# Partial Differential Equations – Math 442 C13/C14
# Fall 2009
# Homework 4 — due October 9

1. Consider the boundary value problem

$$A'' + \lambda A = 0, \quad A'(0) + aA(0) = 0, \quad A(L) = 0.$$

(a) Show that if $a < 0$, then there is no negative eigenvalue.

(b) Under which conditions is there a zero eigenvalue?

(c) Show there are infinitely many positive eigenvalues for any value of $a$.

**Bonus:** We showed in (a) that if $a < 0$ then there is no negative eigenvalue. If turns out that for some positive $a$, this problem has a negative eigenvalue (and for some others it does not). Write down a condition on $a$ which determines whether such an eigenvalue exists.

*Solution.*

(a) Let's say we have a negative eigenvalue $\lambda < 0$. Then if we say $A(x) = e^{rx}$, we have

$$\begin{aligned} r^2 + \lambda &= 0, \\ r &= \pm\sqrt{-\lambda}. \end{aligned}$$

Since $\lambda < 0$, this means the roots are real, let us write them as $r = \pm b$ where $b > 0$. Then we have

$$A(x) = C_1 e^{bx} + C_2 e^{-bx}.$$

Plugging in the initial conditions gives

$$\begin{aligned} C_1 b - C_2 b + a(C_1 + C_2) &= 0, \\ C_1 e^{bL} + C_2 e^{-bL} &= 0. \end{aligned}$$

The second equation can be written as $C_2 = -C_1 e^{2bL}$, and plugging this into the first gives

$$C_1(a + b) + C_1 e^{2bL}(b - a) = C_1(b(1 + e^{2bL}) + a(1 - e^{2bL})).$$

This must be zero, but $C_1$ cannot be zero (since then so would $C_2$ be), so we need to solve

$$b(1 + e^{2bL}) + a(1 - e^{2bL}) = 0. \tag{1}$$

However, this equation has no solution! Notice that since $bL > 0$, we know $e^{2bL} > 1$, and thus $1 - e^{2bL} < 0$. Since $a$ is also less than zero, we know that

$$a(1 - e^{2bL}) > 0.$$

But since $b(1 + e^{2bL}) > 0$ as well, it is not possible to solve (1).

(b) If we have a zero eigenvalue, this means we have a solution to $A''(x) = 0$ with those boundary conditions. However, this means that $A(x) = \alpha x + \beta$, and plugging this into the boundary conditions gives

$$\begin{aligned} \alpha + a\beta &= 0, \\ \alpha L + \beta &= 0. \end{aligned}$$

We want to find a nontrivial solution to this system, and if we write it as a matrix equation:

$$\begin{pmatrix} 1 & a \\ L & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and this has a nontrivial solution iff the matrix is singular, i.e. if its determinant is zero. So the condition is that

$$\det \begin{pmatrix} 1 & a \\ L & 1 \end{pmatrix} = 1 - aL = 0,$$

or $aL = 1$.

(c) If $\lambda > 0$, then the solutions to the ODE look like

$$C_1 \cos(\omega x) + C_2 \sin(\omega x),$$

where $\omega^2 = \lambda$. Plugging in the boundary conditions gives

$$\begin{aligned} C_2\omega + aC_1 &= 0, \\ C_1 \cos(\omega L) + C_2 \sin(\omega L) &= 0. \end{aligned}$$

Solving the first equation gives $C_2 = -aC_1/\omega$, and plugging this into the second and doing some algebra gives

$$\tan(\omega L) = \frac{\omega}{a}.$$

The question is, how many roots does this equation have? We cannot answer this analytically, but we can see from the graph that it has to have infinitely many; see an example picture in Figure 1.

