RESEARCH SUMMARY

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1. OVERVIEW

My research area is the topology and geometry of 3-manifolds. I was attracted to it because of the richness it acquired from W. Thurston's revolutionary work starting in the 1970s. Thurston's key insight was that many 3-manifolds admit homogeneous Riemannian metrics, and that one can study the topology of a 3-manifold via this geometry. This profusion of geometry has now been stunningly confirmed by Perelman's recent proof of Thurston's Geometrization Conjecture. As a direct result, while my work has focused on what will initially seem like purely topological problems, in fact I have used a broad range of techniques to attack them, including hyperbolic geometry, number theory, and algebraic geometry, as well as more obviously related areas such as combinatorial group theory and the theory of foliations. These connections to other fields have led me to collaborate with both number theorists and theoretical physicists, and below I'll need to refer to both the Langlands Conjecture and the Classification of Finite Simple Groups, as well as to such topological oddities as "random 3-manifolds". In the rest of this overview, I will outline the division of my work into broad topics, and then discuss my results in detail in later sections.

As in many other areas of geometry, the study of 3-manifolds begins with codimension-one submanifolds, focusing on those surfaces which carry the most topological information. Here, I'll be concerned with those that are topologically essential: a compact orientable surface S embedded in a 3-manifold M is called *incompressible* if the surface S is not a 2-sphere and the homomorphism $\pi_1(S) \to \pi_1(M)$ is injective. For example, the 3-torus $S^1 \times S^1 \times S^1$ contains the incompressible torus $S^1 \times S^1 \times \{\mathrm{pt}\}$. On the other hand, a 3-manifold with finite fundamental group, such as S^3, can't contain an incompressible surface. In addition, there are many 3-manifolds with infinite fundamental group which don't contain incompressible surfaces. However, it is suspected that they *almost* do in the following sense, which was proposed by Waldhausen in the 1960s:

1.1. **Virtual Haken Conjecture.** *Let M be a compact irreducible 3-manifold with infinite fundamental group. Then M has a finite cover which contains an incompressible surface.*

Using number theory, finite group theory, probability, and dynamics, I have worked with (variously) William Thurston, Dylan Thurston, Frank Calegari, and Danny Calegari to gain a better understanding of some the key issues surrounding this conjecture, which is one of the most central in the theory of 3-manifolds (see Section 2 for details).

There are also codimension-one objects with a more dynamical flavor, namely taut foliations and essential laminations. Danny Calegari and I have shown that certain classes of such objects in M are associated to actions of $\pi_1(M)$ on the circle, thus translating the topological data about the lamination into more algebraic information (Section 3).

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One of the basic operations in 3-dimensional topology is Dehn filling: starting with a 3-manifold M with boundary a torus, we can create an infinite family of manifolds without boundary by gluing a solid torus to M along their boundaries. I have worked to understand this process in the context of the Virtual Haken Conjecture, as well as the question of when "simple" manifolds (e.g. those with finite π_1) can result (Section 4).

In a very different direction, I have also worked with Sergei Gukov and Jake Rasmussen to understand the interrelationships between various homology theories of knots in the 3-sphere. In particular, we proposed a framework for connecting those introduced by Khovanov and Rozansky, which come from quantum algebra, to knot Floer homology, which is defined geometrically, counting certain pseudo-holomorphic disks (Section 5).

Finally, Section 6 summarizes some additional results that don't fit into the above categories, including an experimental check of certain consequences of Perelman's proof of the Geometrization Conjecture.

