Introduction to Differential Equations – Math 286 X1
Fall 2009
Homework 5 Solutions

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1. Solve
$$y'' - 3y' + 2y = 4e^{3x}, \quad y(0) = 1, \quad y'(0) = 2.$$

Solution: We first solve the homogeneous problem (note, this will be good for the next three questions). We have
$$y'' - 3y' + 2y = 0.$$
Using the Ansatz $y = e^{rx}$ gives
$$r^2 - 3r + 2 = 0,$$
which has roots $r = 1, 2$. So the general solution to the homogeneous equation is
$$y_h(x) = C_1 e^x + C_2 e^{2x}.$$
Noting that the right-hand side does not appear as a homogeneous solution, we guess
$$y_p(x) = Ae^{3x}.$$
Plugging this in gives
$$9Ae^{3x} - 3(3Ae^{3x}) + 2Ae^{3x} = 4e^{3x},$$
or $A = 2$, and thus
$$y_p(x) = 2e^{3x},$$
giving
$$y(x) = C_1 e^x + C_2 e^{2x} + 2e^{3x}.$$
To find C_1, C_2 we plug in the initial conditions, and we have
$$y(0) = C_1 + C_2 + 2 = 1,$$
$$y'(0) = C_1 + 2C_2 + 6 = 2.$$
Solving this gives $C_1 = 2, C_2 = -3$, so the solution is
$$y(x) = 2e^x - 3e^{2x} + 2e^{3x}.$$

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2. Solve
$$y'' - 3y' + 2y = e^{2x}, \quad y(0) = 1, \quad y'(0) = -1.$$

Solution: The homogeneous problem is as above, so we have
$$y_h(x) = C_1 e^x + C_2 e^{2x}.$$
Since the forcing appears as a homogeneous solution, we need to guess
$$y_p(x) = Axe^{2x}.$$

We evaluate:

$$y_p'(x) = Ae^{2x} + 2Axe^{2x},$$
$$y_p''(x) = 2Ae^{2x} + 2Ae^{2x} + 4Axe^{2x} = 4Ae^{2x} + 4Axe^{2x}.$$

Plugging in gives

$$\begin{aligned} &4Ae^{2x} + 4Axe^{2x} - 3(Ae^{2x} + 2Axe^{2x}) + 2(Axe^{2x}) = \\ &= (4A - 3A)e^{2x} + (4A - 6A + 2A)xe^{2x} \\ &= Ae^{2x}. \end{aligned}$$

