Partial Differential Equations – Math 442 C13/C14
Fall 2009
Homework 2 — due September 18

1. **(Strauss 2.1.1.)** Solve $u_{tt} = c^2 u_{xx}$, $u(x,0) = e^x$, $u_t(x,0) = \sin x$.

2. **(Strauss 2.1.7.)** We define an odd function to be any function f such that $f(-x) = -f(x)$ for all x. Prove that if that the initial conditions ϕ, ψ are odd functions, then so is the solution $u(x,t)$ for any fixed time t.

3. We have defined a "well-posed" problem in class (also see book) typically for PDE, but we can consider if an ODE satisfies these three properties as well. Here you are given a sequence of ODEs and initial conditions; determine which of these problems are well-posed, and which are not[1]:

   (a) $\dfrac{dy}{dt} = 2y, \quad y(0) = 2,$

   (b) $\dfrac{dy}{dx} = \ln x, \quad y(0) = 0.$

4. **(Strauss 2.2.2.)** Let us consider a solution to the wave equation $u_{tt} = u_{xx}$ (we have assumed that $c^2 = 1$). Define the *energy density* $e(x,t) = \frac{1}{2}(u_t^2 + u_x^2)$ and the *momentum density* $p(x,t) = u_t u_x$. Show that

   (a) $\dfrac{\partial e}{\partial t} = \dfrac{\partial p}{\partial x}$ and $\dfrac{\partial p}{\partial t} = \dfrac{\partial e}{\partial x}$,

   (b) e and p both satisfy the wave equation themselves (although with different initial conditions).

5. **(Strauss 2.2.3.)** Show the following invariance properties for solutions of the wave equation. Assume that $u(x,t)$ satisfies the wave equation, then show that each of the transformed solutions *also* satisfy the wave equation:

   (a) **translation:** $u(x-\alpha, t)$ for any α,

   (b) **derivative:** $u_x(x,t)$,

   (c) **dilation:** $u(ax, at)$ for any a

6. **(Strauss 2.3.3.)** Consider a solution to the diffusion equation $u_t = u_{xx}$ for $x \in [0, L]$ and $t > 0$. Define

$$M(T) = \text{maximum of } u(x,t) \text{ on the rectangle } [0,L] \times [0,T],$$
$$m(T) = \text{minimum of } u(x,t) \text{ on the rectangle } [0,L] \times [0,T].$$

   Does $M(T)$ increase or decrease as a function of T? Does $m(T)$ increase or decrease as a function of T? Explain why.

[1] Recall the Existence–Uniqueness Theorem which you saw in ODEs