

Math 286 X1 — Midterm 1 Solutions
September 24, 2009

Question 1, Version A. **(12 points.)** In each of these problems, you should identify which kind of equation it is, then explain a method of solution. If the equation is of a type which we have not yet considered, explain this and say that we don't yet have a method.

a] $\frac{dy}{dx} + x^4y + x^{19} = 0$, b] $y' = x^2y$, c] $y' = x^2 + y$,
d] $y' = x^2 + y^2$, e] $x^4y'' + \sin(x)y' + y = 0$, f] $y'' - 3y' + 2y = 0$.

Solution:

- a. This is a first-order linear equation (here $p(x) = x^4, q(x) = -x^{19}$) and so we use an integrating factor.
 - b. This equation is both separable and first-order linear, so we could separate or use integrating factor.
 - c. This equation is first-order linear, so we use integrating factor.
 - d. We have no method for this equation.
 - e. We have no method to solve this equation fully, however notice that if we did have two solutions to the equation, we would be able to proceed.
 - f. Second-order linear, constant coefficient: use exponential Ansatz, get two solutions from roots, take linear combination.
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Question 1, Version B. **(12 points.)** In each of these problems, you should identify which kind of equation it is, then explain a method of solution. If the equation is of a type which we have not yet considered, explain this and say that we don't yet have a method.

a] $\frac{dy}{dx} + y/x = 1$, b] $y' = xy^3$, c] $y' = x^4 + y^2$,
d] $y' = x^2$, e] $xy'' + e^xy' + x^5y = 0$, f] $y'' - 2y' + 5y = 0$.

Solution:

- a. First-order linear equation (here $p(x) = 1/x, q(x) = 1$) and so we use an integrating factor.
- b. This equation is separable, so separate variables and integrate both sides.
- c. We have no method for this equation.
- d. The right-hand side is a function only of x , just integrate.
- e. We have no method to solve this equation fully, however notice that if we did have two solutions to the equation, we would be able to proceed.

- f. Second-order linear, constant coefficient: use exponential Ansatz, get two solutions from roots, take linear combination.

Question 2, Version A. (10 points.) Let $P(t)$ be a solution to

$$\frac{dP}{dt} = P(P-1)(P-2)e^{P^2 \sin(P^6)}.$$

If we consider the solution with $P(0) = 1/2$, what is $\lim_{t \rightarrow \infty} P(t)$? Same question, but if $P(0) = 3/2$? Same question, but $P(0) = 5/2$?

Solution: Since this is a first-order autonomous equation $P' = f(P)$, we need only know the sign and roots of f to answer these questions. In particular, a graph of f suffices. Also noting that the exponential term is always positive makes our job easier, since we need only graph $P(P-1)(P-2)$ and the signs will be correct. Finally, to get this graph is easy, since it is clearly zero at $P = 0, 1, 2$, and is a cubic with a positive leading order coefficient, so it must look like the graph in Figure 1. From this we see that $P = 1$ is stable and will

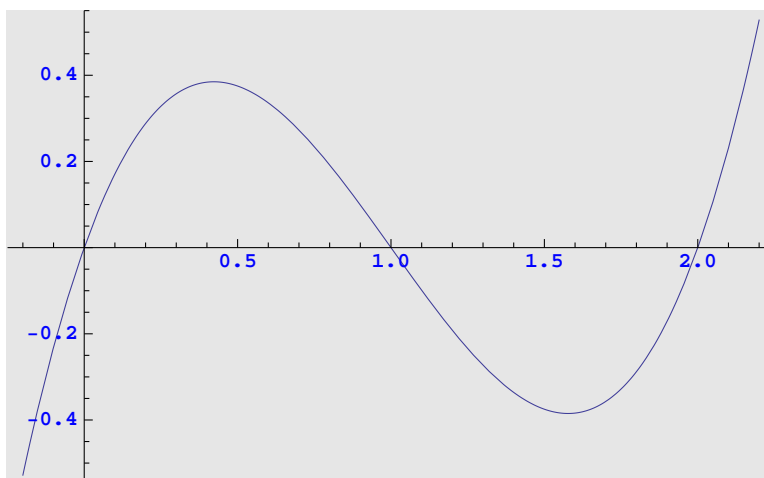


Figure 1: Graph of $P(P-1)(P-2)$.

attract any initial condition between 0 and 2; any IC which is < 0 goes to $-\infty$, and any IC above 2 goes to ∞ . Thus the answers to the three questions will be 1, 1, ∞ .