2. The Virtual Haken Conjecture

First, let me give some motivation for the Virtual Haken Conjecture. Throughout, let M be a compact irreducible 3-manifold with infinite $\pi_1(M)$. (Irreducible just means that every embedded sphere bounds a ball; modulo technicalities, every 3-manifold can be decomposed by connected sum into such manifolds.) Instead of just considering embedded surfaces in M, look at immersions $S \to M$ which are injective on π_1. A natural question is, does M contain such an immersed incompressible surface? The Virtual Haken Conjecture implies that the answer is always yes. Another point of view on the conjecture is group-theoretic. While the fundamental group of a 4-manifold can be any finitely presented group, that of a 3-manifold is quite restricted; a central task of 3-dimensional topology is to discern their special properties. In this context, the Virtual Haken Conjecture implies that $\pi_1(M)$ always contains the fundamental group of a closed surface. For the generic case when M is hyperbolic (i.e. has a Riemannian metric of curvature -1) this fits nicely into a more general question of Gromov: must a 1-ended word-hyperbolic group contain a surface group? In another direction, if we ask for slightly more and require M to have a finite cover N where $H^1(N;\mathbb{Z}) \neq 0$, this becomes a natural problem in the theory of lattices of Lie groups and number theory. Finally, those manifolds that contain incompressible surfaces, called *Haken* manifolds, are the easiest to understand topologically, and this conjecture would reduce other problems to that case.

2.1. **Kinds of covers.**
The first hurdle in attacking the Virtual Haken Conjecture is to show that M has any finite covers at all. To do this, one has to bring geometry into the picture; in fact, a corollary of Perelman's Geometrization Theorem for 3-manifolds is that $\pi_1(M)$ must have many finite-index subgroups [Per1, Per2, Hem]. Work of Lubotzky [Lub] gives many more covers in the generic case when M is hyperbolic, but still many fewer than one expects from strong forms of the Virtual Haken Conjecture.

William Thurston and I have worked to understand the number and kind of covers present. We began with some computer experiments [DT2] and found a plethora of covers of types not forced by [Lub]. To better understand these patterns, we were led to the question: What is a random 3-manifold and what can you say about it? We have settled on a notion of random 3-manifold which uses random walks in the mapping class group of a surface to glue together two handlebodies (a random Heegaard splitting). Our main result confirms our experimental observation that covers where the covering group is a simple group are quite common:

2.2. **Theorem** ([DT3]). *Let Q be a non-abelian finite simple group. Then as the genus of the handlebody goes to infinity, the probability that a random 3-manifold has a Q-cover converges to:*

$$1 - e^{-\mu} \quad \text{where} \quad \mu = \#H_2(Q;\mathbb{Z})/\#\mathrm{Out}(Q).$$

For example, if $Q = A_n$ or $Q = \mathrm{PSL}_2\mathbb{F}_p$, where p is an odd prime, then $\mu = 1$; hence the probability of such a cover is startlingly large, about 0.6.

The proof of Theorem 2.2 boils down to understanding the action of the mapping class group of a surface on a certain finite set. Let S_g be a closed surface of genus g, and let $\mathscr{M}_g$ be its mapping class group. Consider the set $\mathscr{A}_g$ of epimorphisms from $\pi_1(S_g)$ onto our fixed simple group Q, modulo automorphisms of Q. Then $\mathscr{M}_g$ acts on $\mathscr{A}_g$ via the induced automorphisms of $\pi_1(S_g)$, and we showed:

2.3. **Theorem** ([DT3]). *Let Q be a non-abelian finite simple group. Then for all sufficiently large g, the orbits of $\mathscr{A}_g$ under the action of $\mathscr{M}_g$ correspond bijectively to $H_2(Q,\mathbb{Z})/\mathrm{Out}(Q)$. Moreover, the action of $\mathscr{M}_g$ on each orbit is by the full alternating group of that orbit.*

You can view Theorem 2.3 as saying that when the genus is large, the action of $\mathscr{M}_g$ on $\mathscr{A}_g$ is nearly as mixing as possible. As such, it is directly analogous to Goldman's theorem that the action of $\mathscr{M}_g$ on the $\mathrm{SU}(2)$-character variety is ergodic for any genus ≥ 2 [Gol]. Perhaps surprisingly, the proof of Theorem 2.3 uses the Classification of Simple Groups even for concrete cases such as $Q = A_n$.

