Introduction to Differential Equations – Math 286 X1
Fall 2009
Homework 8 Solutions

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1. Solve

$$\mathbf{x}' = \begin{pmatrix} -1 & 1 \\ -4 & -1 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Solution: We first compute the eigenvalues and eigenvectors. The characteristic polynomial is

$$\lambda^2 + 2\lambda + 5 = 0,$$

which has roots

$$\lambda = \frac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 2i.$$

Choosing $\lambda = -1 + 2i$, we have

$$A - (-1+2i)I = \begin{pmatrix} -2i & 1 \\ -4 & -2i \end{pmatrix}$$

and solving

$$\begin{pmatrix} -2i & 1 \\ -4 & -2i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

gives

$$\begin{aligned} -2ix + y &= 0, \\ -4x - 2iy &= 0. \end{aligned}$$

One solution to this is $x = 1, y = 2i$, so our eigenvector can be chosen as

$$\mathbf{v} = \begin{pmatrix} 1 \\ 2i \end{pmatrix}.$$

Our two solutions will be the real and imaginary parts of $e^{\lambda t}\mathbf{v}$, which we compute as

$$e^{(-1+2i)t} \begin{pmatrix} 1 \\ 2i \end{pmatrix} = e^{-t}(\cos(2t) + i\sin(2t)) \begin{pmatrix} 1 \\ 2i \end{pmatrix} = e^{-t} \begin{pmatrix} \cos(2t) + i\sin(2t) \\ 2i\cos(2t) - 2\sin(2t) \end{pmatrix}.$$

Taking real and imaginary parts gives two solutions:

$$\mathbf{x}_1 = e^{-t} \begin{pmatrix} \cos(2t) \\ -2\sin(2t) \end{pmatrix}, \quad \mathbf{x}_2 = e^{-t} \begin{pmatrix} \sin(2t) \\ 2\cos(2t) \end{pmatrix}.$$

Thus the general solution is

$$\mathbf{x}(t) = C_1 e^{-t} \begin{pmatrix} \cos(2t) \\ -2\sin(2t) \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} \sin(2t) \\ 2\cos(2t) \end{pmatrix}.$$

Plugging in $t = 0$ gives

$$\mathbf{x}(0) = \begin{pmatrix} C_1 \\ 2C_2 \end{pmatrix},$$

so $C_1 = 1, C_2 = 1/2$, so the solution is

$$\mathbf{x}(t) = e^{-t} \begin{pmatrix} \cos(2t) \\ -2\sin(2t) \end{pmatrix} + \frac{1}{2} e^{-t} \begin{pmatrix} \sin(2t) \\ 2\cos(2t) \end{pmatrix}.$$

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2. Solve

$$\mathbf{x}' = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Solution: The characteristic polynomial is

$$\lambda^2 - 6\lambda + 9 = 0,$$

which is factored as $(\lambda - 3)^2 = 0$, so it has a double root of 3. We have that

$$A - 3I = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

and to get the eigenvector we get

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

or the equation $y = 0$. A solution to this is $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

This gives us one solution to the ODE, namely

$$e^{3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

We need to now find the generalized eigenvector, which we find by solving the system

$$(A - 3I)\mathbf{v}_2 = \mathbf{v}_1,$$

or

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

or the equation $y = 1$. So we can choose $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then we know another solution to this problem is

$$\mathbf{x}_2 = e^{3t}(t\mathbf{v}_1 + \mathbf{v}_2) = e^{3t}\left(t\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = e^{3t}\begin{pmatrix} t \\ 1 \end{pmatrix}.$$

Thus our general solution is

$$\mathbf{x}(t) = C_1 e^{3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} t \\ 1 \end{pmatrix}.$$

Plugging in $t = 0$ gives

$$\mathbf{x}(0) = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix},$$

so $C_1 = C_2 = 1$, and our solution is

$$\mathbf{x}(t) = e^{3t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{3t}\begin{pmatrix} t \\ 1 \end{pmatrix} = e^{3t}\begin{pmatrix} t+1 \\ 1 \end{pmatrix}.$$

