Surfaces in finite covers of 3-manifolds: The Virtual Haken Conjecture

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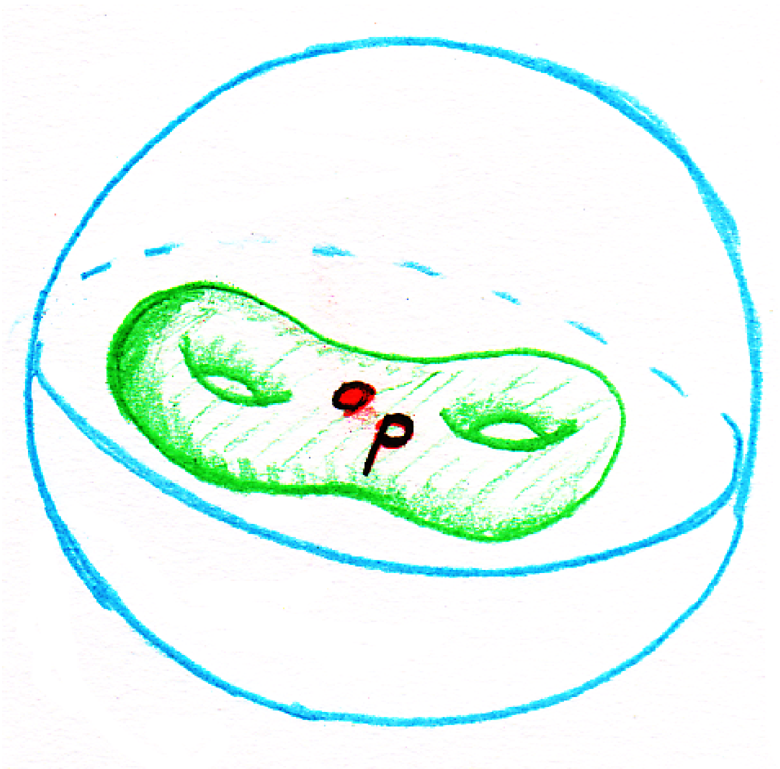
In the 1960s, Waldhausen proposed:

Virtual Haken Conjecture. *Let M be compact 3-manifold. If $\pi_1(M)$ is infinite, then M has a finite cover N which contains an incompressible surface.*

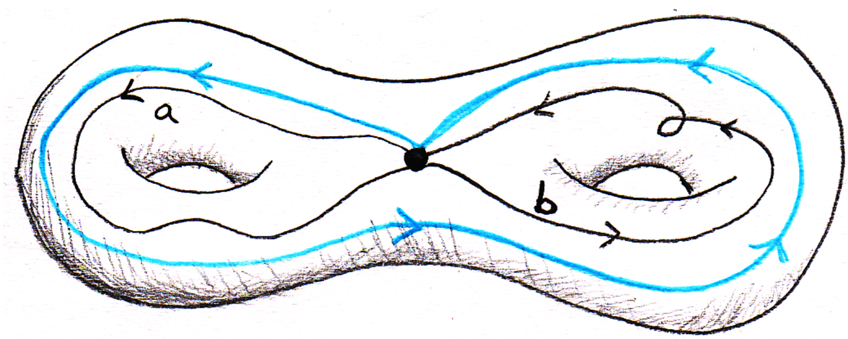
Natural place to start: studying surfaces Σ^2 in M^3. Need to ignore things like:



Convention: All manifolds are orientable.

