

# Research Statement

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My primary research interests lie in the areas of functional analysis and harmonic analysis. I also have interests in certain subjects in stochastic process, e.g. maximal martingale inequalities.

One part of my research is on harmonic analysis for (matrix) operator-valued functions. This includes the theory of Hardy and BMO spaces, maximal inequalities, singular integrals and Littlewood-Paley inequalities for operator-valued functions. The motivation comes from the theory of operator spaces and noncommutative probability. My results are already significant in the case of  $n \times n$  matrix-valued functions for  $n$  large, where I get estimations with constants **independent** of the size  $n$ .

Another part of my research is on harmonic analysis problems associated with semigroups of operators, e.g. Riesz transforms, Fourier multipliers, maximal martingale and ergodic inequalities on nonabelian discrete groups or von Neumann algebras. The proof of classical theories on those objects heavily rely on certain geometric/metric condition of Euclidean spaces, like “doubling measures”, which are not available in the noncommutative case. My research in this part **aims** to find “right” substitutions for such geometric properties by using semigroups of operators, which exist in a very general setting.

## 1. Operator-valued Hardy Spaces and Singular Integrals

The (real) Hardy spaces  $H^p(\mathbb{R})$  are important objects in classical analysis. They are essentially the same as  $L^p(\mathbb{R})$ 's for  $1 < p < \infty$  (known as the Littlewood-Paley-Stein theory); while for  $p = 1, \infty$ , the  $L^p$  spaces have some undesirable properties, and  $H^p$ 's are much better behaved. For many topics in classical analysis,  $H^1$  (resp. BMO) appears as a natural substitute of  $L^1$  (resp.  $L^\infty$ ).

In [M1], I defined  $H^p$  spaces for operator-valued functions by considering the operator-valued Littlewood-Paley G-functions. One analogue of classical results is that my  $H^1$ 's are preduals of the noncommutative BMO spaces defined in recent works on matrix-valued harmonic analysis and noncommutative martingale inequalities. I also obtained desired interpolation results and the  $L^p$ - $H^p$  equivalence theory.

The **noncommutativity** (of production of operators) is the main difficulty of my study and the main difference from the vector-valued cases. To illustrate it, let me describe the BMO space studied in [M1] in the  $n \times n$  matrix valued case. Let  $M_n$  be the algebra of  $n \times n$  matrices. For  $a \in M_n$ , denote by  $\|a\|_{M_n}$  the operator norm of  $a$  on  $\ell_n^2$ . The fact that  $a^*a \neq aa^*$  leads to column and row versions of the classical BMO-norms. The column BMO norm of an  $M_n$ -valued function  $\varphi$  is

$$\|\varphi\|_{\text{BMO}_c} = \sup_{I \subset \mathbb{R}} \left\| \frac{1}{|I|} \int_I (\varphi(t) - \varphi_I)^* (\varphi(t) - \varphi_I) dt \right\|_{M_n}^{\frac{1}{2}},$$

Here  $\varphi_I = \frac{1}{|I|} \int_I \varphi(t) dt$  with  $\int_I$  taking for each coefficient of  $\varphi(t)$ . The row BMO norm is  $\|\varphi\|_{\text{BMO}_r} = \|\varphi^*\|_{\text{BMO}_c}$ . Note that these two norms are **not** equivalent uniformly over  $n$ . In fact, let  $b$  be a scalar valued function with  $\|b\|_{\text{BMO}} = 1$ . Set  $\varphi = \sum_{k=1}^n b e_1 \otimes e_k$ , where  $e_i$  is the canonical basis of  $\ell_n^2$ . It is easy to check that  $\|\varphi\|_{\text{BMO}_c} = 1$  while  $\|\varphi\|_{\text{BMO}_r} = \sqrt{n}$ .

