Introduction to Differential Equations – Math 286 X1
Fall 2009
Homework 3 Solutions

1. (a) Solve exactly:

$$\frac{dx}{dt} = 2x - x^2, \quad x(0) = 3.$$

   (b) Now solve exactly:

$$\frac{dx}{dt} = 2x - x^2, \quad x(0) = 1.$$

   (c) Now forget about exact solutions, and use the qualitative analysis we discussed in class to analyze the ODE

$$\frac{dx}{dt} = 2x - x^2.$$

   What is the long-time behavior ($t \to \infty$) of any solution to this equation with $x(0) > 0$? Is this consistent with the answers to #1,2?

Solution: We first compute the general solution to this equation. We will use separation, as follows:

$$\frac{dx}{2x - x^2} = dt,$$
$$\frac{dx}{x(2-x)} = dt.$$

Writing

$$\frac{1}{x(2-x)} = \frac{A}{x} + \frac{B}{2-x}$$

and multiplying through gives

$$1 = A(2-x) + Bx = 2A + (B-A)x,$$

so we get

$$B - A = 0, \quad 2A = 1,$$

or

$$A = 1/2, \quad B = 1/2.$$

Thus we have

$$\int \frac{1}{2x} + \frac{1}{2(2-x)} = \int dt,$$
$$\frac{1}{2}\ln x - \frac{1}{2}\ln(2-x) = t + C,$$
$$\ln\left(\frac{x}{2-x}\right) = 2t + C,$$
$$\frac{x(t)}{2-x(t)} = Ce^{2t},$$
$$x(t) = Ce^{2t}(2 - x(t)),$$
$$x(t)(1 + Ce^{2t}) = 2Ce^{2t},$$
$$x(t) = \frac{2Ce^{2t}}{1+Ce^{2t}}.$$

Now, we plug in the initial conditions. If $x(0) = 3$, then we get

$$\frac{2C}{1+C} = 3, \quad C = -3,$$

and the solution is

$$x(t) = \frac{-6e^{2t}}{1-3e^{2t}}.$$

In the case where $x(0) = 1$, we get

$$\frac{2C}{1+C} = 1, \quad C = 1,$$

so the solution is

$$x(t) = \frac{2e^{2t}}{1+e^{2t}}.$$

Now, let us graph the right-hand side of this function, see Figure 1.

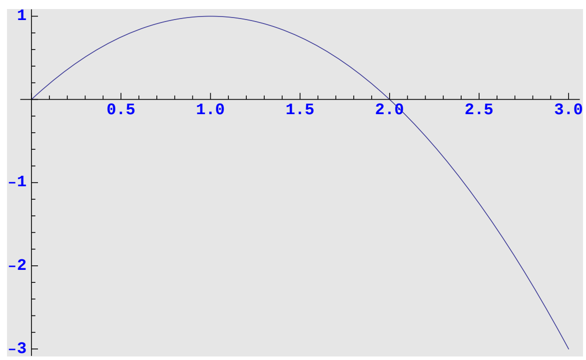

Figure 1: A graph of $2x - x^2$

From this graph we can see that there are two fixed points, at $x = 0, 2$, and that $x = 2$ is an attracting fixed point: all trajectories which start to the right of 0 will end up approaching 2. Therefore

$$\lim_{t\to\infty} x(t) = 2.$$

Looking at the two exact formulas derived above, we see that this is consistent. For we see that

$$\lim_{t\to\infty} \frac{-6e^{2t}}{1-3e^{2t}} = \lim_{t\to\infty} \frac{-6}{e^{-2t}-3} = \frac{-6}{0-3} = 2,$$

and similarly for the other.

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2. Consider the autonomous equation $\frac{dy}{dt} = f(y)$. In each case, either demonstrate a function f (drawing the graph of f is fine, no need for a formula), or argue why no such f can exist:
   (a) the equation has no fixed points
   (b) the equation has exactly one fixed point, and it is stable
   (c) the equation has exactly one fixed point, and it is unstable
   (d) the equation has exactly 3 stable and 2 unstable fixed points
   (e) the equation has 5 stable fixed points and 1 unstable fixed point

Solution:

(a) Any curve which does not intersect the y-axis will do, for example $f(y) = 5 + \sin y$ works.