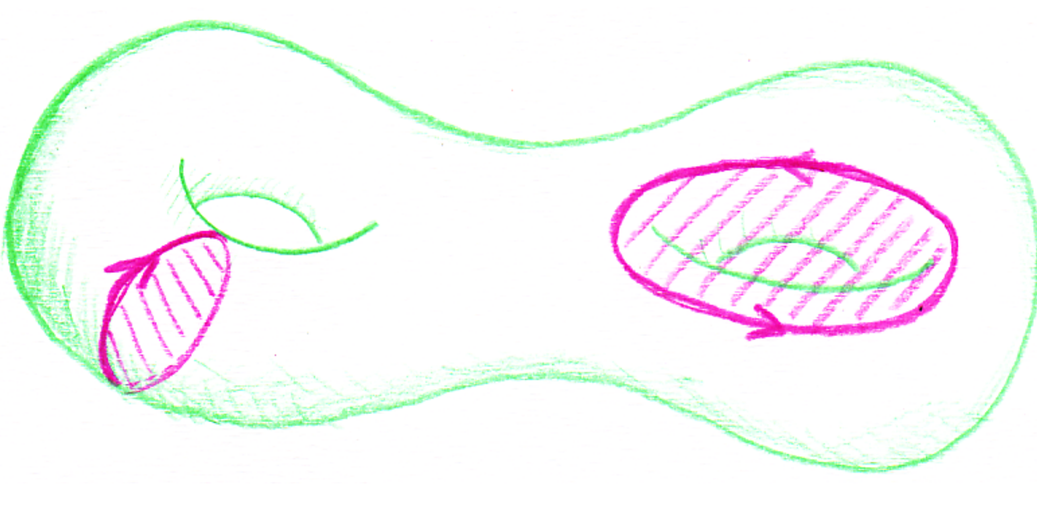
Def. *A surface $\Sigma \neq S^2$ embedded in M^3 is incompressible if $\pi_1(\Sigma) \to \pi_1(M)$ is 1-1.*

Recall that $\pi_1(M)$ is the group of loops in M, up to homotopy:



Ex. $\pi_1(S^3) = 1$.
$\pi_1(T^3) = \mathbb{Z}^3$, where $T = S^1 \times S^1 \times S^1 = \mathbb{R}^3/\mathbb{Z}^3$.
$\pi_1(W) =$
$\langle a, b \mid a^2b^2a^2b^{-1}ab^{-1} = b^2a^2b^2a^{-1}ba^{-1} = 1 \rangle$.

Compressible:

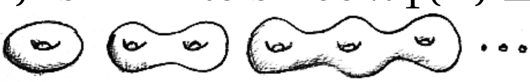


Incompressible: For $\Sigma = S^1 \times S^1 \times \{\text{pt}\} \subset T^3$, the map on π_1 is: $\mathbb{Z} \oplus \mathbb{Z} \hookrightarrow \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$.

Similarly, $\Sigma \times \{\text{pt}\}$ is an incompressible surface in $M^3 = \Sigma \times S^1$.