Question 2, Version B. (10 points.) Let $P(t)$ be a solution to

$$\frac{dP}{dt} = (P^2 - 1)(\sin(P^2) + 17)^2.$$

If we consider the solution with $P(0) = 0$, what is $\lim_{t \rightarrow \infty} P(t)$? Same question, but if $P(0) = 3$? Same question, but $P(0) = -17$?

Solution: This problem is similar to version A. Notice that the last factor is always positive (the minimum value which sine can take is -1 , thus $\sin(P^2) + 17 > 16 > 0$). Therefore

we can ignore this factor to determine the sign of f and need only plot $P^2 - 1$. This is a quadratic with roots at ± 1 , see Figure 2. From this graph we see that $P = -1$ is attracting

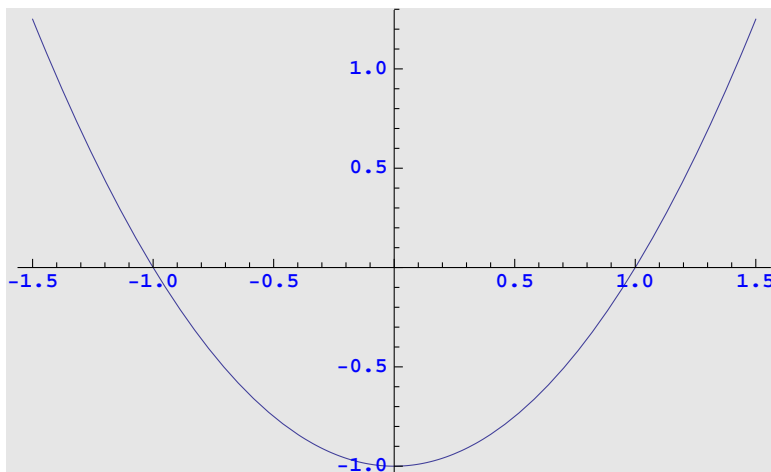


Figure 2: Graph of $P^2 - 1$.

and $P = 1$ is repelling, so that all initial conditions less than 1 go to -1 , but those bigger than 1 go to ∞ . Thus the answers are $-1, \infty, -1$.

Question 2, Version C. (10 points.) Let $P(t)$ be a solution to

$$\frac{dP}{dt} = P(P^2 - 1).$$

Identify all of the equilibrium solutions (also called fixed points) and determine whether they are stable or unstable.

Solution: To answer this question we need to plot the right-hand side. Notice that it has roots at $P = 0, 1, -1$, and since it's a cubic with a positive leading order coefficient the graph must look as in Figure 3. We see from this graph that 0 is the only stable fixed point, while -1 and 1 are unstable.

Question 3, Version A. (20 points.) Solve

$$y' + 3x^2y = e^{-x^3}, \quad y(0) = 4.$$

Solution: This equation is first-order linear, so we use the integrating factor

$$\rho(x) = e^{\int 3x^2 dx} = e^{x^3}.$$

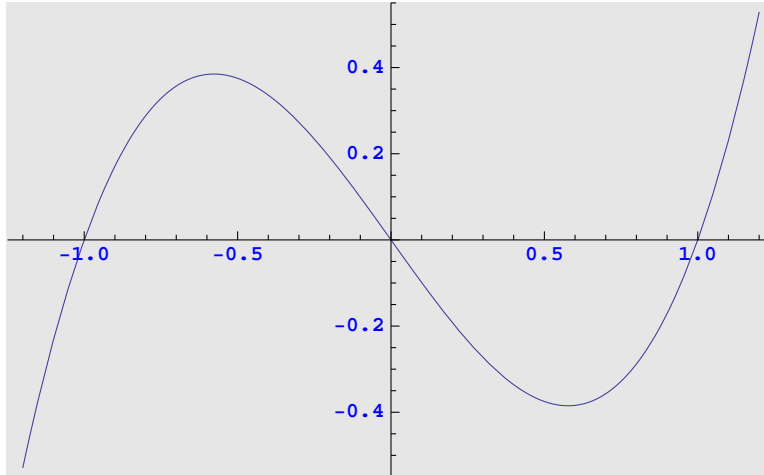


Figure 3: Graph of $P(P^2 - 1)$.

