Partial Differential Equations – Math 442 C13/C14
Fall 2009
Homework 1 Solutions

1. *Determine which of the following operators are linear:*

   (a) $Lu = u_{xx} + u_{xy}$

   (b) $Lu = uu_x$

   (c) $Lu = 4x^2 u_y - 4y^2 u_{yy}$

   Solution: We will see that the first and third are linear and the second is not. For example, we compute:

   (a)
   $$L(u+v) = (u+v)_{xx} + (u+v)_{xy} = u_{xx} + v_{xx} + u_{xy} + v_{xy}$$
   $$= (u_{xx} + u_{xy}) + (v_{xx} + v_{xy}) = Lu + Lv,$$
   and
   $$L(\alpha u) = (\alpha u)_{xx} + (\alpha u)_{xy} = \alpha u_{xx} + \alpha u_{xy} = \alpha(u_{xx} + u_{xy}) = \alpha Lu.$$

   (b) $L(\alpha u) = (\alpha u)(\alpha u)_x = \alpha u(\alpha u_x) = \alpha^2 u_x$ and this is not equal to αLu if $\alpha^2 \neq \alpha$.

   (c)
   $$L(u+v) = 4x^2(u+v)_y - 4y^2(u+v)_{yy} = 4x^2(u_y + v_y) - 4y^2(u_{yy} + v_{yy})$$
   $$= 4x^2 u_y - 4y^2 u_{yy} + 4x^2 v_y - 4y^2 v_{yy} = Lu + Lv,$$
   and
   $$L(\alpha u) = 4x^2(\alpha u)_y - 4y^2(\alpha u)_{yy} = 4\alpha x^2 u_y - 4\alpha y^2 u_x = \alpha(4x^2 u_y - 4y^2 u_{yy}) = \alpha Lu.$$

2. *(Strauss, 1.1.4) Show that the difference of two solutions to $Lu = g$ is a solution to $Lu = 0$, when L is any linear operator.*

   Solution: Assume that $Lu = g$ and $Lv = g$. Define $w = u - v$, then
   $$Lw = L(u - v) = Lu - Lv = g - g = 0,$$
   where the second equality comes from L being linear.

3. *Solve:*
   $$2u_x + 3u_t = 0,$$
   $$u(x,0) = x^2.$$

   Solution: From the formula derived in class, we know the solution is of the form
   $$u(x,t) = f(3x - 2t),$$
   for some undetermined function f. Using the initial condition, we obtain
   $$u(x,0) = f(3x) = x^2,$$
   so that $f(s) = s^2/9$, or
   $$u(x,t) = (3x - 2t)^2/9.$$

4. *Consider the heat equation with initial condition given as*

$$u_t = u_{xx},$$
$$u(x,0) = \alpha x + \beta,$$

where α, β are real numbers. Make an educated guess for the solution to this PDE and check that it is correct. Interpret this as a statement about the evolution of temperature in a 1D object.

Solution: There are at least two ways to come up with a guess.

If we think of the temperature of an object as being a linear function, notice then that at every point, we expect the heat coming into the point to be the same as the heat leaving, so while there is a heat flux at every point, the actual temperature stays fixed. Thus we expect this temperature not to change, and we would guess $u(x,t) = \alpha x + \beta$.

Another approach is to notice that if we plug the initial condition into the right hand side of the equation, then we get zero, which means that u_t is also zero, which means that u should not change in time. Then we would guess that $u(x,t) = \alpha x + \beta$ for all t.

Either way, though, we can check that $u(x,t) = \alpha x + \beta$ is a solution to the system. Clearly it satisfies the initial condition $u(x,0) = \alpha x + \beta$, and moreover it satisfies the PDE ($u_t = 0$ and $u_{xx} = 0$). So it is a solution. (We will show later in class that it is the *only* solution.)

5. *Determine the type of the following equations:*

   (a) $u_{xx} + u_{xy} + u_{yy} + 3u_y = 0$,

   (b) $9u_{xx} - u_y = 0$.

   (c) Now, for the equation $u_{xx} + 3u_{xy} + \alpha u_{yy} = 0$, *determine which values of α make the equation elliptic.*

Solution:

(a) Using the notation of class (or the book), we have $\alpha_{11} = 1, \alpha_{12} = \frac{1}{2}, \alpha_{22} = 1$, and thus $\alpha_{11}\alpha_{22} = 1$ and $\alpha_{12}^2 = \frac{1}{4}$, so the equation is elliptic.