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3. Solve

$$\mathbf{x}' = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Solution: The characteristic polynomial is

$$\lambda^2 - 5\lambda + 6 = 0,$$

which has roots $\lambda = 2, 3$. We have

$$A - 2I = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},$$

so the eigenvector for $\lambda = 2$ is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, and

$$A - 3I = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix},$$

so the eigenvector for $\lambda = 3$ is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Thus the general solution is

$$\mathbf{x}(t) = C_1 e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Plugging in $t = 0$ gives

$$\mathbf{x}(0) = C_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

which gives the system

$$\begin{aligned} C_1 + C_2 &= 1, \\ -C_1 &= 1, \end{aligned}$$

which has solution $C_1 = -1, C_2 = 2$. Thus our solution is

$$\mathbf{x}(t) = -e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + 2e^{3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

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4. For each of these matrices A, compute e^{tA}:

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad A = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}, \quad A = \begin{pmatrix} 7 & -4 \\ 8 & -5 \end{pmatrix}.$$

Solution: In each case, we need to compute the eigenvalues and eigenvectors of the matrix first. So we proceed:

(a) The characteristic polynomial is

$$\lambda^2 - 4\lambda + 4 = 0,$$

which has a double root of 2. We have

$$A - 2I = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

so that all vectors are eigenvectors and we can choose what we like. Choose

$$\mathbf{v}_1 = \left(\begin{array}{c} 1 \\ 0 \end{array} \right), \quad \mathbf{v}_2 = \left(\begin{array}{c} 0 \\ 1 \end{array} \right),$$

and this gives

$$S = \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right).$$

Since S is the identity matrix, $S^{-1} = S$, so we have

$$e^{tA} = S \left(\begin{array}{cc} e^{2t} & 0 \\ 0 & e^{2t} \end{array} \right) S^{-1} = I \left(\begin{array}{cc} e^{2t} & 0 \\ 0 & e^{2t} \end{array} \right) I = \left(\begin{array}{cc} e^{2t} & 0 \\ 0 & e^{2t} \end{array} \right).$$

(b) the characteristic polynomial is

$$\lambda^2 - \lambda - 2 = 0,$$

which has roots $\lambda = 2, -1$. For $\lambda = 2$, we have

$$A - 2I = \left(\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right),$$

and

$$\left(\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right) \left(\begin{array}{c} x \\ y \end{array} \right) = \left(\begin{array}{c} 0 \\ 0 \end{array} \right)$$

gives

$$y = 0,$$

so we have $\mathbf{v}_1 = \left(\begin{array}{c} 1 \\ 0 \end{array} \right)$. For $\lambda = -1$, we have

$$A + I = \left(\begin{array}{cc} 3 & 0 \\ 0 & 0 \end{array} \right),$$

and we have $\mathbf{v}_1 = \left(\begin{array}{c} 0 \\ 1 \end{array} \right)$. Thus we have

$$S = \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right).$$

Similarly to above, we have

$$S \left(\begin{array}{cc} e^{2t} & 0 \\ 0 & e^{-t} \end{array} \right) S = \left(\begin{array}{cc} e^{2t} & 0 \\ 0 & e^{-t} \end{array} \right)$$

since S is the identity.

(c) The characteristic polynomial is

$$\lambda^2 - 2\lambda - 3 = 0,$$

which has roots $\lambda = 3, -1$. We have

$$A - 3I = \left(\begin{array}{cc} 4 & -4 \\ 8 & -8 \end{array} \right)$$

and our eigenvector can be $\left(\begin{array}{c} 1 \\ 1 \end{array} \right)$, and

$$A + I = \left(\begin{array}{cc} 8 & -4 \\ 8 & -4 \end{array} \right),$$

so our eigenvector can be $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Thus

$$S = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix},$$

and

$$S^{-1} = \frac{1}{\det S} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}.$$

Thus

$$e^{tA} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -e^{-t} + 2e^{3t} & e^{-t} - e^{3t} \\ -2e^{-t} + 2e^{3t} & 2e^{-t} - e^{3t} \end{pmatrix}.$$