Multiplying through gives

$$\begin{aligned} e^{x^3} y'(x) + 3x^2 e^{x^3} y(x) &= 1, \\ \frac{d}{dx} (e^{x^3} y(x)) &= 1, \\ e^{x^3} y(x) &= x + C, \\ y(x) &= x e^{-x^3} + C e^{-x^3}. \end{aligned}$$

We need to satisfy the initial condition, plugging in gives

$$y(0) = C,$$

or $C = 4$, so the answer is

$$y(x) = x e^{-x^3} + 4 e^{-x^3}.$$

Question 3, Version B. **(20 points.)** Solve

$$x y' + 3y = 5x^2, \quad y(1) = 2.$$

Solution: This is a first-order linear equation, so we will use an integrating factor. We first need to divide through by x to obtain the equation

$$y' + \frac{3}{x} y = 5x.$$

The integrating factor is

$$\rho(x) = e^{\int 3/x dx} = e^{3 \ln(x)} = e^{\ln(x^3)} = x^3.$$

Multiplying through gives

$$\begin{aligned}x^3y'(x) + 3x^2y(x) &= 5x^4, \\ \frac{d}{dx}(x^3y(x)) &= 5x^4, \\ x^3y(x) &= x^5 + C, \\ y(x) &= x^2 + Cx^{-3}.\end{aligned}$$

Plugging in the initial condition gives

$$y(1) = 1 + C,$$

so $C = 1$, and we have

$$y(x) = x^2 + x^{-3}.$$

Question 3, Version C. (20 points.) Solve

$$y' - y = e^x, \quad y(0) = 2.$$

Solution: This is a first-order linear equation and we use an integrating factor. The integrating factor will be

$$\rho = e^{\int -1 dx} = e^{-x},$$

so we have

$$\begin{aligned}e^{-x}y'(x) - e^{-x}y(x) &= 1, \\ \frac{d}{dx}(e^{-x}y(x)) &= 1, \\ e^{-x}y(x) &= x + C, \\ y(x) &= xe^x + Ce^x.\end{aligned}$$

Plugging in $x = 0$ gives

$$y(0) = C,$$

or $C = 2$, so we get

$$y(x) = xe^x + 2e^x.$$

Question 4, Version A. (20 points.) Solve

$$\frac{dy}{dx} = \sin(x)y^2, \quad y(0) = 2.$$

Solution: This is a separable equation, so we separate to get

$$\frac{dy}{y^2} = \sin(x) dx,$$
$$-\frac{1}{y(x)} = -\cos(x) + C.$$

We can plug in the initial condition now and get

$$-\frac{1}{2} = -1 + C,$$

or $C = 1/2$, and so we have

$$-\frac{1}{y(x)} = -\cos(x) + \frac{1}{2},$$
$$\frac{1}{y(x)} = \cos(x) - \frac{1}{2},$$
$$y(x) = \frac{1}{\cos(x) - 1/2}.$$

Question 4, Version B. **(20 points.)** Solve

$$\frac{dy}{dx} = x^4 y, \quad y(0) = 4.$$

Solution: This equation is separable; separating gives

$$\frac{dy}{y} = x^4 dx,$$
$$\ln |y| = \frac{x^5}{5} + C,$$
$$y(x) = C e^{x^5/5}.$$

Plugging in the initial condition gives $C = 4$, so the solution is

$$y(x) = 4e^{x^5/5}.$$

Question 4, Version C. **(20 points.)** Solve

$$\frac{dy}{dt} = \cos(t)y, \quad y(0) = 5.$$

Solution: This equation is separable; separating gives

$$\begin{aligned}\frac{dy}{y} &= \cos(t) dt, \\ \ln |y| &= \sin(t) + C, \\ y(t) &= Ce^{\sin(t)}.\end{aligned}$$

Plugging in the initial condition gives $C = 5$, so the solution is

$$y(t) = 5e^{\sin(t)}.$$

Question 5, Version A. (**18 points.**) The top of the bell tower of Altgeld Hall is 40m above the ground. How fast do you need to throw an object from ground level so that it goes this high? (Assume you're throwing from exactly 0m height, and that the acceleration due to gravity is -10m/s^2 .)