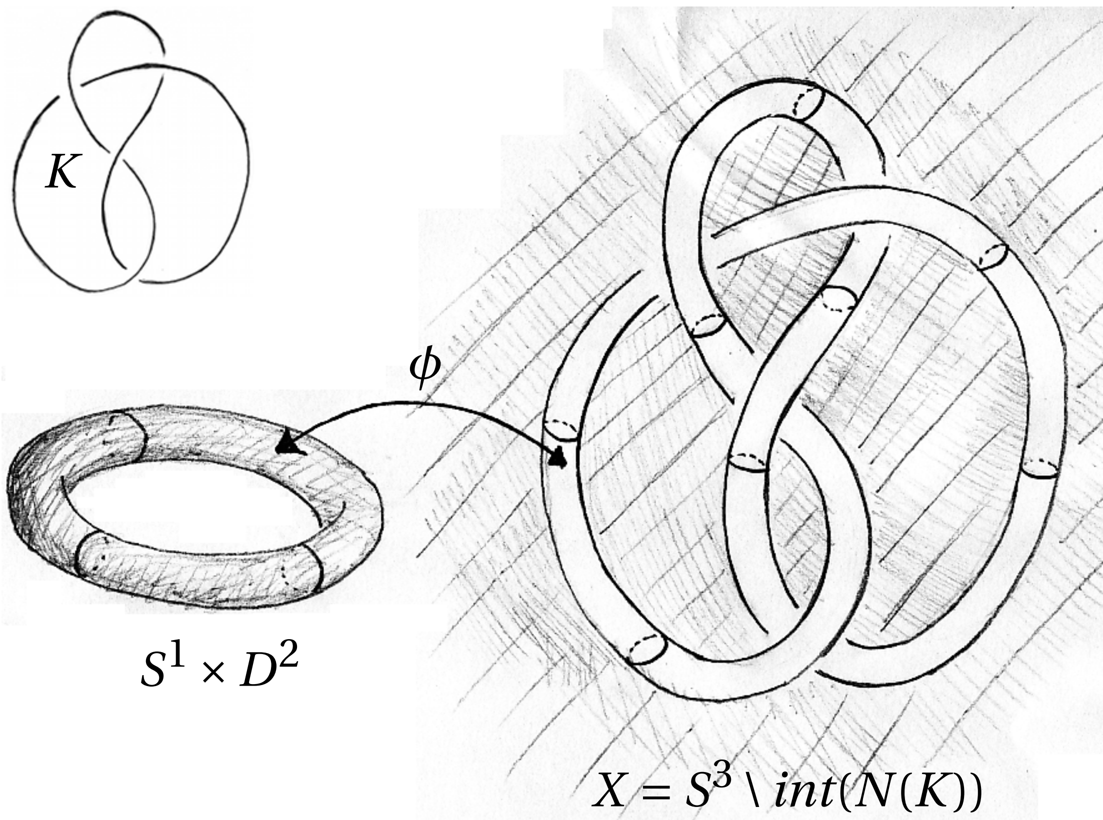
Def. *A compact M^3 is Haken if it is irreducible and contains an incompressible surface.*

Irreducible: Every embedded S^2 bounds a ball, that is, M is not a connected sum.
An arbitrary M^3 is of the form $M_1 \# M_2 \# \cdots \# M_n$ where the M_k can't be further decomposed.

If M is Haken, then $\pi_1(M)$ is infinite since $\pi_1(\Sigma) \leq \pi_1(M)$ and Σ is among: 

Haken: T^3 Non-Haken: π_1 finite, e.g. S^3.

π_1 condition is not sufficient: Given a knot K in S^3, Dehn surgery creates infinitely many compact 3-manifolds via $M = X \cup_\phi (S^1 \times D^2)$



All but 4 Dehn surgeries on the figure-8 knot are non-Haken 3-manifolds with infinite π_1.

Virtual Haken Conjecture. *Let M be an irreducible compact 3-manifold. If $\pi_1(M)$ is infinite, then M has a finite cover N which is Haken.*

Closely related question: Does M contain an *immersed* incompressible surface? Equivalently, does $\pi_1(M)$ contain the fundamental group of some surface?

A lot of evidence for this conjecture including:

- True for all the manifolds coming from the figure-8 knot. [D-Thurston 2003].
- Weaker results for surgery on any knot, e.g. [Cooper-Long 1997, Cooper-Walsh 2006].
- True for all 11,000 examples in a census of simple 3-manifolds. In one case, a cover of degree 5,050 was needed! [D-Thurston 2003].

Rest of talk:

- Make the conjecture weaker and prove it.
- Make the conjecture stronger and disprove it.

Real point of talk:

- Role of geometry is crucial for this seemingly topological question (Thurston/Perelman).
- Study of 3-manifolds uses many different parts of mathematics.

Virtual Haken Conjecture. *Let M be an irreducible compact 3-manifold. If $\pi_1(M)$ is infinite, then M has a finite cover N which is Haken.*

Conj. *Let M be an irreducible compact 3-manifold. If $\pi_1(M)$ is infinite, then M has a* non-trivial *finite cover.*

Equivalently, $\pi_1(M)$ has a subgroup H with $1 < [\pi_1(M) : H] < \infty$.

This seemingly simple conjecture was only proved in 2003!

