

Introduction to Differential Equations – Math 286 X1
Fall 2009
Homework 9 — due November 11

1. Determine the fundamental period of the following functions:

- (a) $\cos(2t)$
- (b) $\sin(2\pi t)$
- (c) $\sin^2(t)$
- (d) $\cos(t) + \sin(t)$

Solution: We know from trig that the fundamental period of $\cos(t)$ or $\sin(t)$ is 2π . Clearly, then, if we consider $\cos(\alpha t)$ or $\sin(\alpha t)$ for any real α , then this should have fundamental period $2\pi/\alpha$. To see this, notice that $\cos(\alpha t)$ has a period of $2\pi/\alpha$:

$$\cos(\alpha(t + 2\pi/\alpha)) = \cos(\alpha t + 2\pi) = \cos(\alpha t),$$

and, moreover, if p is a period for $\cos(\alpha t)$, then we have

$$\cos(\alpha(t + p)) = \cos(\alpha t + \alpha p) = \cos(\alpha t)$$

for all t , and therefore $\alpha p = 2k\pi$ for some integer k , and thus $p = 2k\pi/\alpha$ for some integer k . The smallest of these is choosing $k = 1$, or $2\pi/\alpha$.

Once we know all this, solving (a,b) is straightforward, and we obtain π and 1, respectively.

For (c), we see that clearly any period of $\sin(t)$ is also a period of $\sin^2(t)$, i.e. if we have a p with $\sin(t + p) = \sin(t)$ for all t , then clearly, also $\sin^2(t + p) = \sin^2(t)$ for all t . However, it is possible that $\sin^2(t)$ has a smaller period, because if we require that

$$\sin^2(t + p) = \sin^2(t)$$

then this means that

$$\sin(t + p) = \pm \sin(t).$$

If it is possible to solve this equation with the minus sign with a p smaller than 2π , then we have a smaller fundamental period. But note that

$$\sin(t + \pi) = -\sin(t)$$

for all t , and thus

$$\sin^2(t + \pi) = \sin^2(t)$$

and we have a smaller fundamental period.

Part (d) is a bit trickier. We need a p which solves

$$\cos(t + p) + \sin(t + p) = \cos(t) + \sin(t)$$

for all t . Now, of course, any multiple of 2π will work, but can we do it with a smaller p ? We use the trig identities

$$\begin{aligned}\cos(A + B) &= \cos A \cos B - \sin A \sin B, \\ \sin(A + B) &= \cos A \sin B + \sin A \cos B.\end{aligned}$$

Thus we have

$$\begin{aligned}\cos(t+p) + \sin(t+p) &= \cos t \cos p - \sin t \sin p + \cos t \sin p + \sin t \cos p \\ &= (\cos p + \sin p) \cos t + (\cos p - \sin p) \sin t.\end{aligned}$$

Thus we need

$$\begin{aligned}\cos p + \sin p &= 1, \\ \cos p - \sin p &= 0,\end{aligned}$$

or $\cos p = 1, \sin p = 0$. This is solved by exactly $p = 2k\pi$, and thus 2π is the fundamental period.

2. Is the function $f(t) = \cos(t) + \cos(4t)$ periodic? If yes, demonstrate this by finding a period of the function. Same questions for $g(t) = \cos(t) + \cos(\pi t)$.

Solution: Yes it is, and the way to see that is to note that the periods of $\cos(t)$ are $2k\pi$, and the periods of $\cos(4t)$ are $l\pi/2$, where k, l are integers. These sets of numbers share a common element, namely 2π (choose $k = 1, l = 4$) and thus $f(t)$ is periodic with period 2π .

On the other hand, this will not work for g ; notice that the periods of $\cos(t)$ are $2k\pi$, but the periods of $\cos(\pi t)$ are $2l$, and

$$2k\pi = 2l$$

only if $\pi = l/k$, which would mean π is rational, which it is not.

3. Let $f(t)$ be a 2π -periodic function defined by

$$f(t) = \begin{cases} 3, & -\pi < t < 0, \\ -4, & 0 < t < \pi, \\ 132, & t = 0, \pi. \end{cases}$$

Compute its Fourier series.

Solution: We know that the 132 in the formula will not matter at all, so we can ignore it. Here we have $L = \pi$.

We use the formula for

$$\begin{aligned}A_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt) dt \\ &= \frac{1}{\pi} \left(\int_{-\pi}^0 3 \cos(nt) dt + \int_0^{\pi} -4 \cos(nt) dt \right), \\ &= \frac{1}{\pi} \left(\frac{3}{n} \sin(nt) \Big|_{t=-\pi}^{t=0} + \frac{-4}{n} \sin(nt) \Big|_{t=0}^{t=\pi} \right) \\ &= \frac{1}{\pi} (0 - 0 + 0 - 0) = 0,\end{aligned}$$

but this formula only works for $n > 0$ since we divided by n . For

$$A_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) dt = \frac{1}{\pi} i(-\pi) = -1.$$