Solution: The acceleration due to gravity is $g = -10\text{m/s}^2$. We know that acceleration is the derivative of velocity, so we have

$$\frac{dv}{dt} = g, \text{ or } v(t) = gt + v(0),$$

and since velocity is the derivative of position, then

$$\frac{dx}{dt} = gt + v(0), \text{ or } x(t) = \frac{g}{2}t^2 + v(0)t + x(0).$$

We assume that $x(0) = 0$, and we are looking for $v(0)$. We first need to determine at which time the object reaches its maximum height, and this will be when the velocity is zero, so we set

$$gt + v(0) = 0,$$

giving

$$t = -\frac{v(0)}{g}.$$

Plugging this into the equation for $x(t)$ gives

$$x(-v(0)/g) = \frac{(v(0))^2}{2g} - \frac{(v(0))^2}{g} = -\frac{(v(0))^2}{2g}$$

So we set this equal to 40m, and we get

$$\begin{aligned}-\frac{(v(0))^2}{2g} &= 40 \text{ m}, \\ (v(0))^2 &= 800 \text{ m}^2/\text{s}^2, \\ v(0) &= \sqrt{800} \text{ m/s}.\end{aligned}$$

Question 5, Version B. (18 points.) Let us say that we have two populations of bacteria whose growth is determined by the linear population model, but that they grow at different rates. Call these rates k_1, k_2 . We observe that population #1 starts off with 1000 bacteria and has 3000 bacteria one day later. Population #2 starts off with 500 bacteria and has 2500 bacteria one day later. Determine k_1 and k_2 . Now, will the size of population #2 ever surpass that of population #1? If not, say why not, but if so, determine when it happens.

Solution: We will use the linear growth model, so population #1 grows like

$$\frac{dP_1}{dt} = k_1 P_1,$$

and population #2 like

$$\frac{dP_2}{dt} = k_2 P_2.$$

First consider population #1. We can solve these equations to obtain

$$P_1(t) = P_1(0)e^{k_1 t}, \quad P_2(t) = P_2(0)e^{k_2 t}$$

The problem tells us

$$P_1(0 \text{ days}) = 1000, \quad P_1(1 \text{ day}) = 3000,$$

or

$$\begin{aligned} 3000 &= 1000e^{k_1(1 \text{ day})}, \\ 3 &= e^{k_1(1 \text{ day})}, \\ k_1(1 \text{ day}) &= \ln 3, \\ k_1 &= \ln 3 \text{ day}^{-1}. \end{aligned}$$

Plugging in the values to the other equation gives

$$k_2 = \ln 5 \text{ day}^{-1}.$$

Just looking at the numbers, we see that population #2 grows faster than population #1, but started smaller, so it should overtake at some point. We see if there is a point where the populations are equal:

$$\begin{aligned} 1000e^{k_1 t} &= 500e^{k_2 t}, \\ 2 &= e^{(k_2 - k_1)t}, \\ (k_2 - k_1)t &= \ln 2, \\ t &= \frac{\ln 2}{k_2 - k_1} \end{aligned}$$

Since $k_2 > k_1$, this gives a positive time, so population #2 will overtake population #1 in the future, in particular in

$$\frac{\ln 2}{\ln 5 - \ln 3} \text{ days.}$$

Question 5, Version C. (**18 points.**) Assume that a car's acceleration is proportional to the difference between 300 km/hr and its velocity. What is the maximum velocity it will ever attain? Now say that the car can go from rest to 100 km/hr in 5 seconds. How fast will it be going in 10 seconds?

Solution: We use $v(t)$ to denote the velocity of the car, then the problem tells us that

$$\frac{dv}{dt} = k(300 \text{ km/hr} - v)$$

for some unknown k . Notice that this is a first-order autonomous equation, so to get long-time behavior all we need to do is graph the right-hand side as a function of v . But this is just a line with negative slope and one root, at $v = 300$ km/hr. Thus there is a single attracting fixed point which attracts all initial conditions, so the maximum speed is 300 km/hr.