Geometrization (Thurston/Perelman):
A compact M^3 can be cut along spheres and incompressible tori into pieces which admit geometric structures. That is, each piece admits a homogeneous Riemannian metric modeled on one of

$$\mathbb{E}^3,\ S^3,\ \mathbb{H}^3,\ S^2\times\mathbb{R},\ \mathbb{H}^2\times\mathbb{R},\ \text{Nil},\ \text{Sol},\ \widetilde{\text{SL}_2\mathbb{R}}.$$

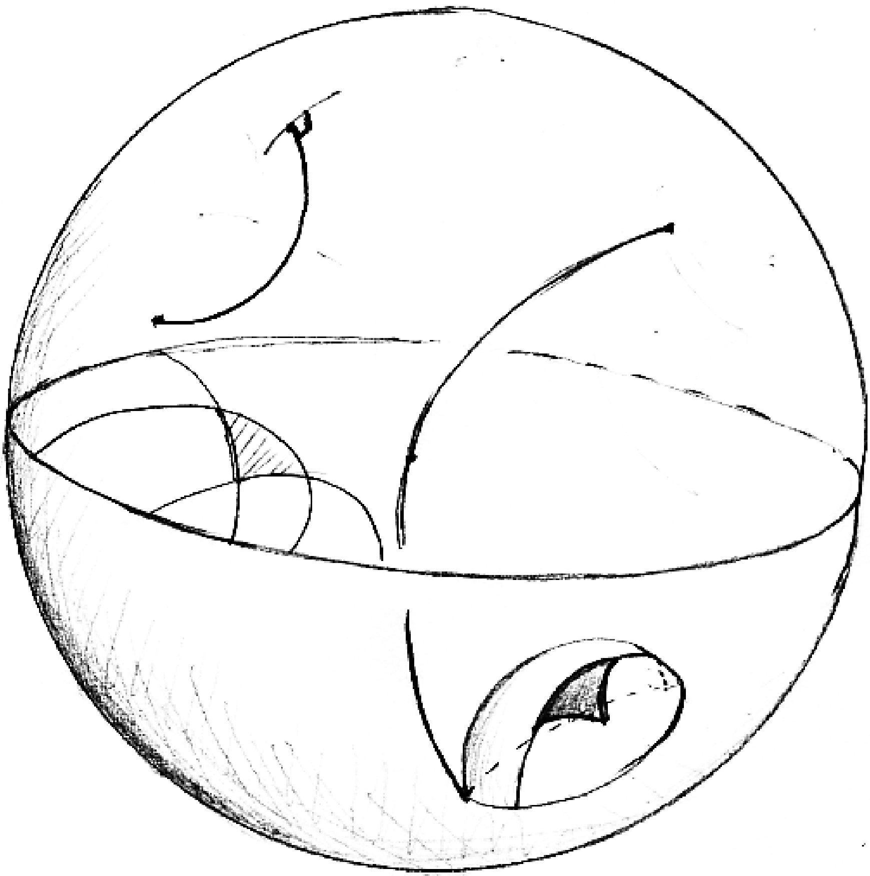
Ex: T^3 is Euclidean as $= \mathbb{E}^3/\mathbb{Z}^3$, whereas $S^2\times S^1$ has a $S^2\times\mathbb{R}$ geometry.