For B_n , we compute

$$\begin{aligned}
 B_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt) dt \\
 &= \frac{1}{\pi} \left(\int_{-\pi}^0 3 \sin(nt) dt + \int_0^{\pi} -4 \sin(nt) dt \right), \\
 &= \frac{1}{\pi} \left(\frac{-3}{n} \cos(nt) \Big|_{t=-\pi}^{t=0} + \frac{4}{n} \cos(nt) \Big|_{t=0}^{t=\pi} \right) \\
 &= \frac{1}{n\pi} (-3(1 - (-1)^n) + 4((-1)^n - 1)),
 \end{aligned}$$

which is 0 when n is even, but $-14/n$ when n is odd. Therefore the Fourier series of f is

$$-\frac{1}{2} + \sum_{n \text{ odd}} \frac{-14}{n} \sin(nt).$$

4. Let $f(t)$ be a 2π -periodic function defined by $f(t) = |t|$ for $t \in [-\pi, \pi]$ and extended periodically elsewhere. Compute its Fourier series.

Solution: We first note that f is even and therefore all of the B_n are zero. Moreover, we can use the Fourier cosine series coefficient formula for A_n and save a bit of writing, so we have

$$\begin{aligned}
 A_n &= \frac{2}{\pi} \int_0^{\pi} f(t) \cos(nt) dt \\
 &= \frac{2}{\pi} \int_0^{\pi} t \cos(nt) dt, \\
 &= \frac{2}{\pi} \left(\frac{t \sin(nt)}{n} \Big|_{t=0}^{t=\pi} - \int_0^{\pi} \frac{1}{n} \sin(nt) dt \right) \\
 &= \frac{2}{\pi} \left(\frac{t \sin(nt)}{n} + \frac{\cos(nt)}{n^2} \Big|_{t=0}^{t=\pi} \right) = \frac{2}{\pi} \begin{cases} 0, & n \text{ even,} \\ \frac{-2}{n^2}, & n \text{ odd,} \end{cases}
 \end{aligned}$$

where, again, this does not work if $n = 0$. For A_0 we compute

$$A_0 = \frac{2}{\pi} \int_0^{\pi} t dt = \pi.$$

Thus the Fourier series is

$$\frac{\pi}{2} + \sum_{n \text{ odd}} \frac{-4}{n^2} \cos(nt).$$

5. Define f to be the function with period 3 defined as

$$f(t) = t^2, \quad -3/2 < t < 3/2.$$

Compute its Fourier series.

Solution: Again note that t^2 is even so we need not compute B_n . We have

$$A_n = \frac{2}{3} \int_{-3/2}^{3/2} t^2 \cos(2\pi nt/3) dt.$$

To simplify notation, we first compute

$$\int t^2 \cos(n\pi t/L) dt,$$

and after two integrations by parts, we obtain

$$\int t^2 \cos(n\pi t/L) dt = -\frac{t^2 L}{n\pi} \sin(n\pi t/L) + \frac{2tL^2}{n^2\pi^2} \cos(n\pi t/L) - \frac{2L^3}{n^3\pi^3} \sin(n\pi t/L).$$

Computing the definite integral (evaluating all of these terms at L and $-L$) gives

$$0 - 0 + \frac{2L^3}{n^2\pi^2} \cos(n\pi) + \frac{2L^3}{n^2\pi^2} \cos(-n\pi) = (-1)^n \frac{4L^3}{n^2\pi^2}.$$

Of course, we also have to do the separate calculation

$$A_0 = \frac{1}{L} \int_{-L}^L t^2 dt = \frac{1}{L} \left. \frac{t^3}{3} \right|_{t=-L}^{t=L} = \frac{2}{3} L^2.$$

Plugging in $L = 3/2$ gives

$$A_0 = \frac{3}{2},$$
$$A_n = (-1)^n \frac{27}{2n^2\pi^2}.$$

So we have a Fourier series of

$$\frac{3}{4} + \sum_{n=1}^{\infty} (-1)^n \frac{27}{2n^2\pi^2} \cos(2n\pi t/3).$$

6. Prove that

$$\int_{-\pi}^{\pi} \cos(nt) \sin(mt) dt = 0$$

for any integers n, m . **Hint:** Think about even and odd functions.

Solution: \cos is even and \sin is odd, therefore their product is odd, therefore the integral over any symmetric interval is zero.

7. Prove that

$$\int_{-\pi}^{\pi} \cos(nt) \cos(mt) dt = \begin{cases} 0, & m \neq n \\ \pi, & m = n. \end{cases}$$

Hint: For $m \neq n$, use the trig identity

$$\cos A \cos B = \frac{1}{2} (\cos(A+B) + \cos(A-B)).$$

Why does this calculation fail when $m = n$?

Solution: We compute

$$\begin{aligned}\int_{-\pi}^{\pi} \cos(nt) \cos(mt) &= \frac{1}{2} \int_{-\pi}^{\pi} \cos((m+n)t) + \cos((m-n)t) dt \\ &= \frac{1}{2} \left. \frac{\sin((m+n)t)}{m+n} + \frac{\sin((m-n)t)}{m-n} \right|_{t=-\pi}^{t=\pi} \\ &= \frac{1}{2}(0 - 0 + 0 - 0) = 0,\end{aligned}$$

but notice that we divided by $m - n$, so this formula is invalid for $m = n$.