Therefore $A = 1$ and

$$y_p(x) = xe^{2x}.$$

This gives a general solution of

$$y(x) = C_1e^x + C_2e^{2x} + xe^{2x}.$$

We compute that

$$y'(x) = C_1e^x + 2C_2e^{2x} + e^{2x} + 2xe^{2x},$$

and we have

$$y(0) = C_1 + C_2 = 1,$$
$$y'(0) = C_1 + 2C_2 + 1 = -1,$$

which has solution $C_1 = 4, C_2 = -3$, so

$$y(x) = 4e^x - 3e^{2x} + xe^{2x}.$$

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3. Solve

$$y'' - 3y' + 2y = 4\sin(x), \quad y(0) = 2, \quad y'(0) = 2.$$

Solution: The homogeneous solution is

$$y_h(x) = C_1e^x + C_2e^{2x}.$$

The forcing does not appear in the homogeneous solution, so we guess

$$y_p(x) = A\sin(x) + B\cos(x).$$

This gives

$$y_p'(x) = A\cos(x) - B\sin(x),$$
$$y_p''(x) = -A\sin(x) - B\cos(x).$$

Plugging in gives

$$-A\sin(x) - B\cos(x) - 3(A\cos(x) - B\sin(x)) + 2(A\sin(x) + B\cos(x)),$$

or

$$(-A + 3B + 2A)\sin(x) + (-B - 3A + 2B)\cos(x),$$

giving the system

$$A + 3B = 4,$$
$$-3A + B = 0.$$

This system has solution $A = 2/5, B = 6/5$, or

$$y_p(x) = \frac{2}{5}\sin(x) + \frac{6}{5}\cos(x).$$

Therefore the general solution is

$$y(x) = C_1 e^x + C_2 e^{2x} + \frac{2}{5}\sin(x) + \frac{6}{5}\cos(x).$$

We have

$$y'(x) = C_1 e^x + 2C_2 e^{2x} + \frac{2}{5}\cos(x) - \frac{6}{5}\sin(x),$$

so

$$y(0) = C_1 + C_2 + 6/5 = 2,$$
$$y'(0) = C_1 + 2C_2 + 2/5 = 2,$$

which has solution $C_1 = 0, C_2 = 4/5$, so our solution is

$$y(x) = \frac{4}{5}e^{2x} + \frac{2}{5}\cos(x) - \frac{6}{5}\sin(x).$$

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4. Solve

$$y'' - 4y' + 4y = e^x, \quad y(0) = 1, \quad y'(0) = 3.$$

Solution: We solve the homogeneous equation, this will be good for the next three problems. We have

$$y'' - 4y' + 4y = 0,$$

plugging in $y = e^{rx}$ gives

$$r^2 - 4r + 4 = 0,$$

which has a double root $r = 2, 2$. So our homogeneous solution is

$$y_h(x) = C_1 e^{2x} + C_2 x e^{2x}.$$

The forcing is not a homogeneous solution, so we guess

$$y_p(x) = Ae^x,$$

which, plugging in, gives

$$Ae^x - 4Ae^x + 4Ae^x = Ae^x,$$

so $A = 1$. Our general solution is

$$y(x) = C_1 e^{2x} + C_2 x e^{2x} + e^x.$$

We compute

$$y'(x) = 2C_1 e^{2x} + C_2 e^{2x} + 2C_2 x e^{2x} + e^x,$$

so

$$
\begin{aligned}
y(0) &= C_1 + 1 = 1, \\
y'(0) &= 2C_1 + C_2 + 1 = 3,
\end{aligned}
$$

which has solution $C_1 = 0, C_2 = 2$, or

$$y(x) = 2xe^{2x} + e^x.$$

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5. Solve

$$y'' - 4y' + 4y = 5e^{2x}, \quad y(0) = 3, \quad y'(0) = 2.$$

Solution: We have the homogeneous solution

$$y_h(x) = C_1e^{2x} + C_2xe^{2x}.$$

The forcing does appear in the homogeneous solution, so we guess

$$y_p(x) = Ax^2e^{2x}.$$

This gives

$$
\begin{aligned}
y_p'(x) &= 2Axe^{2x} + 2Ax^2e^{2x}, \\
y_p''(x) &= 2Ae^{2x} + 4Axe^{2x} + 4Axe^{2x} + 4Ax^2e^{2x} = 2Ae^{2x} + 8Axe^{2x} + 4Ax^2e^{2x}.
\end{aligned}
$$

Plugging in gives

$$
\begin{aligned}
&2Ae^{2x} + 8Axe^{2x} + 4Ax^2e^{2x} - 4(2Axe^{2x} + 2Ax^2e^{2x}) + 4(Ax^2e^{2x}) = \\
&= 2Ae^{2x} + (8A - 8A)xe^{2x} + (4A - 8A + 4A)x^2e^{2x},
\end{aligned}
$$