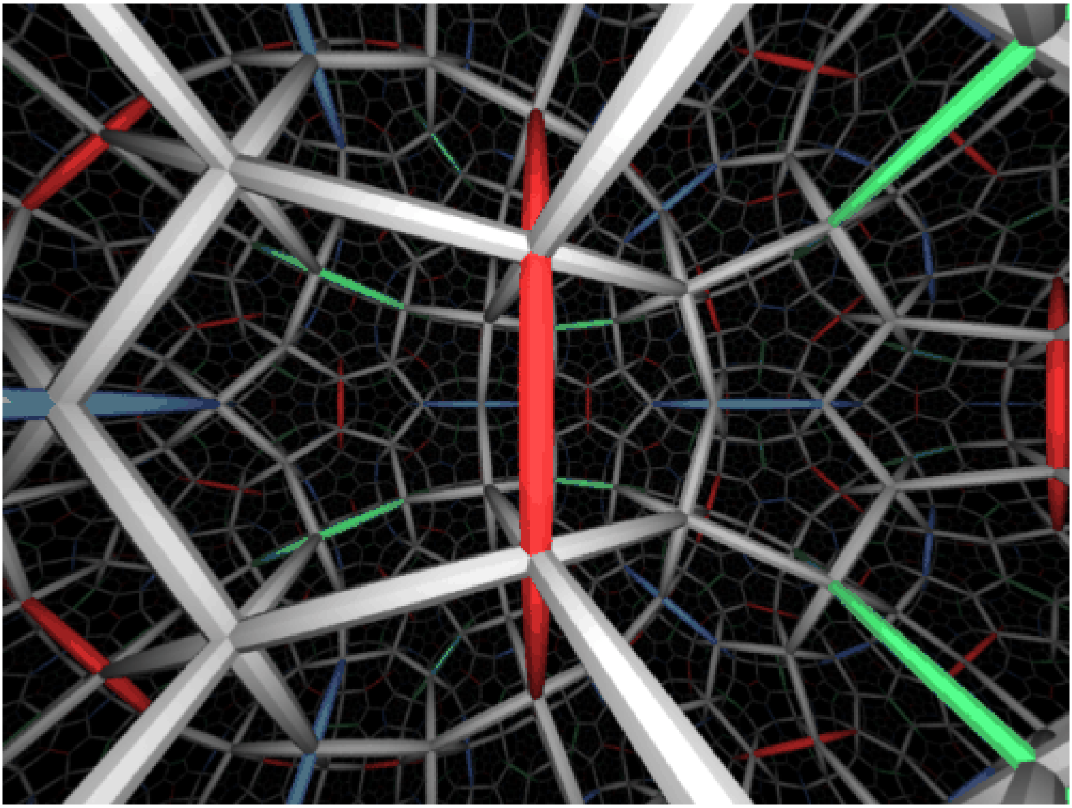
The case $\mathbb{H}^3$ of hyperbolic geometry is the generic one; manifolds with the other geometries have been classified, and we know the VHC holds for them.

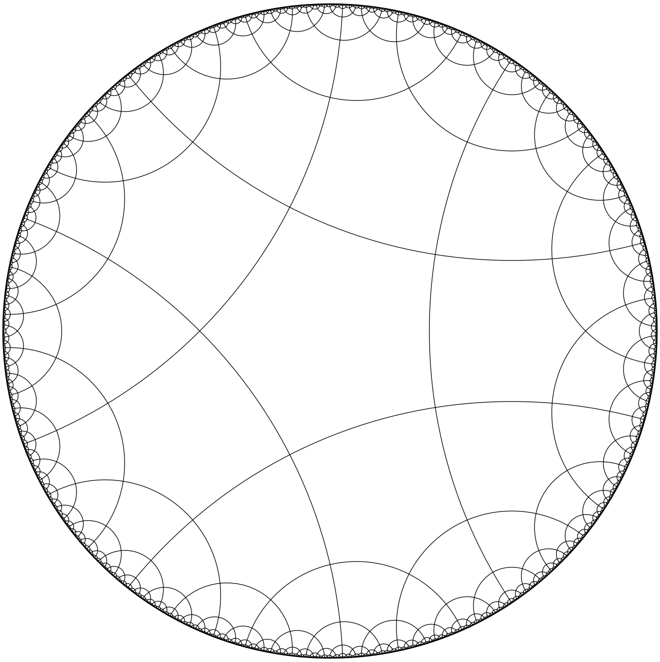
From now on, M will be a *hyperbolic* 3-manifold, i.e. one with a metric of constant sectional curvature -1. Equivalently, $M = \mathbb{H}^3/\Gamma$, where $\Gamma \leq \mathrm{Isom}^+(\mathbb{H}^3) = \text{Möbius}(\hat{\mathbb{C}}) = \mathrm{PSL}_2(\mathbb{C})$.