Another difficulty of my research in this direction is the absence of maximal functions. There is no way to define a least upper bound even for a pair of  $2 \times 2$  matrices. However, maximal  $L^p$  norms are still possible due to the work of Pisier and Junge (see [J], [P]). Based on their work, I obtained a Hardy-Littlewood maximal inequality for operator-valued function (see [M1]). Its applications include a Lebesgue differentiation theorem and a convergence theorem for Poisson integrals for operator-valued functions. **Not all** the classical results have nice analogues in the operator-valued setting. For example, the classical Carleson embedding theorem fails to have a satisfactory operator-valued analogues. (See [NPTV]). Another example is the subtle behavior of paraproducts with an operator valued symbol. In [M2], I proved that even the  $L_\infty$ -boundedness of the symbol does not imply the  $L_2$  boundedness of the paraproduct contrary to the fact that the BMO-boundedness of the symbol is already sufficient in the scalar-valued case.

Continuing on this line, I studied extrapolation properties for operator-valued dyadic paraproducts and singular integrals in [M3] and [MP] (joint with J. Parcet).

## 2. Harmonic Analysis Problems associated with Semigroups of Operators.

The extension of classical harmonic analysis to the context of semigroups of operators has drawn many mathematicians' attention after Stein's pioneering work (see [St1], [Cowl]). In the abstract setting, we normally miss certain good properties associated with the geometric structure of Euclidean spaces, for example, the doubling measure condition. However,  $L^p$ -spaces and semigroups of operators can be studied in a very general setting. For example, given an unbounded operator  $A$  on a  $L^2$ -space with a conditionally negative kernel,  $(e^{tA})_{t \geq 0}$  is always a semigroup of positive operators.

My research in this direction strives to use certain behaviors of semigroups of operators to substitute for geometric properties present in the proof of classical analysis. For example, when studying functions on the real line  $\mathbb{R}$ , people often consider certain integrations on cones and cubes for their analytic extensions to  $\mathbb{R} \times \mathbb{R}_+$ . Let us consider an arbitrary measure space  $(\Omega, \sigma, \mu)$  instead of  $\mathbb{R}$ . It is now hard to define "cones" and "cube" on  $\Omega \times \mathbb{R}_+$  since we even do not have a metric on  $\Omega$ . Assume that there exists a semigroup of operators  $(P_y)_{y \geq 0}$  defined simultaneously on  $L^p(\Omega, \sigma, \mu)$  satisfying standard assumptions. It is natural to view the operators  $\int_0^\infty P_y(\cdot) dy$  as **alternatives to** classical integrations on cones, " $\int_0^\infty \int_{|x| < y} \cdot dx dy$ " and to view  $P_t \int_0^t(\cdot) dy$  as **alternatives to** integrations on cubes, " $\int_0^t \int_{|x| < t} \cdot dx dy$ ". The question is, when these "alternatives" are suitable (i.e. to guarantee analogues of classical results)?

An interesting observation in my recent research is a quasi monotone property satisfied by all subordinated Poisson semigroups  $(P_y)_y$  that  $P_{2y}f \leq 2P_yf$  for any  $y$  and  $f \geq 0$ . This property seems to be a favorable **alternative** to the doubling measure assumption. Using this property, my recent research takes initial steps towards building up the correspondence between the validity of analogues of certain classical results and behaviors of underlying semigroups of operators. In [M4], I study tent spaces  $\mathcal{T}_p$  associated with semigroups of operators and find sufficient and necessary conditions on the underlying semigroups such that the classical duality result between  $\mathcal{T}_p$  holds. Apply to Hardy spaces, I obtain an  $H^1$ -BMO duality result associated to certain semigroups of operators for arbitrary measure spaces, and more general, for semifinite von Neuman algebras. The BMO norms studied in [M4] are defined as  $\sup_t \| |P_t|f - P_t f \|^2_{\frac{1}{2}}$ , since intervals are no longer available. Further studies on the BMO spaces lead to the boundedness of noncommutative Riesz transforms and a large class of Fourier multipliers on nonabelian discrete groups (see [JM1], [JM2]).

My study has a strong motivation and takes big advantage of noncommutative probability, e.g. noncommutative martingales. The applications include new examples of Rieffel's quantum metric spaces (see [Rie]) and optimal constants of the noncommutative maximal ergodic inequalities proved by Junge/Xu (see [JX]). Moreover, even when restricted to the commutative setting, some of my results are still new. See [M5] for a nice connection between BMO spaces and dyadic BMO spaces on Euclidean spaces, which is found when I studied the subject in [M1]; see [JM2] (Corollary 5.3 and Remark 5.5) for better/optimal constants for the  $L_p$  boundedness of Stein-Cowling's universal Fourier multipliers.

## 3. Research plan in the future.

I. *Fourier multipliers/Singular Integrals in the non-commutative setting.* Given a (nonabelian) discrete group  $G$ , let  $\lambda_g$  be the left regular representation. Let  $\phi$  be a scalar valued function on  $G$ . A Fourier multiplier  $M_\phi$  with symbol  $\phi$  is a map sending  $\lambda_g$  to  $\phi(g)\lambda_g$ . Which conditions on  $\phi$  imply the boundedness of  $M_\phi$  on  $L^p(VN(G))$ ? More general, what are analogues of Calderón-Zygmund theory on singular integral in the noncommutative setting?

(i) Fourier multipliers from conditionally negative functions on nonabelian discrete groups. Under the support of the NSF grant DMS-0901009, I already started a program to understand Riesz transforms/Fourier multipliers on discrete groups, where conditionally negative functions exists. This existence is equivalent to the so-call Haagerup property and has been proved for all amenable groups and many non-amenable groups as well. We already had a joint paper (in preparation) with Junge and Parcet.

(ii)  $H^1$ -BMO duality on von Neumann algebras. A satisfactory  $H^1$ -BMO duality theory in the context of general semigroups will be a great help in studying noncommutative Fourier multipliers/singular integrals. [M4] is a first effort on this problem, where two additional conditions on the underlying semigroups are

assumed. I plan to find more concrete examples satisfying these conditions and applications to the program described in (i).

(iii) Dimension-free estimate on commutative problems. Stein asked for the possibility of infinite-dimensional formulation of part of harmonic analysis problems in  $\mathbb{R}^n$  (see the end of [St2]). Certain desired results of Fourier multipliers in the noncommutative setting will imply dimension free results in the commutative case. See a result (Corollary 5.3) in [JM2] which improved the known commutative results. On the other hand, in the study of fourier multipliers on noncommutative discrete groups (with Junge and Parcet), we found an exciting clue to reduce the questions to the corresponding Fourier multipliers on Euclidean spaces, where the dimension of the constructed Euclidean spaces is crucial and can be infinite in many cases. This means that noncommutative theories will need dimension free estimate from the commutative case. This motivates my research plan to find dimension free estimate for Fourier multipliers/Singular integrals on Euclidean spaces. As mentioned previously, I have had an alternative for the missing of a uniform doubling measure constant, which is one of the main difficulty for dimension free estimates. Namely, for every subordinated standard positive semigroups  $P_t$ , we have  $P_{2t}f \leq 2P_t f$  for positive  $f$ 's.

**II Littlewood-Paley-Stein theory for operator-valued functions.** There are many interesting problems in this direction. Generalization of Rubio de Francia's Littlewood-Paley theory ([Ru]) to the operator-valued setting will yield a new sufficient condition of the boundedness of the **Schur multipliers**. Joint with Q. Xu, I have obtained some partial results in this project.

There are several natural  $H^1$  norms for operator-valued functions. An essential way to understand the noncommutative  $H^p$  theory is to compare various these norms and find sharp estimations in the  $n \times n$  matrix-valued case. This will also shed some light on operator space structures of Hardy spaces and BMO spaces.

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