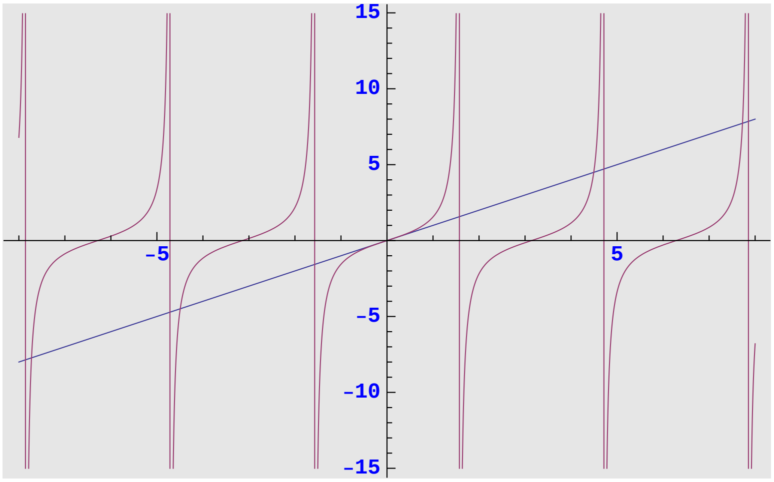

Figure 1: Graphs of $\tan(x)$ and $x$

To prove this analytically, choose any nonnegative integer $k$ and consider the interval $[(k + 1/2)\pi/L, (k + 3/2)\pi/L]$. We know from properties of tan that

$$\lim_{t \to \frac{\pi(k+1/2)}{L}+} \tan(x) = -\infty, \qquad \lim_{t \to \frac{\pi(k+3/2)}{L}-} \tan(x) = \infty,$$

so for $x$ slightly larger than $(k+1/2)\pi/L$, $\tan(x) < x/a$ and for $x$ slightly smaller than $(k+3/2)\pi/L$, $\tan(x) > x/a$. By the Intermediate Value Theorem, the functions must be share at least one point in the strip. Since the curves intersect in each of these strips, there are infinitely many intersections.

**Bonus:** Repeating some of part (a), we know we need to solve (1) for $b$, given a positive $a$, so we have

$$\tanh(2bL) = \frac{a}{b}.$$

These curves will always intersect as long as $a > 0$. Notice that $\tanh(0) = 0$ and $\lim_{b\to\infty} \tanh(2bL) = 1$, whereas $\lim_{b\to 0+} a/b = \infty$ and $\lim_{b\to\infty} a/b = 0$, so the curves must cross. So the condition is that for any $a > 0$, there is a negative eigenvalue.

2. **(Strauss 4.3.2.)** Consider the eigenvalue problem with Robin boundary conditions

$$A'' + \lambda A = 0, \quad A'(0) - \alpha_0 A(0) = 0, \quad A'(L) + \alpha_L A(L) = 0.$$

(a) Show that zero is an eigenvalue if and only if $\alpha_0 + \alpha_L = -\alpha_0\alpha_L L$.

(b) Compute the eigenfunction corresponding to this eigenvalue.

*Solution.*

(a) If we have a zero eigenvalue, then we have $A'' = 0$ or

$$A(x) = \alpha x + \beta.$$

Plugging in the boundary conditions gives

$$\begin{aligned} \alpha - \alpha_0\beta &= 0, \\ \alpha + \alpha_L(\alpha L + \beta) &= 0. \end{aligned}$$

We want to find a nontrivial solution for $\alpha, \beta$ in this equation, or, as above, we need the matrix

$$\begin{pmatrix} 1 & -\alpha_0 \\ 1 + \alpha_L L & \alpha_L \end{pmatrix}$$

to have determinant zero, or

$$\alpha_L + \alpha_0(1 + \alpha_L L) = 0.$$

(b) If this determinant is zero, we know that the two equations we have are redundant, so we can solve either. The simpler to solve is the first, which gives $\alpha = \alpha_0\beta$, and of course we will have one free choice for $\beta$. So one eigenfunction we can choose is

$$A(x) = \alpha_0 x + 1,$$

and we can of course choose any scalar multiple of this.