2.4. **Homology of covers.** I turn now to the question of when a cover N of a hyperbolic 3-manifold M is actually Haken. In particular, how hard does one need to look in order to find a Haken cover? I will focus on the stronger requirement that $H^1(N;\mathbb{Z}) \neq 0$, because this gives a nice algebraic formulation of the conjecture; it is equivalent to asking that $\pi_1(M)$ has a finite-index subgroup with infinite abelianization. Consider a tower of regular finite covers

$$M \leftarrow M_1 \leftarrow M_2 \leftarrow M_3 \leftarrow \cdots$$

which unwraps all of the topology of M in the sense that $\bigcap_{n=1}^{\infty} \pi_1(M_n) = \{1\}$. We say such a tower of covers *exhausts* M. A natural question is

2.5. **Question.** *If a tower of covers exhausts M must $H^1(M_n;\mathbb{Z}) \neq 0$ for some n?*

It is reasonable to hope that the answer would be yes for at least some classes of 3-manifolds, in which case one could attack the Virtual Haken Conjecture with any such tower of covers. However, Frank Calegari and I showed instead that:

2.6. **Theorem** ([CD4]). *Assume the Generalized Riemann Hypothesis and the Langlands Conjecture for GL_2. Then there is a tower of covers M_n exhausting a hyperbolic 3-manifold where $H^1(M_n;\mathbb{Z}) = 0$ for all n.*

The example, which is arithmetic in nature, is completely explicit. The proof works by using that nontrivial cohomology of an arithmetic hyperbolic 3-manifold gives rise to an automorphic representation of the corresponding adélic group. In turn, the Langlands conjecture posits such representations have corresponding Galois representations $\mathrm{Gal}(\overline{\mathbb{Q}}/K) \to \mathrm{GL}_2 E$. In our case, if one of the covers had $H^1 \neq 0$ then the resulting Galois representation is highly restricted, and in fact we showed that such Galois representations simply do not exist. Subsequently, Boston and Ellenberg analyzed our examples using the theory of pro-p groups, and were able to unconditionally show that each of the covers has $H^1 = 0$ [BE]. Thus the answer to Question 2.5 is definitively no.

In the context of random 3-manifolds, William Thurston and I investigated whether the profusion of covers discussed above have $H^1 \neq 0$ with positive probability. Unfortunately, this appears not to be the case. While we were only able to prove it in the following limited case, experimental evidence strongly indicates that for a fixed covering group Q the probability of having $H^1 \neq 0$ is 0.

2.7. **Theorem** ([DT3]). *Let Q be a finite abelian group. The probability that a 3-manifold obtained from a random Heegaard splitting of genus 2 has a Q-cover N with $H^1(N;\mathbb{Z}) \neq 0$ is* 0.

The moral seems to be that one needs to look at covers that are "large" in some sense when compared to the complexity of the base manifold. Currently, I am working to understand this better in terms of sparse random matrices, with the goal of tying it into the recent work of Lackenby on the Virtual Haken Conjecture [Lac1, Lac2, Lac3].

2.8. **Fibering over the circle.** W. Thurston conjectured a stronger version of the Virtual Haken Conjecture, asking for a cover which is not just Haken but which *fibers over the circle* (i.e. is a surface bundle over the circle). This conjecture is much more mysterious than the other variants, as there is far less evidence for it. To measure the distance between it and the standard Virtual Haken Conjecture, Dylan Thurston and I have worked to understand how common fibering is for random 3-manifolds. We focused on tunnel-number one 3-manifolds, which are the simplest kind of 3-manifold with boundary a torus. These manifolds always have $H^1 \neq 0$, and thus a nontrivial map to S^1; hence there is some chance of them being fibered. However, our main result is:

2.9. **Theorem** ([DT1]). *A tunnel-number one 3-manifold fibers over the circle with probability 0.*

While the question of whether a 3-manifold M fibers over the circle might seem fundamentally geometric, Stallings showed that it can be reduced to an algebraic question about $\pi_1(M)$, namely whether the induced map $\pi_1(M) \to \mathbb{Z}$ has finitely generated kernel [Sta]. Thus it makes sense to ask whether a group "fibers", and in this context we found a remarkable contrast between 3-manifold groups and finitely presented groups in general. The fundamental group of a tunnel-number one 3-manifold has a presentation with two generators and one relator. For abstract groups of this form, we showed that the probability of "fibering" is strictly positive and less than 1 (experimentally, it is about 0.94).

This special behavior for 3-manifold groups is a consequence of the relator coming from an embedded curve on a surface, which gives it a recursive nature where the same pattern repeats on varying scales. The proof of Theorem 2.9 depends on K. Brown's algorithm for computing the Bieri-Neumann-Strebel invariant of a 2-generator group [Bro] placed into the context of the interval exchange techniques used in [AHT]. Analyzing the dynamics of this algorithm is done making use of Kerckhoff's work on normality for interval exchanges [Ker], and Mirzakhani's theorem which says that non-separating simple closed curves have positive density among all multicurves [Mir].

Also in the context of this stronger version of the Virtual Haken Conjecture, Danny Calegari and I gave some restrictions on the kinds of covers you need in order to find one that fibers over the circle. Our theorem applies to certain 3-manifolds with one torus boundary component, and we used the relationship between the classical Alexander polynomial and the variety of representations of a 3-manifold group into $\mathrm{PSL}_2\mathbb{C}$. A concrete corollary is:

2.10. **Theorem** ([CD1]). *The complement of the knot 5_2 in S^3 is not commensurable to the complement of any fibered knot in a $\mathbb{Z}/2$-homology sphere. More generally, this is true of any non-fibered two-bridge knot $K(p,q)$ where $q < p < 40$.*

3. Essential Laminations

A codimension-one object in a 3-manifold M with a more dynamical flavor is a foliation of M by (usually noncompact) surfaces. If one restricts to *taut foliations*, where there is a transverse loop intersecting every surface, then the existence of such a foliation tells you a lot about M. More generally, an *essential lamination* is a foliation of a closed subset of M with suitable incompressibility properties. Taut foliations and incompressible surfaces are both examples of essential laminations. Much work has been done to understand what the presence of these objects tells you about the topology and geometry of the underlying manifold.

Danny Calegari and I have worked to extract other kinds of information from an essential lamination. Our main result is:

3.1. **Theorem** ([CD2]). *Let M be an atoroidal 3-manifold containing a tight essential lamination with solid torus guts. Then $\pi_1(M)$ has a faithful action on S^1.*

This theorem was motivated by Thurston's theorem that if M has a taut foliation, then $\pi_1(M)$ has a faithful action on S^1. However, the two results are disjoint; the above technical hypotheses exclude foliations. Our construction involves flattening the complementary regions of the lamination into ideal polygons in the hyperbolic plane, whereas Thurston's comes from constructing a *universal circle* from the circles at infinity of the surfaces of the foliation. Thurston never published his universal circle theorem, but in [CD2] we gave a complete proof based on a new, purely topological, proof of the key lemma.

Given our theorem, it is natural to ask if all hyperbolic 3-manifolds have faithful circle actions. For the smallest known hyperbolic 3-manifold we showed:

3.2. **Theorem** ([CD2]). *Any action of the fundamental group of the Weeks manifold on S^1 has finite image; in particular, it is not faithful. This is the first example of a rank-1 lattice which is known not to act faithfully on the circle.*

Combining this with Thurston's theorem gives another proof of the result of [RSS] that there exist hyperbolic 3-manifolds without taut foliations. Subsequently, Floer homology was used to give many more such examples [KMOS, OS1]. I am currently working on the "inverse problem" for taut foliations using these Floer homology techniques; that is, I want to understand when the action of $\pi_1(M)$ on something that looks like a leaf-space actually comes from a foliation.

4. Dehn Filling

Let M be a 3-manifold with boundary a torus. Pick a simple closed curve γ in ∂M. Create a new 3-manifold $M(\gamma)$ without boundary by taking a solid torus $D^2 \times S^1$ and gluing its boundary to ∂M in such a way that γ bounds a disc in the solid torus. The manifold $M(\gamma)$

is called a *Dehn filling* of M, and depends only on the homotopy class of γ. A general technique for exploring a conjecture about 3-manifolds is to try to prove it for most of the manifolds obtained by Dehn filling on any fixed manifold M. For example, back in the 1970s, W. Thurston proved that for any M, all but finitely many Dehn fillings on M satisfy his Geometrization Conjecture. In the context of the Virtual Haken Conjecture, W. Thurston and I showed that if M has certain kinds of Seifert fibered Dehn fillings, then one can "transfer" the Virtual Haken property of those fillings to other fillings [DT2]. While not all M have such fillings, they do occur in many of the standard examples. Rather than precisely stating our result, which generalizes work of Boyer and Zhang [BZ1], let me just give a corollary:

4.1. **Theorem** ([DT2]). *Let M be the exterior of the figure-8 knot in S^3. Then every Dehn filling of M satisfies the Virtual Haken Conjecture.*

For other twist knots, a corollary of our theorem is that the Virtual Haken Conjecture is true for all but finitely many fillings. The proof in general uses the Gelfond-Baker theory of linear forms and logarithms to understand the congruence quotients of triangle groups; it involves a number-theoretic constant, which, while effective, is not explicitly known. In the case of the figure-8 knot, additional results of Holt and Plesken apply [HP], allowing a sharper statement.

4.2. **Character varieties and Dehn filling.** In earlier work, I studied when M has a Dehn filling $M(\gamma)$ with cyclic fundamental group. There, I used the variety of representations of $\pi_1(M)$ into $\mathrm{PSL}_2\mathbb{C}$, which is connected to the hyperbolic geometry of M; such techniques were part of the proof of the landmark Cyclic Surgery Theorem [CGLS]. For a space Y, let $X(Y)$ denote the algebraic variety of representations of $\pi_1(Y)$ into $\mathrm{PSL}_2\mathbb{C}$, modulo conjugation. Using a fancy version of Mostow rigidity due to Gromov, Thurston, and Goldman, I proved:

4.3. **Theorem** ([Dun1]). *Let M be a 3-manifold with ∂M a torus whose interior is hyperbolic. Let X_0 be the irreducible component of $X(M)$ containing the holonomy representation of the hyperbolic structure. Consider the map $i^*\colon X(M) \to X(\partial M)$ induced by the inclusion $i\colon \partial M \to M$. Then the restriction of i^* to X_0 is a birational isomorphism onto its image.*

The above theorem was one of the key tools for proving the following theorem about Dehn surgery on knots in S^3. A manifold M is called *small* if every incompressible surface in M has non-empty boundary.

4.4. **Theorem** ([Dun1]). *Let M be the exterior of a knot in S^3 and assume that M is small and hyperbolic. Suppose M has a non-trivial cyclic Dehn filling with slope $r \in \mathbb{Q}$. Then there is an incompressible surface in M with the following rare property: the boundary slope is not an integer and and lies in $(r-1, r+1)$.*

5. Homology theories of knots in S^3

A fundamental task in 3-dimensional topology is to understand embedded circles in the 3-sphere, that is, the study of knots. A large number of knot invariants have been discovered, many of which associate a polynomial to each knot. Recently, homology theories of knots have been introduced whose Euler characteristics, in an appropriate sense, are these knot invariants. Sergei Gukov, Jake Rasmussen, and I have studied the relationships between these various homology theories and proposed a cohesive framework for unifying two seemingly disparate types of them [DGR].

First, I need to discuss the polynomial invariants themselves. Associated to a knot K in S^3 is the two-variable HOMFLY-PT polynomial $P(a,q)$, which has a simple combinatorial description. This polynomial unifies the invariants coming from the quantum groups associated to the Lie algebras $sl(N)$, which are denoted by $P_N(q)$ and are equal to $P(a = q^N, q)$. Here, the original Jones polynomial J is just P_2. The HOMFLY-PT polynomial also encodes the classical Alexander polynomial as $P(a = 1, q)$.

As mentioned, a number of different knot homology theories have been discovered that are related to these polynomial invariants. Although the details of these theories differ, the basic idea is that for a knot K, there should be doubly-graded homology groups $H_{i,j}(K)$ whose graded Euler characteristic with respect to one of the gradings gives a particular knot polynomial. Such a theory is referred to as a *categorification* of the knot polynomial. For example, the Jones polynomial J is the graded Euler characteristic of the *Khovanov Homology* $H^{Kh}_{i,j}(K)$ [Kho]; that is,

$$J(q) = P(q^2, q) = \sum_{i,j} (-1)^j q^i \dim H^{Kh}_{i,j}(K).$$

This theory was generalized by Khovanov and Rozansky [KR1] to categorify the quantum $sl(N)$ polynomial invariant $P_N(q)$. Another such theory is knot Floer homology, $HFK_{i,j}(K)$, introduced in [OS2, Ras1], which categorifies the Alexander polynomial.