(b) We have $\alpha_{11} = 9$ and $\alpha_{12} = \alpha_{22} = 0$, and thus we have $0 \cdot 9 = 0^2$, so that the equation is parabolic. In fact, writing this equation as $u_y = 9u_{xx}$ and thinking of y as time, we have the heat equation exactly.

(c) Here we have $\alpha_{11} = 1, \alpha_{12} = \frac{3}{2}$. So we compare $\alpha \cdot 1$ and $(3/2)^2 = 9/4$. Thus if $\alpha > 9/4$, the equation is elliptic.

6. *We define the operator ∂_x by the equation $\partial_x u = \dfrac{\partial u}{\partial x}$, ∂_{xy} by $\partial_{xy} u = \frac{\partial^2 u}{\partial x \partial y}$, and similarly for other independent variables. Moreover, when we concatenate operators, we take it to mean composition, i.e.*

$$LMu := L(M(u)).$$

   (a) Show that $\partial_x^2 = \partial_{xx}$.

   (b) Show that

$$(\alpha\partial_x + \beta\partial_y)^2 = \alpha^2\partial_{xx} + 2\alpha\beta\partial_{xy} + \beta^2\partial_{yy},$$

   i.e. that they multiply like polynomials.

   (c) From this, prove that if α, β, γ are complex numbers, then there exist λ_1, λ_2 complex such that

$$\alpha\partial_{xx} + \beta\partial_{xy} + \gamma\partial_{yy} = \alpha(\partial_x - \lambda_1\partial_y)(\partial_x - \lambda_2\partial_y).$$

   Compute λ_1, λ_2 in terms of α, β, γ.

Solution: We compute the effect of each operator on a function and see that it is the same.

(a) We check:

$$\partial_x^2 u = \partial_x \partial_x u = \partial x \frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2} = \partial_{xx} u.$$

This is true for any function u and we are done.

(b) Now we compute:

$$\begin{aligned}
(\alpha\partial_x + \beta\partial_y)^2 u &= (\alpha\partial_x + \beta\partial_y)(\alpha\partial_x + \beta\partial_y)u \\
&= (\alpha\partial_x + \beta\partial_y)(\alpha u_x + \beta u_y) \\
&= \alpha\partial_x(\alpha u_x + \beta u_y) + \beta\partial_y(\alpha u_x + \beta u_y) \\
&= \alpha^2 u_{xx} + \alpha\beta u_{yx} + \beta\alpha u_{xy} + \beta^2 u_{yy} \\
&= \alpha^2 u_{xx} + 2\alpha\beta u_{xy} + \beta^2 u_{yy} \\
&= (\alpha^2\partial_{xx} + 2\alpha\beta\partial_{xy} + \beta^2\partial_{yy})u.
\end{aligned}$$

(c) There are a couple of ways to do this, but the most straightforward might be to compute the right-hand side first, and we have

$$\begin{aligned}
\alpha(\partial_x - \lambda_1\partial_y)(\partial_x - \lambda_2\partial_y)u &= \alpha(\partial_x - \lambda_1\partial_y)(u_x - \lambda_2 u_y) \\
&= \alpha\left[\partial_x(u_x - \lambda_2 u_y) - \lambda_1\partial_y(u_x - \lambda_2 u_y)\right] \\
&= \alpha\left[u_{xx} - (\lambda_1 + \lambda_2)u_{xy} + \lambda_1\lambda_2 u_{yy}\right].
\end{aligned}$$

So this will work, if we have

$$\begin{aligned}
\alpha(\lambda_1 + \lambda_2) &= -\beta, \\
\alpha\lambda_1\lambda_2 &= \gamma
\end{aligned}$$

Solving the second equation for λ_1 gives $\lambda_1 = \gamma/\alpha\lambda_2$; plugging this into the first gives

$$\frac{\gamma}{\lambda_2} + \alpha\lambda_2 = -\beta,$$

or

$$\alpha\lambda_2^2 + \beta\lambda_2 + \gamma = 0.$$

(Notice this similarity of this equation and the original left-hand side of the equation.) We then have two solutions for this quadratic polynomial, namely

$$\lambda_2 = \frac{-\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha},$$

and it's not hard to see that if we pick λ_2 to be either of these roots, this will make λ_1 the other one.