# Introduction to Differential Equations – Math 286 X1
# Fall 2009
# Homework 2 Solutions

1. Solve each of the following differential equations:

   (a) $y' + 3xy = 0$

   (b) $y' + 3y = 3x$

   (c) $\frac{dy}{dt} = \cos(t)y$

   (d) $x^2\frac{dy}{dx} - y = 3$

   **Solution:**

   (a) This is a first-order linear equation, so we use an integrating factor. The integrating factor is $\rho(x) = e^{3x^2/2}$, so we have

   $$\begin{aligned} e^{3x^2/2}y'(x) + 3xe^{3x^2/2}y(x) &= 0, \\ \frac{d}{dx}(e^{3x^2/2}y(x)) &= 0, \\ e^{3x^2/2}y(x) &= C, \\ y(x) &= Ce^{-3x^2/2}. \end{aligned}$$

   (We could also use separation.)

   (b) Again first-order linear, and our integrating factor is $\rho(x) = e^{3x}$. So we have

   $$\begin{aligned} e^{3x}y' + 3e^{3x}y &= 3xe^{3x}, \\ \frac{d}{dx}(e^{3x}y) &= 3xe^{3x}, \\ e^{3x}y &= xe^{3x} - \frac{1}{3}e^{3x} + C, \\ y(x) &= x - \frac{1}{3} + Ce^{-3x}. \end{aligned}$$

   (c) We separate variables to get

   $$\frac{dy}{y} = \cos(t)\,dt,$$

   or

   $$\log|y| = \sin(t) + C,$$

   or

   $$y(t) = Ce^{\sin t}.$$

   (d) We first divide by $x^2$ and get

   $$y' - \frac{1}{x^2}y = \frac{3}{x^2}.$$

   Our integrating factor should be $e^{1/x}$ and so we get

   $$e^{1/x}y' - \frac{1}{x^2}e^{1/x}y = \frac{3}{x^2}e^{1/x},$$

Integrating gives

$$e^{1/x}y(x) = -3e^{1/x} + C,$$

or

$$y(x) = -3 + Ce^{1/x}.$$

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2. Solve each of the following IVPs:

   (a) $y' - 2y = 0, \quad y(0) = 0$

   (b) $\dfrac{dy}{dx} = x^2 y, \quad y(1) = 2$

   (c) $(1+x)y' + y = 3, \quad y(0) = -1$

   (d) $xy' + (3x+1)y = 5, \quad y(2) = 4$

   **Solution:**

   (a) This one we can solve many ways, but we see that the solution is $y(x) = Ce^{2x}$. Plugging in the initial condition gives $C = 0$, so the solution is $y(x) \equiv 0$.

   (b) We solve by separation to get

   $$\begin{aligned} \frac{dy}{y} &= x^2\,dx, \\ \ln|y| &= \frac{x^3}{3} + C, \\ y(x) &= Ce^{x^3/3}. \end{aligned}$$

   Plugging in the initial condition gives $y(1) = Ce^{1/3} = 2$, so $C = 2e^{-1/3}$, and thus the solution is

   $$y(x) = 2e^{(x^3-1)/3}.$$

   (c) We see that the left-hand side is already in the form of a product rule, so we can write

   $$\begin{aligned} \frac{d}{dx}((1+x)y(x)) &= 3 \\ (1+x)y(x) &= 3x + C \\ y(x) &= \frac{3x}{1+x} + \frac{C}{1+x}. \end{aligned}$$

   (If we didn't make this observation, then, first divide by the front coefficient to get

   $$y' + \frac{1}{1+x}y = \frac{3}{1+x}.$$

   The integrating factor will be

   $$\rho(x) = e^{\int \frac{1}{1+x}} = e^{\ln|1+x|} = 1 + x$$

   and multiplying through gives the original equation.) In any case, plugging in $x = 0$ gives

   $$y(x) = C = -1,$$

   so the solution is

   $$y(x) = \frac{3x-1}{1+x}.$$

(d) Divide through by the front coefficient to get

$$y' + (3 + \frac{1}{x})y = \frac{5}{x}.$$

The integrating factor is

$$e^{\int 3+\frac{1}{x}\,dx} = e^{3x+\ln x} = e^{3x}e^{\ln x} = xe^{3x}.$$

Multiplying through gives

$$\begin{aligned} xe^{3x}y' + (3xe^{3x} + e^{3x})y &= 5e^{3x}, \\ \frac{d}{dx}(xe^{3x}y) &= 5e^{3x}, \\ xe^{3x}y(x) &= \frac{5}{3}e^{3x} + C, \\ y(x) &= \frac{5}{3x} + \frac{C}{xe^{3x}}. \end{aligned}$$

Plugging in $x = 2$ gives

$$y(2) = \frac{5}{6} + \frac{C}{2e^6} = 4,$$

or

$$C = \frac{19}{3}e^6.$$

This gives

$$y(x) = \frac{5}{3x} + \frac{19}{3xe^{3x-6}}.$$

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3. (Problem 1.5.36 from book.) A tank initially contains 60 gal of pure water. Salt water containing 1 lb of salt per gallon enters the tank at 2 gal/min, and the perfectly mixed solution leaves the tank at 3 gal/min. Thus the tank is empty after 1 hour. Find the amount of salt in the tank after $t$ minutes. Determine the maximum amount of salt ever in the tank.