so we have $2A = 5$ or $A = 5/2$, so

$$y_p(x) = \frac{5}{2}x^2e^{2x}.$$

The general solution is then

$$y(x) = C_1e^{2x} + C_2xe^{2x} + \frac{5}{2}x^2e^{2x}.$$

We compute

$$y'(x) = 2C_1e^{2x} + C_2e^{2x} + 2C_2xe^{2x} + 5xe^{2x} + 5x^2e^{2x},$$

giving

$$
\begin{aligned}
y(0) &= C_1 = 3, \\
y'(0) &= 2C_1 + C_2 = 2,
\end{aligned}
$$

which has solution $C_1 = 3, C_2 = -4$, so

$$y(x) = 3e^{2x} - 4xe^{2x} + \frac{5}{2}x^2e^{2x}.$$

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6. Solve

$$y'' - 4y' + 4y = -4e^x \sin(x), \quad y(0) = 1, \quad y'(0) = 2.$$

Solution: We have the homogeneous solution

$$y_h(x) = C_1 e^{2x} + C_2 x e^{2x}.$$

The forcing does appear in the homogeneous solution, so we guess

$$y_p(x) = Ae^x \sin(x) + Be^x \cos(x).$$

We compute

$$\begin{aligned} y_p'(x) &= Ae^x \sin(x) + Ae^x \cos(x) + Be^x \cos(x) - Be^x \sin(x) \\ &= (A - B)e^x \sin(x) + (A + B)e^x \cos(x). \end{aligned}$$

We also have

$$\begin{aligned} y_p''(x) &= (A - B)e^x \sin(x) + (A - B)e^x \cos(x) + (A + B)e^x \cos(x) - (A + B)e^x \sin(x) \\ &= -2Be^x \sin(x) + 2Ae^x \cos(x). \end{aligned}$$

Plugging in gives

$$\begin{aligned} &-2Be^x \sin(x) + 2Ae^x \cos(x) - 4((A - B)e^x \sin(x) + (A + B)e^x \cos(x)) + 4(Ae^x \sin(x) + Be^x \cos(x)) \\ &= (-2B - 4(A - B) + 4A)e^x \sin(x) + (2A - 4(A + B) + 4B)e^x \cos(x) \\ &= 2Be^x \sin(x) - 2Ae^x \cos(x). \end{aligned}$$

This gives $A = 0, B = -2$, or

$$y_p(x) = -2e^x \cos(x).$$

Thus the general solution is

$$y(x) = C_1 e^{2x} + C_2 x e^{2x} - 2e^x \cos(x).$$

We also have

$$y'(x) = 2C_1 e^{2x} + C_2 e^{2x} + 2C_2 x e^{2x} - 2e^x \cos(x) + 2e^x \sin(x).$$

Thus

$$\begin{aligned} y(0) &= C_1 - 2 = 1, \\ y'(0) &= 2C_1 + C_2 - 2 = 2, \end{aligned}$$

which has solution $C_1 = 3, C_2 = -2$. Thus our solution is

$$y(x) = 3e^{2x} - 2xe^{2x} - 2e^x \cos(x).$$

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7. Solve

$$y'' + 2y' + 2y = 2e^x, \quad y(0) = 2, \quad y'(0) = 1.$$

Solution: We first solve the homogeneous equation

$$y'' + 2y' + 2y = 0.$$

Making the Ansatz $y = e^{rx}$ gives

$$r^2 + 2r + 2 = 0,$$

whose roots are

$$r = \frac{-2 \pm \sqrt{-4}}{2} = -1 \pm i.$$

Thus the homogeneous solution will be

$$y_h(x) = C_1 e^{-x} \cos(x) + C_2 e^{-x} \sin(x).$$

Since the forcing does not appear here, we guess

$$y_p(x) = Ae^x.$$

Plugging in gives

$$Ae^x + 2Ae^x + 2Ae^x = 5Ae^x,$$

so $A = 2/5$ and

$$y_p(x) = \frac{2}{5}e^x.$$

Thus the general solution is

$$y(x) = C_1 e^{-x} \cos(x) + C_2 e^{-x} \sin(x) + \frac{2}{5}e^x,$$

and

$$y'(x) = -C_1 e^{-x} \cos(x) - C_1 e^{-x} \sin(x) - C_2 e^{-x} \sin(x) + C_2 e^{-x} \cos(x) + \frac{2}{5}e^x.$$

