# Introduction to Differential Equations – Math 286 X1
# Fall 2009
# Homework 10 — due never

1. Write down the general solution for the heat equation given by

$$u_t = 2u_{xx}, 0 \leq x \leq 4\pi, t > 0,$$
$$u(0,t) = u(4\pi, t) = 0.$$

**Solution:** We make the Ansatz

$$u(x,t) = A(x)B(t),$$

and plugging this into the PDE gives

$$A(x)B'(t) = 2A''(x)B(t),$$

or

$$\frac{B'(t)}{2B(t)} = \frac{A''(x)}{A(x)} = -\lambda.$$

This gives the two equations

$$A''(x) + \lambda A(x) = 0,$$
$$B'(t) + k\lambda B(t) = 0.$$

Plugging in the boundary conditions gives

$$A(0)B(t) = A(4\pi)B(t) = 0,$$

or

$$A(0) = A(4\pi) = 0.$$

Now, as we know from before, the eigenvalue problem for $A$ with these boundary conditions only has positive eigenvalues, so we set $\omega = \sqrt{\lambda}$ and we know then that

$$A(x) = C_1 \cos(\omega x) + C_2 \sin(\omega x).$$

Plugging in the boundary conditions gives

$$A(0) = C_1 = 0,$$
$$A(4\pi) = C_1 \cos(4\pi\omega) + C_2 \sin(4\pi\omega) = 0,$$

and the solution to this is $C_1 = 0$, $4\pi\omega = n\pi$, where $n$ is an integer, or

$$\omega_n = \frac{n}{4}.$$

Thus the solution to the $A$ problem is

$$A_n(x) = \sin(nx/4), \quad \lambda_n = \frac{n^2}{16}, \quad n = 1, 2, 3, \ldots.$$

Solving the $B$ equation gives

$$B_n(t) = C_n e^{-2\lambda_n t},$$

and thus the general solution is

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-n^2 t/8} \sin(nx/4).$$

2. Write down the general solution for the wave equation given by

$$\begin{aligned} u_{tt} &= 4u_{xx}, 0 \leq x \leq 1, \\ u(0,t) &= u(1,t) = 0. \end{aligned}$$

**Solution:** We separate variables as in the previous problem, and we get

$$A(x)B''(t) = 4A''(x)B(t),$$

or

$$\frac{A''(x)}{A(x)} = \frac{B''(t)}{4B(t)} = -\lambda,$$

or the system

$$\begin{aligned} A''(x) + \lambda A(x) &= 0, \\ B''(t) + 4\lambda B(t) &= 0. \end{aligned}$$

The boundary conditions we now get on the $A$ equation are $A(0) = A(1) = 0$, and in a similar manner to above, we get

$$A_n(x) = \sin(n\pi t), \quad \lambda_n = n^2\pi^2, \quad n = 1, 2, 3, \ldots.$$

The $B$ equation becomes

$$B_n''(t) + 4n^2\pi^2 B_n(t) = 0,$$

which has solution

$$B_n(t) = C_n \cos(2n\pi t) + D_n \sin(2n\pi t).$$

Thus our general solution is

$$u(x,t) = \sum_{n=1}^{\infty} (C_n \cos(2n\pi t) + D_n \sin(2n\pi t)) \sin(n\pi t).$$

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3. Write down the specific solution to the heat equation given by

$$\begin{aligned} u_t &= 2u_{xx}, 0 \leq x \leq 1, t > 0, \\ u(0,t) &= u(1,t) = 0, \\ u(x,0) &= x(1-x). \end{aligned}$$

**Solution:** We follow a similar procedure to the problem #1, and we obtain the equations

$$\begin{aligned} A''(x) + \lambda A(x) &= 0, \\ B'(t) + 2\lambda B(t) &= 0, \end{aligned}$$

and the boundary conditions on the $A$ equation are $A(0) = A(1) = 0$. The solutions to this equation will be

$$A_n(x) = \sin(n\pi x), \lambda_n = n^2\pi^2.$$

(Notice that these are *exactly* the same as in the previous problem, although we're solving the heat equation instead of wave. Of course, the solutions to the $B$ equation will be different.)