**Solution:** First note that the tank is losing a net of 1 gal/min of liquid, which means that it will be empty in 60 minutes. More specifically, the number of gallons in the tank at time $t$ is $V(t) = 60 - t$.

Second, we want to write down a differential equation for the salt at time $t$, which we will denote by $S(t)$. Clearly, the rate of change of salt will be the rate coming in minus the rate going out. The amount coming in is 1 lb/ gal * 2 gal/min = 2 lb/min. The rate going out is more complicated, since it depends on the concentration of the salt in the tank at the time. But notice that if there is $S(t)$ pounds of salt at time $t$, and there are $60 - t$ gallons in the tank at time $t$, then the concentration is thus $S(t)/(60 - t)$ lb/gallon in the tank. The rate of liquid leaving is 3 gal/min, so the rate out is $3S(t)/(60 - t)$. Thus we have

$$\frac{dS}{dt} = 2 - \frac{3S(t)}{60 - t},$$

or

$$S' + \frac{3}{60 - t}S = 2.$$

We will use the integrating factor

$$e^{\int \frac{3}{60-t}\,dt} = e^{-3\ln(60-t)} = e^{\ln((60-t)^{-3})} = (60 - t)^{-3}.$$

Then we have

$$\begin{aligned}(60-t)^{-3}S' + 3(60-t)^{-4}S &= 2(60-t)^{-3},\\ \frac{d}{dt}((60-t)^{-3}S(t)) &= 2(60-t)^{-3},\\ (60-t)^{-3}S(t) &= (60-t)^{-2} + C,\\ S(t) &= C(60-t)^3 + (60-t).\end{aligned}$$

Now, we also need an initial condition, but notice that there is no salt in the tank at the start, so that $S(0) = 0$. Plugging in gives

$$S(0) = C * 60^3 + 60 = 0,$$

or

$$C = -\frac{1}{60^2} = -\frac{1}{3600}.$$

Thus the solution for all $t$ between 0 and 60 minutes is given by

$$S(t) = -\frac{(60-t)^3}{3600} + (60-t).$$

(We can check that $S(60) = 0$, as it should.) Now we want to know the maximum value of $S(t)$ for $t$ between 0 and 60. We first compute its derivative

$$S'(t) = \frac{-(60-t)^2}{1200} + 1.$$

Setting this equal to zero gives

$$(60-t)^2 = 1200$$

or

$$t = 60 \pm \sqrt{1200}.$$

Since we only care about $t$ less than 60, we want to take the solution with the minus and not the plus. Thus the value we are looking for is $S(60 - \sqrt{1200}) = 20(\sqrt{3} - 1/\sqrt{3}) \approx 23.094$.

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4. Find constants $A, B$ so that

$$y(x) = A\sin x + B\cos x$$

is a solution of

$$y' + y = 4\sin x.$$

Now, find constants $A, B, C$ so that

$$y(x) = A\sin x + B\cos x + Ce^{-x}$$

is a solution to

$$y' + y = 4\sin x, \quad y(0) = 4.$$

**Solution:** We plug in. We start off with

$$\begin{aligned}y(x) &= A\sin x + B\cos x,\\ y'(x) &= A\cos x - B\sin x\end{aligned}$$

So

$$y' + y = (A+B)\cos x + (A-B)\sin x,$$

so we have

$$A + B = 0,$$
$$A - B = 1.$$

The solution to this is $A = 1/2, B = -1/2$, and our solution is

$$y(x) = \frac{1}{2}(\sin x - \cos x).$$

As for the second, we first check to find $A, B$:

$$y(x) = A\sin x + B\cos x + Ce^{-x},$$
$$y'(x) = A\cos x - B\sin x - Ce^{-x}.$$

We get

$$y' + y = (A + B)\cos x + (A - B)\sin x + (C - C)e^{-x},$$

and the $C$'s cancel, thus we get the same solution for $A, B$ as before. Our solution is then

$$y(x) = \frac{1}{2}(\sin x - \cos x) + Ce^{-x}.$$

Plugging in $x = 0$ gives

$$y(0) = C - \frac{1}{2} = 4,$$

so $C = 9/2$, and the full solution is

$$y(x) = \frac{1}{2}(\sin x - \cos x) + \frac{9}{2}e^{-x}$$