We have

$$\begin{aligned} y(0) &= C_1 + 2/5 = 2, \\ y'(0) &= -C_1 + C_2 + 2/5 = 1, \end{aligned}$$

which has solution $C_1 = 8/5, C_2 = 11/5$, so the solution is

$$y(x) = \frac{8}{5}e^{-x} \cos(x) + \frac{11}{5}e^x \sin(x) + 2e^x.$$

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8. Solve

$$y'' + 2y' + 2y = -\sin(x), \quad y(0) = 1, \quad y'(0) = -2.$$

Solution: The solution to the homogeneous equation is

$$y_h(x) = C_1 e^{-x} \cos(x) + C_2 e^{-x} \sin(x).$$

Since the forcing does not appear here, we guess

$$y_p(x) = A\sin(x) + B\cos(x).$$

We have

$$\begin{aligned} y_p'(x) &= A\cos(x) - B\sin(x), \\ y_p''(x) &= -A\sin(x) - B\cos(x). \end{aligned}$$

Plugging in gives

$$\begin{aligned}&-A\sin(x)-B\cos(x)+2(A\cos(x)-B\sin(x))+2(A\sin(x)+B\cos(x))\\&=(-A-2B+2A)\sin(x)+(-B+2A+2B)\cos(x),\end{aligned}$$

or

$$\begin{aligned}A-2B&=-1,\\2A+B&=0,\end{aligned}$$

which has solution $A=-1/5, B=2/5$, so we have

$$y_p(x)=-\frac{1}{5}\sin(x)+\frac{2}{5}\cos(x).$$

Thus the general solution is

$$y(x)=C_1e^{-x}\cos(x)+C_2e^{-x}\sin(x)-\frac{1}{5}\sin(x)+\frac{2}{5}\cos(x),$$

with

$$y'(x)=-C_1e^{-x}\cos(x)-C_1e^{-x}\sin(x)-C_2e^{-x}\sin(x)+C_2e^{-x}\cos(x)-\frac{1}{5}\cos(x)-\frac{2}{5}\sin(x).$$

Thus we have

$$\begin{aligned}y(0)&=C_1+2/5=1,\\y'(0)&=-C_1+C_2-1/5=-2,\end{aligned}$$

which has solution $C_1=3/5, C_2=-6/5$, so our solution is

$$y(x)=\frac{3}{5}e^{x}\cos(x)-\frac{6}{5}e^{x}\sin(x)-\frac{1}{5}\sin(x)+\frac{2}{5}\cos(x).$$

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9. Solve

$$y''+2y'+2y=3e^{-x}\cos(x),\quad y(0)=0,\quad y'(0)=-1.$$

Solution: The solution to the homogeneous equation is

$$y_h(x)=C_1e^{-x}\cos(x)+C_2e^{-x}\sin(x).$$

The forcing appears as a homogeneous solution, so we have to guess

$$y_p(x)=Axe^{-x}\cos(x)+Bxe^{-x}\sin(x).$$

We compute

$$\begin{aligned}y_p'(x)&=Ae^{-x}\cos(x)-Axe^{-x}\cos(x)-Axe^{-x}\sin(x)+Be^{-x}\sin(x)\\&\quad-Bxe^{-x}\sin(x)+Bxe^{-x}\cos(x)\\&=Ae^{-x}\cos(x)+Be^{-x}\sin(x)+(B-A)xe^{-x}\cos(x)+(-A-B)xe^{-x}\sin(x).\end{aligned}$$

Then

$$\begin{aligned}
y_p''(x) &= -Ae^{-x}\cos(x) - Ae^{-x}\sin(x) - Be^{-x}\sin(x) + Be^{-x}\cos(x) \\
&\quad + (B-A)e^{-x}\cos(x) + (A-B)xe^{-x}\cos(x) + (A-B)xe^{-x}\sin(x) \\
&\quad + (-A-B)e^{-x}\sin(x) + (A+B)xe^{-x}\sin(x) + (-A-B)xe^{-x}\cos(x) \\
&= (-A+B+(B-A))e^{-x}\cos(x) + (-A-B-A-B)e^{-x}\sin(x) \\
&\quad + (A-B-A-B)xe^{-x}\cos(x) + (A-B+A+B)xe^{-x}\sin(x) \\
&= (2B-2A)e^{-x}\cos(x) + (-2A-2B)e^{-x}\sin(x) - 2Bxe^{-x}\cos(x) + 2Axe^{-x}\sin(x).
\end{aligned}$$

