(1) (Kelvin inversion) Prove that if \( u(x) \) is harmonic in \( \Omega \subset \mathbb{R}^n, n \geq 2 \), then
\[
v(x) = \frac{1}{|x|^{n-2}} u(x^*)
\]
is harmonic in \( \Omega^* \) defined as \( \{ y^* : y \in \Omega \} \), where we define \( x^* = x/|x| \) to be the reflection of \( x \) in the unit sphere. \textit{Hint.} Compute \( \Delta v \).

\textit{Note 1.} Check for yourself that \((x^*)^* = x\) and that \(0^* = \infty, \infty^* = 0\). Thus the Kelvin inversion turns problems for Laplace’s equation on unbounded domains into problems on bounded domains.

\textit{Note 2.} The function \( v \) is called the \textit{Kelvin inversion} of \( u \). Observe that in two dimensions, the Kelvin inversion of \( u(z) \) is simply \( u(1/z) \) where \( z \) is the complex conjugate of \( z \in \mathbb{C} \cong \mathbb{R}^2 \), because you can check that \( z^* = 1/z \).

(2) Show the Green function for the unit disk in \( \mathbb{R}^2 \) can be written in the simple form
\[
G(z, w) = \frac{1}{2\pi} \log \left| \frac{z - w}{1 - z \overline{w}} \right|
\]
where we regard \( z \) and \( w \) as complex numbers (in other words, writing the vector \((x_1, x_2)\) as a complex number \(z = x_1 + ix_2\) and so on).

(3) McOwen 4.2.1(a)

\textit{Remark.} This gives a simple formula for the Poisson kernel of the unit disk in \( \mathbb{R}^2 \).

(4) (Negativity of the Green function.) Show \( G(x, y) < 0 \) for all \( x, y \in \Omega, x \neq y \).

\textit{Notation.} Many authors call \(-G\) the Green function, so that their Green function is positive.

(5) (Symmetry of the Green function.) Let \( \Omega \) be a bounded smooth domain in \( \mathbb{R}^n, n \geq 2 \). Assume that for each \( x \in \Omega \) a “corrector” function exists, that is, a harmonic function \( w_x(y) \) with \( w_x \in C^2(\overline{\Omega}) \) and \( w_x(y) = -K(x-y) \) for \( y \in \partial \Omega \). Define the Green function
\[
G(x, y) = K(x-y) + w_x(y),
\]
so that \( G(x, y) = 0 \) when \( x \in \Omega, y \in \partial \Omega \).

Prove the Green function is symmetric: \( G(x, y) = G(y, x) \) whenever \( x, y \in \Omega, x \neq y \).

\textit{Hints.} Fix \( x, y \in \Omega, x \neq y \), and define \( u(z) = G(x, z) \) and \( v(z) = G(y, z) \) for \( z \in \Omega \), so that \( u \in C^2(\overline{\Omega} \setminus \{x\}), v \in C^2(\overline{\Omega} \setminus \{y\}) \). Use Green’s second formula on the domain formed from \( \Omega \) by removing balls of radius \( \epsilon \) centered at \( x \) and \( y \).