-k\lambda_n t},$$

and our general solution is

$$u(x,t) = \sum_n C_n e^{-k\lambda_n t} \sin(n\pi x/L).$$

Plugging in $t = 0$ gives

$$x = \sum_n C_n \sin(n\pi x/L),$$

and thus C_n are the Fourier sine series coefficients of x, or

$$C_n = \frac{2}{L}\int_0^L x \sin(n\pi x/L)\,dx = \frac{2(-1)^{n+1}L}{\pi n}.$$

Therefore we have

$$u(x,t) = \sum_n \frac{2(-1)^{n+1}L}{\pi n} e^{-kn^2\pi^2 t/L^2} \sin(n\pi x/L).$$

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5. **(Strauss 5.3.8.)** Let f and g satisfy the same Robin boundary condition at $x = 0$ and the same Robin boundary condition at $x = L$ (i.e., we assume that

$$f'(0) + \alpha f(0) = g'(0) + \alpha g(0) = f'(L) + \beta f(L) = g'(L) + \beta g(L) = 0.)$$

Prove then that

$$(f'(x)g(x) - f(x)g'(x))|_{x=0}^{x=L} = 0.$$

Deduce from this that eigenfunctions of a Robin BVP are orthogonal.

Solution: We have

$$f'(0) = -\alpha f(0), \quad g'(0) = -\alpha g(0), \quad f'(L) = -\beta f(L), \quad g'(L) = -\beta g(L).$$

Plugging this into the left hand side gives

$$-\beta f(L)g(L) - f(L)(-\beta g(L)) - (-\alpha f(0)g(0) - f(0)(-\alpha g(0))) = 0.$$

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6. Prove that if f has period p, then

$$\int_a^{p+a} f(y)\,dy$$

is independent of a.

Solution: One way to do this is to write

$$\int_a^{p+a} = \int_0^p - \int_0^a + \int_p^{p+a}, \tag{1}$$

and we have that

$$\int_p^{p+a} f(y)\,dy = \int_0^a f(y-p)\,dy = \int_0^a f(y)\,dy,$$

(the first is from the u-substitution $z = y + p$ and the second is from periodicity) and thus the second and third terms in (1) cancel.

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7. Consider the infinite list of functions

$$\{1, \cos(x), \cos(2x), \ldots, \cos(nx), \ldots, \sin(x), \sin(2x), \ldots, \sin(nx), \ldots\}.$$

Show that this is an orthogonal set of functions on the set $[-\pi, \pi]$, i.e. if we define the inner product

$$\langle f, g\rangle := \int_{-\pi}^{\pi} f(x)g(x)\,dx,$$

then if we choose any two different functions from that list, then their inner product is zero.

Solution: We first notice that

$$\int_{-\pi}^{\pi} \sin(nx)\cos(mx)\,dx = 0$$

since sin is odd and cos is even and thus their product is odd, and the integral of an odd function on a symmetric interval is always zero.

Now, consider

$$\int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,dx = \int_{-\pi}^{\pi} \cos((n-m)x) - \cos((n+m)x)\,dx$$
$$= \frac{-\sin((n-m)x)}{n-m} + \frac{\sin((n+m)x)}{n+m}\bigg|_{x=-\pi}^{x=\pi},$$

as long as $n \neq m$, and this is clearly zero by periodicity of sin. We also have

$$\int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,dx = \int_{-\pi}^{\pi} \cos((n-m)x) + \cos((n+m)x)\,dx$$
$$= \frac{-\sin((n-m)x)}{n-m} - \frac{\sin((n+m)x)}{n+m}\bigg|_{x=-\pi}^{x=\pi},$$

as long as $n \neq m$, and this is again zero by periodicity of sin.

Finally, to compute the inner products of the functions with themselves, choose $n > 0$, and notice that

$$\sin^2(nx) + \cos^2(nx) = 1,$$

and thus

$$\int_{-\pi}^{\pi} \sin^2(nx) + \cos^2(nx)\,dx = 2\pi.$$

Moreover, since sine and cosine are the same function under a phase shift, we know from the previous problem that

$$\int_{-\pi}^{\pi} \sin^2(nx) = \int_{-\pi}^{\pi} \cos^2(nx)$$

and therefore they each equal π. For $n = 0$ we have

$$\int_{-\pi}^{\pi} \cos^2(0x)\,dx = \int_{-\pi}^{\pi} dx = 2\pi.$$

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8. Let $\{f_n(x)\}$ be any sequence of functions which converge to $f(x)$ uniformly on $[a, b]$. Prove then that $f_n(x)$ converge to f in the L^2 sense as well. Show a counterexample to demonstrate that the converse is false, i.e. that we can have L^2 converge but not uniform. (The term used for this is that uniform convergence is *stronger* than L^2).

Solution: To prove the first statement, notice that

$$\int_a^b |f_n(x)|^2 \, dx \leq \int_a^b \left| \max_{x\in[a,b]} f(x) \right|^2 \, dx = \left| \max_{x\in[a,b]} f(x) \right|^2 (b-a).$$

Then if

$$\lim_{n\to\infty} \max_{x\in[a,b]} f(x) = 0,$$

then clearly

$$\lim_{n\to\infty} \int_a^b |f_n(x)|^2 \, dx \leq (b-a) \left(\lim_{n\to\infty} \max_{x\in[a,b]} f(x) \right)^2 = 0$$

as well. Also, notice that it is nonnegative by definition, and therefore the limit is zero.