Aaaand plugging in we get

$$\begin{aligned}
&(2B-2A)e^{-x}\cos(x) + (-2A-2B)e^{-x}\sin(x) - 2Bxe^{-x}\cos(x) + 2Axe^{-x}\sin(x) \\
&\quad + 2(Ae^{-x}\cos(x) + Be^{-x}\sin(x) + (B-A)xe^{-x}\cos(x) + (-A-B)xe^{-x}\sin(x)) \\
&\quad + 2(Axe^{-x}\cos(x) + Bxe^{-x}\sin(x)) \\
&= (2B-2A+2A)e^{-x}\cos(x) + (-2A-2B+2B)e^{-x}\sin(x) \\
&\quad + (-2B+2(B-A)+2A)xe^{-x}\cos(x) + (2A+2(-A-B)+2B)xe^{-x}\sin(x) \\
&= 2Be^{-x}\cos(x) - 2Ae^{-x}\sin(x),
\end{aligned}$$

so we have $A = 0, B = 3/2$, or

$$y_p(x) = \frac{3}{2}xe^{-x}\sin(x).$$

Thus the general solution is

$$y(x) = C_1e^{-x}\cos(x) + C_2e^{-x}\sin(x) + \frac{3}{2}xe^{-x}\sin(x).$$

We have

$$\begin{aligned}
y'(x) &= -C_1e^{-x}\cos(x) - C_1e^{-x}\sin(x) - C_2e^{-x}\sin(x) + C_2e^{-x}\cos(x) \\
&\quad + \frac{3}{2}e^{-x}\sin(x) - \frac{3}{2}xe^{-x}\sin(x) + \frac{3}{2}xe^{-x}\cos(x).
\end{aligned}$$

This gives

$$\begin{aligned}
y(0) &= C_1 = 0, \\
y'(0) &= -C_1 + C_2 = -1,
\end{aligned}$$

which has solution $C_1 = 0, C_2 = -1$, so

$$y(x) = -e^{-x}\sin(x) + \frac{3}{2}xe^{-x}\sin(x).$$

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10. Solve

$$y'' + 2y' + 2y = 3x - 4, \quad y(0) = 0, \quad y'(0) = 1.$$

Solution: The solution to the homogeneous equation is

$$y_h(x) = C_1e^{-x}\cos(x) + C_2e^{-x}\sin(x).$$

Since the forcing does not appear here, we guess

$$y_p(x) = Ax + B.$$

Plugging in we have

$$2A + 2(Ax + B) = 3x - 4,$$

or $A = 3/2, B = -7/2$, so

$$y_p(x) = \frac{3}{2}x - \frac{7}{2}.$$

This gives a general solution of

$$y(x) = C_1 e^{-x}\cos(x) + C_2 e^{-x}\sin(x) + \frac{3}{2}x - \frac{7}{2},$$

and

$$y'(x) = -C_1 e^{-x}\cos(x) - C_1 e^{-x}\sin(x) - C_2 e^{-x}\sin(x) + C_2 e^{-x}\cos(x) + \frac{3}{2},$$

so we have

$$\begin{aligned} y(0) &= C_1 - 7/2 = 0, \\ y'(0) &= -C_1 + C_2 + 3/2 = 1, \end{aligned}$$

which has solution $C_1 = 7/2, C_2 = 3$, so

$$y(x) = \frac{7}{2}e^{-x}\cos(x) + 3e^{-x}\sin(x) + \frac{3}{2}x - \frac{7}{2}.$$