Here $\mathbb{H}^3 = \left\{\mathbf{x} \in \mathbb{R}^3 \middle| \, |\mathbf{x}| < 1\right\}$ with the metric where

$$ds_{\mathbb{H}^3} = 2/(1 - |\mathbf{x}|^2)\, ds_{\mathbb{E}^3}$$







Thm (Perelman 2003). *Let M be a compact 3-manifold. If $\pi_1(M)$ is infinite, then M has a non-trivial finite cover. Equivalently, $\pi_1(M)$ has a finite-index proper subgroup.*

Proof. Reduce to the case when M is hyperbolic. As M is compact, $\pi_1(M)$ is finitely generated and also $\pi_1(M) \leq \mathrm{PSL}_2(\mathbb{C})$. A finitely generated group of matrices has many finite index subgroups by [Mal'tsev 1940s]. Idea: For $\mathrm{PSL}_2(\mathbb{Z})$ we build the needed subgroup Λ by considering:

$$1 \to \Lambda \to \mathrm{PSL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{Z}/(p\mathbb{Z})) \to 1.$$

Though this theorem is a simple topological/group theoretic statement, all known proofs rely on Geometrization and thus start with Hamilton's Ricci flow…

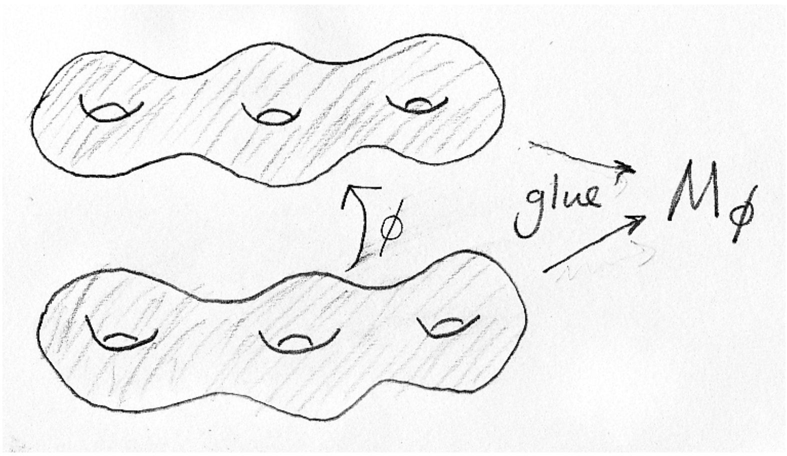
Generalizations:

[Lubotzky 1995] Many more subgroups than just the congruence ones.

[D-Thurston 2006] Studied random Heegaard splittings. For a finite simple group Q, the number of Q-covers is Poisson distributed with mean

$$\mu = |H_2(Q;\mathbb{Z})| / |\mathrm{Out}(Q)|.$$

E.g. the probability of an A_n cover is $1 - e \approx 0.6$.



Conj. *M^3 compact hypebolic. Then M has a finite cover N where $H_2(N;\mathbb{Z}) \cong H^1(N;\mathbb{Z}) \neq 0$.*

Equivalently, $\pi_1(M)$ has a finite-index subgroup H where $H \twoheadrightarrow \mathbb{Z}$.

A tower of regular finite covers

$$M \leftarrow M_1 \leftarrow M_2 \leftarrow M_3 \leftarrow \cdots$$

exhausts M if $\bigcap \pi_1(M_n) = 1$.

Conj. *If M_n exhaust M, then $H^1(M_n;\mathbb{Z}) \neq 0$ for some n.*

Thm (Calegari-D 2006). *There exists an M with exahustion M_n where $H^1(M_n) = 0$ for all n.*

Proof conditional on Langlands for GL_2 and the Generalized Riemann Hypothesis!

Thankfully, Boston-Ellenberg (2006) were able to analyze these examples unconditionally, using our picture: