Math 553 — Fall 2004 — Final Exam

Total points: 175. Show ALL your working and make your explanations as full as possible. Electronic devices are not allowed on this exam; neither are books or notes.

Green Formulas:
\[
\int_{\Omega} [v \Delta u + \nabla v \cdot \nabla u] \, dx = \int_{\partial \Omega} v \frac{\partial u}{\partial \nu} \, dS
\]
\[
\int_{\Omega} [v \Delta u - u \Delta v] \, dx = \int_{\partial \Omega} \left[ v \frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu} \right] \, dS
\]

Darboux Formula:
\[
\left( \frac{\partial^2}{\partial r^2} + \frac{n-1}{r} \frac{\partial}{\partial r} \right) M_u(x, r) = M_{\Delta u}(x, r)
\]
1: (35 points) First order equations

Let $G(z) = \frac{1}{3}(1 - z)^3$.

(a) Solve the conservation law $G(u)_x + u_y = 0$ for $x \in \mathbb{R}, 0 < y < 3$, given initial data

$$u(x, 0) = \begin{cases} 1 & \text{for } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

(A well–labelled sketch of the characteristics is a good way to present your solution.)

(b) Then find an equation for the shock slope $x'(y)$ when $y > 3$. 
2: (30 points) Heat equation

Let $\Omega$ be a bounded domain in $\mathbb{R}^n, n \geq 2$, with smooth boundary. Assume $u(x, t) \in C^\infty(\overline{\Omega} \times [0, T])$ solves the following initial value problem for the heat equation:

$$
\begin{align*}
    u_t &= \Delta u, \quad x \in \Omega, \quad t > 0, \\
    u(x, 0) &= g(x), \quad x \in \Omega, \\
    u(x, t) &= 0, \quad x \in \partial \Omega, \quad t > 0,
\end{align*}
$$

where $g \in C_0^\infty(\Omega)$ is given, $g \not\equiv 0$. Assume $g(x) \leq 0$ for all $x \in \Omega$.

(a) Show that $u(x, t) \leq 0$ for all $x \in \Omega, 0 < t < T$.

(b) Actually $u(x, t) < 0$ for all $x \in \Omega, 0 < t < T$. Assuming this, explain why it means the heat equation allows “infinite propagation speed” of disturbances.
3: (15 points) Nonhomogeneous wave equation in one dimension

Use Duhamel’s principle to find an explicit solution of

\[ u_{tt} = u_{xx} + e^x, \quad x \in \mathbb{R}, \quad t > 0, \]

\[ u(x, 0) = 0, \]

\[ u_t(x, 0) = 0. \]
4: (30 points) Consider the following two functions on $\mathbb{R}^2$:

\[ u(x, y) = \begin{cases} 
2x^2 + 5 & \text{for } x > 0 \\
0 & \text{for } x < 0 
\end{cases} \]

\[ v(x, y) = \begin{cases} 
2x^2 + y^2 + 5 & \text{for } x > 0 \\
0 & \text{for } x < 0 
\end{cases} \]

Show one of these functions is a weak solution of the equation $w_{xy} = 0$, and the other function is not.

*Hint.* Start by writing out the definition of a weak solution.
5: (35 points) Let $f$ be a smooth function with compact support in $\mathbb{R}^3$.

(a) Write down the fundamental solution of the Laplacian (solving $\Delta K = \delta$ in $\mathbb{R}^3$ where $\delta$ is the Dirac delta function).

(b) Write down a formula for $u(x)$ solving Poisson’s equation $\Delta u = f$ in $\mathbb{R}^3$ with $u(x) \to 0$ as $|x| \to \infty$.

(c) Show formally that $\Delta u = f$ weakly.

(d) See next page.
Assume $f$ is radially symmetric and is supported in the unit ball. (Radially symmetric means that $f(x) = F(r)$ for some function $F$, where $|x| = r$.) Suppose also that $\int_{\mathbb{R}^3} f(x) \, dx = 1$.

Show $u(x) = K(x)$ for all $|x| > 1$.

*Hint.* Spherical coordinates in the formula for $u$.

*Aside.* Part (d) means physically that the gravitational potential of a radially symmetric planet is the same (outside the planet) as if all the mass were concentrated at the center.
6: (30 points) Consider Poisson’s equation \( \Delta u = f \) in a domain \( \Omega \subset \mathbb{R}^n \) with smooth boundary, with boundary condition \( u = g \) on \( \partial \Omega \).

State and prove a result about “continuous dependence of \( u \) on \( f \) and \( g \)”.

