Introduction to Differential Equations – Math 286 X1
Fall 2009
Homework 7 Solutions

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1. Let

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Compute $A + B$. Compute AB and BA. Does $AB = BA$?

Solution: We have

$$A + B = \begin{pmatrix} 2 & 3 \\ 4 & 3 \end{pmatrix}, \quad AB = \begin{pmatrix} 3 & -1 \\ 7 & -1 \end{pmatrix}, \quad BA = \begin{pmatrix} 4 & 6 \\ -2 & -2 \end{pmatrix}.$$

We see that $AB \neq BA$.

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2. Let

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

Compute AB and BA. Does $AB = BA$?

Solution: We compute

$$AB = \begin{pmatrix} 2 & 6 \\ 4 & 8 \end{pmatrix}, \quad BA = \begin{pmatrix} 2 & 6 \\ 4 & 8 \end{pmatrix},$$

so $AB = BA$.

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3. Let

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

Find a matrix B so that $AB = BA$. Now find a second one. Check that they work.

Solution: There are several ways to go here. We could choose $B = A$, and of course if A and B are equal it does not matter in which order we multiply them. We could also choose $B = I$, since $AI = IA = A$. Pushing these ideas further, if we choose $B = \alpha A$ for any number α, then

$$AB = A(\alpha A) = \alpha AA = \alpha A^2, \quad BA = (\alpha A)A = \alpha A^2.$$

Similarly, we can choose $B = \alpha I$, since

$$AB = A(\alpha I) = \alpha AI = \alpha A, \quad BA = \alpha IA = \alpha A.$$

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4. For each of the following, determine whether or not the matrix has an inverse, and if it does, compute it:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ -1 & -3 \end{pmatrix}.$$

Solution:

(a) Determinant is -2, so inverse exists. We have

$$A^{-1} = \frac{1}{-2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}.$$

(b) Determinant is -2, so inverse exists. We have

$$A^{-1} = \frac{1}{-2}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}.$$

(c) Determinant is 0, so no inverse.

(d) Determinant is -8, so inverse exists. We have

$$A^{-1} = \frac{1}{-8}\begin{pmatrix} 1 & -3 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -1/8 & 3/8 \\ 3/8 & -1/8 \end{pmatrix}.$$

(e) Determinant is 0, so no inverse.

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5. Solve the system

$$\begin{aligned} x_1' &= -x_1 + 3x_2, \\ x_2' &= -2x_1 + 4x_2, \\ x_1(0) = 2&, \quad x_2(0) = 3. \end{aligned}$$

Solution: We write this in matrix form as

$$\mathbf{x}' = A\mathbf{x}, \quad A = \begin{pmatrix} -1 & 3 \\ -2 & 4 \end{pmatrix}.$$

We need to find the eigenvalues and eigenvectors of A. We have

$$A - \lambda I = \begin{pmatrix} -1-\lambda & 3 \\ -2 & 4-\lambda \end{pmatrix}$$

and

$$\det(A - \lambda I) = (-1-\lambda)(4-\lambda) + 6 = \lambda^2 - 3\lambda + 2.$$

The roots are $\lambda = 1, 2$.

For $\lambda_1 = 1$, we solve

$$(P - I)\mathbf{x} = \begin{pmatrix} -2 & 3 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which gives

$$-2x + 3y = 0,$$

and we choose $v_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

For $\lambda_2 = 2$, we solve

$$(P - I)\mathbf{x} = \begin{pmatrix} -3 & 3 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which gives

$$-3x + 3y = 0,$$

and we choose $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Thus our general solution is

$$\mathbf{x}(t) = C_1 e^t \begin{pmatrix} 3 \\ 2 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Plugging in $t = 0$ gives

$$\mathbf{x}(0) = C_1 \begin{pmatrix} 3 \\ 2 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2C_1 + C_2 \\ 3C_1 + C_2 \end{pmatrix},$$

so we solve

$$\begin{aligned} 3C_1 + C_2 &= 2, \\ 2C_1 + C_2 &= 3. \end{aligned}$$

This gives $C_1 = -1, C_2 = 5$, so our solution is

$$-e^t \begin{pmatrix} 3 \\ 2 \end{pmatrix} + 5e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -3e^t + 5e^{2t} \\ -2e^t + 5e^{2t} \end{pmatrix}.$$

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6. Solve the system

$$\mathbf{x}' = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Solution: We first find the eigenvalues and eigenvectors. The trace of the matrix is 0 and the determinant is -4, so the characteristic polynomial is

$$\lambda^2 + 4 = 0,$$

which has roots $\lambda = \pm 2$.

For $\lambda_1 = 2$, we have

$$A - 2I = \begin{pmatrix} -1 & 3 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which gives

$$-x + 3y = 0,$$

and we choose $v_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

For $\lambda_1 = -2$, we have

$$A + 2I = \begin{pmatrix} 3 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which gives

$$3x + 3y = 0,$$

and we choose $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Thus our general solution is

$$\mathbf{x}(t) = C_1 e^{2t} \begin{pmatrix} 3 \\ 1 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Plugging in $t = 0$ gives

$$\mathbf{x}(0) = \begin{pmatrix} 3C_1 + C_2 \\ C_1 - C_2 \end{pmatrix},$$

so we solve

$$\begin{aligned} 3C_1 + C_2 &= 1, \\ C_1 - C_2 &= -1. \end{aligned}$$

This gives $C_1 = 0, C_2 = 1$, so we have

$$\mathbf{x}(t) = e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

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7. Solve the system

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Solution: We first find the eigenvalues and eigenvectors. The trace of the matrix is 2 and the determinant is -1, so the characteristic polynomial is

$$\lambda^2 - 2\lambda - 1 = 0,$$

which has roots

$$\lambda = \frac{2 \pm \sqrt{4+4}}{2} = 1 \pm \sqrt{2}.$$

For $\lambda_1 = 1 + \sqrt{2}$, we have

$$A - (1+\sqrt{2})I = \begin{pmatrix} -\sqrt{2} & 1 \\ 2 & -\sqrt{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which gives

$$-\sqrt{2}x + y = 0,$$

and we choose $v_1 = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$.

For $\lambda_1 = 1 - \sqrt{2}$, we have

$$A - (1-\sqrt{2})I = \begin{pmatrix} \sqrt{2} & 1 \\ 2 & \sqrt{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which gives

$$\sqrt{2}x + y = 0,$$

and we choose $v_2 = \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}$. Thus our general solution is

$$\mathbf{x}(t) = C_1 e^{(1+\sqrt{2})t} \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} + C_2 e^{(1-\sqrt{2})t} \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}.$$

Plugging in $t = 0$ gives

$$\mathbf{x}(0) = \begin{pmatrix} C_1 + C_2 \\ \sqrt{2}(C_1 - C_2) \end{pmatrix},$$

so we solve

$$\begin{aligned} C_1 + C_2 &= 1, \\ \sqrt{2}(C_1 - C_2) &= 0. \end{aligned}$$

This gives $C_1 = 1/2, C_2 = 1/2$, so we have

$$\mathbf{x}(t) = \begin{pmatrix} \frac{1}{2}(e^{(1+\sqrt{2})t} + e^{(1-\sqrt{2})t}) \\ \frac{\sqrt{2}}{2}(e^{(1+\sqrt{2})t} - e^{(1-\sqrt{2})t}) \end{pmatrix}.$$