The solutions to the $B_n$ equation will be

$$B_n(t) = C_n e^{-2n^2\pi^2 t},$$

and then the general solution is

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-2n^2\pi^2 t} \sin(n\pi x).$$

Now, to solve for the particular initial condition, we plug in $t = 0$ and we get

$$x(1-x) = \sum_{n=1}^{\infty} C_n \sin(n\pi x),$$

and we know that the coefficients $C_n$ can be obtained by a Fourier sine series. Thus we have the formula

$$C_n = 2\int_0^1 x(1-x)\sin(n\pi x)\,dx.$$

The most efficient thing to do is integrate the terms separately. We first compute

$$\begin{aligned}\int x\sin(n\pi x)\,dx &= -\frac{x\cos(n\pi x)}{n\pi} + \int \frac{\cos(n\pi x)}{n\pi}\,dx \\ &= -\frac{x\cos(n\pi x)}{n\pi} + \frac{\sin(n\pi x)}{n^2\pi^2}.\end{aligned}$$

Thus

$$\int_0^1 x\sin(n\pi x)\,dx = -\frac{\cos(n\pi)}{n\pi} - 0 + 0 - 0 = \frac{(-1)^{n+1}}{n\pi}.$$

We now compute

$$\begin{aligned}\int x^2\sin(n\pi x)\,dx &= -\frac{x^2\cos(n\pi x)}{n\pi} + \int \frac{2x\cos(n\pi x)}{n\pi}\,dx \\ &= -\frac{x^2\cos(n\pi x)}{n\pi} + \frac{2x\sin(n\pi x)}{n^2\pi^2} - \int \frac{2\sin(n\pi x)}{n^2\pi^2} \\ &= -\frac{x^2\cos(n\pi x)}{n\pi} + \frac{2x\sin(n\pi x)}{n^2\pi^2} + \frac{2\cos(n\pi x)}{n^3\pi^3},\end{aligned}$$

and thus we have

$$\int_0^1 x^2\sin(n\pi x)\,dx = -\frac{\cos(n\pi)}{n\pi} - 0 + 0 - 0 + \frac{2\cos(n\pi)}{n^3\pi^3} - \frac{2}{n^3\pi^3} = \frac{(-1)^{n+1}}{n\pi} + 2\frac{(-1)^n - 1}{n^3\pi^3}.$$

Therefore

$$C_n = 2\int_0^1 x(1-x)\sin(n\pi x)\,dx = -4\frac{(-1)^n - 1}{n^3\pi^3}.$$

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4. In each of the following cases, you are given a list of functions and a domain. Determine whether or not the list of functions is an orthogonal list. (Recall that we say that a list of functions is an orthogonal list if we can choose any pair of different functions in that list, and they are orthogonal.)

   (a) $\{1, x, x^2, x^3, \ldots, x^n, \ldots\}$, domain $[0,1]$,

(b) $\{\sin(x), \sin(2x), \dots, \sin(nx), \dots\}$, domain $[0, \pi]$,

(c) $\{\sin(x), \sin(2x), \dots, \sin(nx), \dots\}$, domain $[0, \pi/2]$,

(d) $\{\cos(x), \cos(2x), \dots, \cos(nx), \dots\}$, domain $[0, \pi]$.

**Solution:**

(a) Take any two functions in this list and check:

$$\int_0^1 x^n x^m \, dx = \int_0^1 x^{n+m} \, dx = \frac{x^{n+m+1}}{n+m+1}\bigg|_{x=0}^{x=1} = \frac{1}{n+m+1}.$$

This is not zero, so these functions are not orthogonal.

(b) Take two functions in this list and compute

$$\int_0^\pi \sin(mx)\sin(nx)\,dx.$$

Use the trig identity

$$\sin(A)\sin(B) = \frac{1}{2}(\cos(A-B) - \cos(A+B)),$$

and we obtain

$$\begin{aligned}\int_0^\pi \sin(mx)\sin(nx)\,dx &= \frac{1}{2}\int_0^\pi \cos((m-n)x) - \cos((m+n)x)\,dx \\ &= \frac{1}{2}\,\frac{\sin((m-n)x)}{m-n} - \frac{\sin((m+n)x)}{m+n}\bigg|_{x=0}^{x=\pi} = \frac{1}{2}(0-0+0-0) = 0.\end{aligned}$$

(c) We get the same thing, except the values at which we evaluate are different, namely we obtain

$$\frac{1}{2}\,\frac{\sin((m-n)x)}{m-n} - \frac{\sin((m+n)x)}{m+n}\bigg|_{x=0}^{x=\pi/2} = \frac{1}{2}\left(\frac{\sin((m-n)\pi/2)}{m-n} + \frac{\sin((m+n)\pi/2)}{m+n}\right).$$

If $m+n$ and $m-n$ are even, then this is zero, but if they are odd, then it is not. For example, choose $m=2, n=1$, then we get

$$1 - \frac{1}{3} = \frac{2}{3}.$$

Thus this is not an orthogonal list.

(d) This works the same as in (b).