(b) Any curve which intersects the axis once with negative slope will do, for example $f(y) = 2 - 3y$.

(c) Any curve which intersects the axis once with positive slope will do, for example $f(y) = 2 + 3y$.

(d) Any curve which crosses the axis exactly five times, which is also positive to the left of the points, and negative to the right, will work.

(e) This is not possible.

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3. We consider a population of deer and represent the size of the population at time t by $P(t)$. Assume that the rate of growth of the population is proportional to $\sqrt{P}$. We also know that the population at $t = 0$ is 36 deer, and it is increasing at the rate of 12 deer/month. How many deer will there be in one year? In three years? How long will it take to get one million deer?

Solution: From the word problem, we see that the law for the population growth is

$$\frac{dP}{dt} = k\sqrt{P(t)},$$

where k is some unknown constant. Plugging in $t = 0$ to both sides of this equation gives

$$12 \text{ deer/month} = k\sqrt{36 \text{ deer}} = 6k\sqrt{\text{deer}},$$

so we get

$$k = 2\frac{\sqrt{\text{deer}}}{\text{month}}.$$

We can solve the equation as well, since it is separable, we have:

$$\begin{aligned}
\frac{dP}{\sqrt{P}} &= k\,dt, \\
2P^{1/2} &= kt + C, \\
P^{1/2} &= 2kt + C, \\
P(t) &= (2kt + C)^2.
\end{aligned}$$

Using the information about k from above, and using the initial condition again, gives

$$\begin{aligned}
36 \text{ deer} &= \left(4\frac{\sqrt{\text{deer}}}{\text{month}}0 + C\right)^2, \\
6\sqrt{\text{deer}} &= C.
\end{aligned}$$

Putting this together gives

$$P(t) = \left(4\frac{\sqrt{\text{deer}}}{\text{month}}t + 6\sqrt{\text{deer}}\right)^2 = \left(\frac{4}{\text{month}}t + 6\right)^2 \text{ deer}.$$

So, if we plug in t in *months*, we get the answer in deer. Thus, to answer the questions, we need to compute $P(12), P(36)$ and solve $P(t) = 10^6$ deer. So we have

$$\begin{aligned}
P(12 \text{ months}) &= (48 + 6)^2 \text{ deer} = 2,916 \text{ deer}, \\
P(36 \text{ months}) &= (144 + 6)^2 \text{ deer} = 22,500 \text{ deer}.
\end{aligned}$$

Solving, we get

$$\begin{aligned}\left(\frac{4}{\text{month}}t+6\right)^2 &= 10^6,\\ \frac{4}{\text{month}}t+6 &= 1000,\\ \frac{4}{\text{month}}t &= 994,\\ t &= \frac{994}{4}\text{ months} = 248\frac{1}{2}\text{ months},\end{aligned}$$

(a bit more than 20 years).

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4. In each of these IVPs, determine whether or not the Existence–Uniqueness Theorem (Theorem 1.3.1 in the book, or in class) guarantees a unique solution:

(a) $\dfrac{dy}{dx} = 2x^6y^2, \quad y(2) = 3,$

(b) $\dfrac{dy}{dx} = 2x^6y^{-2}, \quad y(2) = 3,$

(c) $\dfrac{dy}{dx} = 2x^6y^{-2}, \quad y(1) = 0,$

(d) $\dfrac{dy}{dx} = \ln y, \quad y(0) = 0,$

(e) $\dfrac{dy}{dx} = \ln y, \quad y(1) = 1.$

Solution:

(a) The right-hand side is continuous with continuous derivative everywhere, thus any rectangle works, thus any initial condition works. So, yes.

(b) The function becomes discontinuous, but only at $y = 0$, so any rectangle which avoids $y = 0$ will work. Clearly, we can draw a rectangle around $(2, 3)$ without hitting the x-axis, so yes.

(c) See previous, we cannot draw a rectangle around $(0, 0)$ which avoids the x-axis, so no.

(d) In this case, the function is fine for $y > 0$ but not for $y \leq 0$, so if we can draw a rectangle completely in the upper half-plane around the initial condition, then we are good. However, we clearly cannot draw such a rectangle around $(0, 0)$, so no.

(e) See previous, here we can draw a rectangle in the upper half-plane around $(1, 1)$, so yes.