Introduction to Differential Equations – Math 286 X1
Fall 2009
Homework 6 Solutions

1. Consider a mass–spring system where the mass is 4 kg, and the spring constant is 1 kg s^{-2}.

   (a) Denote the position of the mass at time t by $x(t)$. Write down the equation which determines this position.

   (b) Compute the natural frequency of this system.

   (c) Write down the general solution for this system.

   (d) Force the system with the sinusoidal forcing $f(t) = 2\cos(3t)$ N. Write down the general solution for this system. What is the input/output response as defined in class? (Remember that I/O response is defined with an absolute value!!)

   (e) In each of parts (c,d), compute the specific solution we obtain if we assume that the initial position of the system is 0m and initial velocity of the system is 0 m/s.

   (f) Now imagine that we are allowed to force this system with an amplitude of 2N, but we can use any frequency we like, i.e. we choose $f(t) = 2\cos(\omega t)$ N, where ω is chosen by us. Assume that our system will break if the amplitude of the solution ever reaches 100m. Compute the range of ω which will cause this system to break.

   Solution:

   (a) The equation for an unforced mass–spring system is

   $$mx'' + cx' + kx = 0,$$

   and we have $m = 4$ kg, $c = 0$ kg/s, $k = 1$ kg/s^2, so the equation is

   $$4x'' + x = 0,$$

   which will now be in units of Newtons.

   (b) Plugging in the Ansatz $x = e^{rt}$ gives

   $$4r^2 + 1 = 0,$$

   or

   $$r = \pm\sqrt{\frac{1}{4}} = \pm\frac{1}{2}.$$

   So the natural frequency is $\omega_0 = 1/2$.

   (c) The general solution is

   $$x(t) = C_1\cos(t/2) + C_2\sin(t/2).$$

   (d) Since the forcing does not appear in the homogeneous solution, we guess

   $$x_p(t) = A\cos(3t),$$

   and we get

   $$-4\cdot 9A\cos(3t) + A\cos(3t) = -35A\cos(3t),$$

   so we have $A = -2/35$ or

   $$x_p(t) = -\frac{2}{35}\cos(3t).$$

   The general solution is then

   $$x(t) = C_1\cos(t/2) + C_2\sin(t/2) - \frac{2}{35}\cos(3t).$$

(e) In part (c), we plug in $x(0) = x'(0) = 0$ and we get $C_1 = C_2 = 0$, so the solution is $x(t) \equiv 0$. In part (d), we have

$$x(0) = C_1 - 2/35, x'(0) = C_2,$$

so we have $C_1 = 2/35, C_2 = 0$, or

$$x(t) = \frac{2}{35}(\cos(t/2) - \cos(3t)).$$

(f) We know the input/output response is

$$\frac{1}{|m(\omega_0^2 - \omega^2)|} = \frac{1}{4|1/4 - \omega^2|} = \frac{1}{|1 - 4\omega^2|}.$$

The system will break when the I/O response is larger than 50, since this would give an output amplitude of 100 m, so we need to solve

$$\frac{1}{|1 - 4\omega^2|} > 50,$$

which we solve as

$$\begin{aligned}
|1 - 4\omega^2| &< \frac{1}{50}, \\
-\frac{1}{50} < 1 - 4\omega^2 &< \frac{1}{50}, \\
-\frac{51}{50} < -4\omega^2 &< -\frac{49}{50}, \\
\frac{51}{200} > \omega^2 &> \frac{49}{200}, \\
\sqrt{\frac{51}{200}} > \omega &> \sqrt{\frac{49}{200}}.
\end{aligned}$$

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2. Take the same system and add friction (right now leave it as an unspecified c).

   (a) Write down the equation which governs the entire system when it is not forced.

   (b) Add a forcing to this system which has the natural frequency (as computed above). What is the input/output response? (Remember that this formula is different when $c > 0$!)

   (c) Finally, fix $c = 1$ kg/s and assume that we force with an unspecified frequency, i.e. $f(t) = F_0 \cos(\omega t)$. Find the value of ω which maximizes the input/output response.

   Solution:

   (a) When the system is unforced, the equation which governs this system is given by

   $$4x'' + cx' + kx = 0.$$

   (b) The general formula for the I/O response with friction is

   $$IOR = \frac{1}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} = \frac{1}{\sqrt{(1 - 4\omega^2)^2 + (c\omega)^2}}.$$

   If we force with the natural frequency from above (i.e. choose $\omega = 1/2$), then

   $$IOR = \frac{1}{(c/2)^2} = \frac{4}{c^2}.$$

(c) If we now fix c but leave ω unspecified, we have

$$IOR = \frac{1}{\sqrt{(1-4\omega^2)^2 + (\omega)^2}}.$$

We want to maximize the function

$$f(\omega) = \frac{1}{\sqrt{(1-4\omega^2)^2 + \omega^2}} = \frac{1}{\sqrt{1-7\omega^2+16\omega^4}} = (1-7\omega^2+16\omega^4)^{-1/2}.$$

We first want to find all of the critical points of this function, i.e. solve $f'(\omega) = 0$ or ∞. But we have

$$f'(\omega) = -1/2(-14\omega + 64\omega^3)(1-7\omega^2+16\omega^4)^{-3/2}.$$

We know, in fact, that $1-7\omega^2+16\omega^4 > 0$ since

$$1-7\omega^2+16\omega^4 = (1-4\omega^2)^2 + (\omega)^2,$$

and a sum of squares is nonnegative, and can only be zero if both terms are zero, but that is not possible. Thus we want to solve

$$-14\omega + 64\omega^3 = 0,$$

or

$$\omega^3 = -\frac{7}{32}\omega.$$

This has three roots,

$$\omega = 0, \pm\sqrt{\frac{7}{32}}.$$

We need to show that $\sqrt{7/32}$ is a maximum for $f(\omega)$. But notice that

$$f(\sqrt{7/32}) = \frac{8}{\sqrt{15}}, \quad f(0) = 1, \quad \lim_{\omega\to\infty} f(\omega) = 0.$$

We could check that $f''(\sqrt{7/32}) < 0$, but we don't need to: the function is lower to the left and to the right, and it's the only critical point. Therefore it is the maximum. (We could also have chosen $\omega = -\sqrt{7/32}$, but this is all the same since f is an even function.