3. Solve the equation

$$\begin{aligned} u_t &= k u_{xx}, \quad x \in [0, \infty), \quad t > 0, \\ u(x,0) &= \begin{cases} 1, & x \in (0,1), \\ 0, & x > 1, \end{cases} \\ u(0,t) &= 0. \end{aligned}$$

*Solution.* We use the formula as derived in class (equation (6) in §3.1):

$$u(x,t) = \frac{1}{\sqrt{4\pi k t}} \int_0^\infty \left( e^{-(x-y)^2/4kt} - e^{-(x+y)^2/4kt} \right) \phi(y)\, dy,$$

where $\phi(y) = 1$ for $y < 1$ and 0 otherwise. So we can also write

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_0^1 \left( e^{-(x-y)^2/4kt} - e^{-(x+y)^2/4kt} \right) dy.$$

Let us consider the first term alone,

$$\frac{1}{\sqrt{4\pi kt}} \int_0^1 e^{-(x-y)^2/4kt} \, dy.$$

Using the change of variables $s = (x-y)/\sqrt{4kt}$, we obtain

$$\frac{1}{\sqrt{\pi}} \int_{\frac{x-1}{\sqrt{4kt}}}^{\frac{x}{\sqrt{4kt}}} e^{-s^2} \, ds,$$

or

$$\frac{1}{2} \left( \operatorname{erf}\left( \frac{x}{\sqrt{4kt}} \right) - \operatorname{erf}\left( \frac{x-1}{\sqrt{4kt}} \right) \right).$$

Now consider the second term,

$$\frac{1}{\sqrt{4\pi kt}} \int_0^1 e^{-(x+y)^2/4kt} \, dy.$$

Using the change of variables $s = (x+y)/\sqrt{4kt}$, we obtain

$$\frac{1}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{4kt}}}^{\frac{x+1}{\sqrt{4kt}}} e^{-s^2} \, ds,$$

or

$$\frac{1}{2} \left( \operatorname{erf}\left( \frac{x+1}{\sqrt{4kt}} \right) - \operatorname{erf}\left( \frac{x}{\sqrt{4kt}} \right) \right).$$

Adding these together gives

$$-\frac{1}{2} \left( \operatorname{erf}\left( \frac{x+1}{\sqrt{4kt}} \right) - 2\operatorname{erf}\left( \frac{x}{\sqrt{4kt}} \right) + \operatorname{erf}\left( \frac{x-1}{\sqrt{4kt}} \right) \right).$$

4. Consider the Schrödinger equation with Neumann boundary conditions:

$$iu_t = u_{xx}, \quad \frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0.$$

Write out the general series solution for this equation as we have done for the heat and wave equations, i.e. separate variables, get ODEs in $x$ and $t$, solve these problems, and take the linear combination.

*Solution.* We separate variables as usual, writing

$$u(x,t) = A(x)B(t).$$

Working this out gives us

$$A'' + \lambda A = 0, \quad A'(0) = A'(L) = 0, \quad B' + i\lambda B = 0.$$

for the $A$ equation. We've solved this in class, and we know that the eigenvalues and eigenfunctions are

$$\lambda_n = \left( \frac{n\pi}{L} \right)^2, \quad A_n(x) = \cos\left( \frac{n\pi}{L} x \right), \quad n = 0, 1, 2, \ldots$$

Solving the $B$ equation gives

$$B_n(t) = B_n(0)e^{i\lambda_n t},$$

which oscillates. Notice of course that we can allow $B_n(0)$ to be complex with no extra difficulties. Thus we have

$$u_n(x,t) = C_n e^{i\lambda_n t} \cos\left(\frac{n\pi}{L}x\right),$$

and we form a general solution by linear combinations:

$$u(x,t) = \sum_{n=0}^{\infty} C_n e^{i\lambda_n t} \cos\left(\frac{n\pi}{L}x\right).$$