While the above homology theories categorify polynomial knot invariants all associated to the HOMFLY-PT polynomial, their constructions are very different. The Khovanov-Rozansky homology is defined in a algebraic/combinatorial manner, whereas knot Floer homology comes from counting pseudo-holomorphic curves in a certain configuration space. Despite this, the main result of our paper was to formulate

5.1. **Conjecture** ([DGR]). *The Khovanov-Rozansky $sl(N)$ homology (for all N) and knot Floer homology can be unified into a single theory, satisfying a long list of axioms.*

We proposed a very detailed picture of what this theory should look like: it is a triply-graded homology theory categorifying the HOMFLY-PT polynomial, together with a certain additional formal structure. The additional structure, namely a family of differentials on the homology, is what extracts the $sl(N)$ and Floer homologies from the HOMFLY-PT theory. Motivation for our conjecture included the recent work of Gukov, Schwarz, and Vafa [GSV], which gives a physics interpretation of the Khovanov-Rozansky homology which naturally suggests stabilization of the $sl(N)$ homologies when N is sufficiently large. At the small N end, the original $sl(2)$ Khovanov homology and knot Floer homology seem to be very closely related. For instance, their total ranks are very often (but not always) equal [Ras2].

Even though we didn't give a definition of this triply-graded theory, our description of its properties is powerful enough to give us many non-trivial predictions about knot homologies that can then be verified by direct computation. Much of our paper [DGR] was devoted to various examples and patterns; these served to illustrate the internal consistency of our axioms. Subsequent work has confirmed several aspects of our conjecture. Jake Rasmussen has shown that the $sl(N)$-homology stabilizes for large N [Ras3], and the limit is the HOMFLY-PT homology introduced by Khovanov and Rozansky [KR2]. Rasmussen also showed that there are spectral sequences corresponding to about half of the differentials we postulated.

6. Miscellaneous Results

Finally, I will briefly mention some additional results that don't fit into the above categories. First, Ian Agol and I used Perelman's work [Per1, Per2] to relate the volume of a hyperbolic manifold with that of the complement of a closed geodesic [AST, Thm. 10.1], improving on earlier work of Agol [A]. I then verified that this new relationship holds in some 25,000 examples, giving an experimental check of one aspect of Perelman's revolutionary work [AST, Appendix]. In [Dun3], I answered a question of Curt McMullen by showing that there exist fibered 3-manifolds where the Alexander and Thurston norms differ. Also, Danny Calegari and I showed that $\mathrm{PSL}_2\mathbb{C}$ contains an ascending HNN-extension of a free group [CD3] by using a particular 3-manifold which fibers over the circle. In my most recent paper, I joined with Garoufalidis, Shumakovitch, and Thistlethwaite to investigate the behavior of the knot invariants discussed in Section 5 under the cut-and-paste operation of genus-2 mutation [DGST].

Turning now to further results on character varieties, in [Dun2] I showed that certain roots of unity associated to the ideal points of $X(M)$ could take on values other than ± 1. The character variety of a 3-manifold has an associated two-variable A-polynomial, which gives another polynomial invariant of knots in S^3. Stavros Garoufalidis and I used a deep theorem of Kronheimer and Mrowka [KM] to show that this polynomial is non-trivial for the complement of a nontrivial knot in S^3 [DG] (this was also shown in [BZ2]). David Boyd has shown that the Mahler measure of this A-polynomial is related to the hyperbolic volume of M. In [Dun5], I gave an example of a manifold with a specific A-polynomial which arose in the work of D. Boyd and F. Rodriguez Villegas; the fact that there is a manifold with this polynomial lets one express the Mahler measure of that polynomial exactly in terms of special values of L-functions.

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