Now, to answer the second question, we need to solve the equation. This is a first-order linear equation, which we write as

$$\frac{dv}{dt} + kv = 300k \text{ km/hr.}$$

We use the integrating factor $\rho(t) = e^{kt}$, and we obtain

$$\begin{aligned} e^{kt}v'(t) + ke^{kt}v(t) &= 300ke^{kt} \text{ km/hr,} \\ \frac{d}{dt}(e^{kt}v(t)) &= 300ke^{kt} \text{ km/hr,} \\ e^{kt}v(t) &= 300e^{kt} \text{ km/hr} + C, \\ v(t) &= 300 \text{ km/hr} + Ce^{-kt}. \end{aligned}$$

Since $v(0) = 0$ km/hr, we have

$$C = 300 \text{ km/hr,}$$

or

$$v(t) = 300(1 - e^{-kt}) \text{ km/hr.}$$

Plugging in $t = 5$ s gives

$$\begin{aligned} v(5 \text{ s}) &= 300(1 - e^{-k(5 \text{ s})}) \text{ km/hr} = 100 \text{ km/hr,} \\ (1 - e^{-k(5 \text{ s})}) &= \frac{1}{3}, \\ e^{-k(5 \text{ s})} &= \frac{2}{3}, \\ -k(5 \text{ s}) &= \ln(2/3), \\ k &= \frac{\ln(3/2)}{5} \text{ s.} \end{aligned}$$

but then

$$\begin{aligned}v(10 \text{ s}) &= 300(1 - e^{-k(10 \text{ s})}) \text{ km/hr}, \\&= 300(1 - e^{-2\ln(3/2)}) \text{ km/hr}, \\&= 300(1 - e^{\ln(4/9)}) \text{ km/hr}, \\&= 300(1 - 4/9) \text{ km/hr} = 300 * 5/9 \text{ km/hr} = \frac{500}{3} \text{ km/hr}.\end{aligned}$$

Question 6, Version A. (20 points.) Solve

$$y'' - 4y' + 3y = 0, \quad y(0) = 2, \quad y'(0) = -2.$$

Solution: This is a second-order linear equation with constant coefficients. We make the exponential Ansatz

$$y(x) = e^{rx},$$

and we get

$$r^2 - 4r + 3 = 0.$$

This equation has two roots, $r = 1, 3$, so two solutions to this equation are

$$e^x, \quad e^{3x}.$$

The general solution is

$$y(x) = C_1 e^x + C_2 e^{3x}.$$

Plugging in initial conditions gives

$$\begin{aligned}y(0) &= C_1 + C_2 = 2, \\y'(0) &= C_1 + 3C_2 = -2.\end{aligned}$$

Subtracting the first equation from the second gives $2C_2 = -4$ or $C_2 = -2$ and thus $C_1 = 4$. Thus the answer is

$$y(x) = 4e^x - 2e^{3x}.$$

Question 6, Version B. (20 points.) Solve

$$y'' - 3y' + 2y = 0, \quad y(0) = 1, \quad y'(0) = -1.$$

Solution: This is a second-order linear equation with constant coefficients. We make the exponential Ansatz

$$y(x) = e^{rx},$$

and we get

$$r^2 - 3r + 2 = 0.$$

This equation has two roots, $r = 1, 2$, so two solutions to this equation are

$$e^x, \quad e^{2x}.$$

The general solution is

$$y(x) = C_1 e^x + C_2 e^{2x}.$$

Plugging in initial conditions gives

$$\begin{aligned} y(0) &= C_1 + C_2 = 1, \\ y'(0) &= C_1 + 2C_2 = -1. \end{aligned}$$

Subtracting the first equation from the second gives $C_2 = -2$ and thus $C_1 = 3$. The answer is

$$y(x) = 3e^x - 2e^{2x}.$$

Question 6, Version C. **(20 points.)** Solve

$$y'' - 6y' + 8y = 0, \quad y(0) = 2, \quad y'(0) = 6.$$

Solution: This is a second-order linear equation with constant coefficients. We make the exponential Ansatz

$$y(x) = e^{rx},$$

and we get

$$r^2 - 6r + 8 = 0.$$

This equation has two roots, $r = 2, 4$, so two solutions to this equation are

$$e^{2x}, \quad e^{4x}.$$

The general solution is

$$y(x) = C_1 e^{2x} + C_2 e^{4x}.$$

Plugging in initial conditions gives

$$\begin{aligned} y(0) &= C_1 + C_2 = 2, \\ y'(0) &= 2C_1 + 4C_2 = 6. \end{aligned}$$

Subtracting twice the first equation from the second gives $2C_2 = 2$ or $C_2 = 1$ and thus $C_1 = 1$. The answer is

$$y(x) = e^{2x} + e^{4x}.$$