*Hint.* Maximum principle. And if you cannot solve the problem as stated, then for half-credit on the problem you can assume \( f \equiv 0 \).
1. The PDE is a first order conservation law, and so $u$ is constant along the characteristics. The projected characteristics have the form $x = G'(u)y + x_0$. Since $G'(z) = -(1 - z)^2$ and $G'(0) = -1$, we find

$$
(G')^{-1}(w) = 1 - \sqrt{-w}, \quad w \leq 0.
$$

Since

$$G'(0) = -1 \quad \text{and} \quad G'(1) = 0,
$$

we can immediately sketch most of the characteristics:

A fan starts at $(0, 0)$ and a shock starts at $(1, 0)$. By the solution to Riemann’s problem, in the fan we have

$$u = (G')^{-1}(x/y) = 1 - \sqrt{-x/y}.
$$

By the jump condition, the shock has slope

$$x'(y) = \frac{G(u_l) - G(u_r)}{u_l - u_r} = \frac{G(1) - G(0)}{1 - 0} = \frac{-1}{3},
$$

and so the shock path is $x = 1 - y/3$, at least up until $y = 3$, when the shock hits the fan at $x = 0$. For $y > 3$ the jump condition is

$$x'(y) = \frac{G(u_l) - G(u_r)}{u_l - u_r} = \frac{G(1 - \sqrt{-x/y}) - G(0)}{1 - \sqrt{-x/y} - 0} = \frac{-1}{3} \frac{(\sqrt{-x/y})^3 - 1}{\sqrt{-x/y} - 1}.
$$

2. (a) Since $u$ solves the heat equation, the weak maximum principle certainly applies. Hence the maximum of $u$ over the closed heat cylinder is attained on the parabolic boundary. But $u = 0$ on the spatial boundary, and $u = g \leq 0$ initially. Hence the maximum over the parabolic boundary is 0, so that $u \leq 0$ everywhere by the weak maximum principle.

(b) Because $g$ has compact support in $\Omega$, there is a ball $B(x_0, \epsilon)$ in $\Omega$ on which $g \equiv 0$. In particular, at $t = 0$ we see $u = 0$ at all points within distance $\epsilon$ of $x_0$. But $u(x_0, t) < 0$ for all $t > 0$, and so the disturbance (that is, the region on which $u \neq 0$) propagates a distance
\( \epsilon \) in arbitrarily small time \( t \). Since \( \epsilon/t \to \infty \) as \( t \to 0 \), we conclude that the heat equation allows infinite propagation speed of disturbances.

3. For Duhamel’s principle, we first seek \( U(x, t; s) \) solving

\[
U_{tt} = U_{xx}, \quad x \in \mathbb{R}, \quad t > 0,
\]

\[
U(x, 0; s) = 0,
\]

\[
U_t(x, 0; s) = e^x.
\]

By D’Alembert’s formula, the solution is

\[
U(x, t; s) = \frac{1}{2} \int_{x-t}^{x+t} e^\xi \, d\xi = e^x \sinh t.
\]

(In this case, \( U \) is independent of \( s \).) Now Duhamel’s principle yields

\[
u(x, t) = \int_0^t U(x, t - s; s) \, ds = e^x \int_0^t \sinh(t - s) \, ds = e^x (\cosh t - 1).
\]

One can verify directly that this is the desired solution.

4. The condition for weak solutions of \( w_{xy} = 0 \) is that the following equation be true for all test functions \( \phi \):

\[
\iint_{\mathbb{R}^2} w \phi_{xy} \, dxdy = 0.
\]

For \( u \) this equation can be written as

\[
\iint_{x>0} (2x^2 + 5) \phi_{xy} \, dxdy = 0
\]

and for \( v \) it can be written as

\[
\iint_{x>0} (x^2 + y^2 + 5) \phi_{xy} \, dxdy = 0.
\]

The two integrals are:

\[
\text{for } u: \quad \iint_0^\infty \int_{-\infty}^{\infty} (2x^2 + 5) \phi_{xy} \, dydx = 0 \quad \text{for } v: \quad \int_0^\infty \int_{-\infty}^{\infty} (x^2 + y^2 + 5) \phi_{xy} \, dydx = 0.
\]

Since \( \int_{-\infty}^{\infty} \phi_{xy} \, dy = |\phi_x|_{-\infty}^{\infty} = 0 \) by compact support, the integral for \( u \) vanishes for all test functions, and so the solution \( u \) is indeed a weak solution.

For the other integral we can certainly use compact support to move the \( y \) derivative off \( \phi \) and onto \( x^2 + y^2 \). Then we can do the \( x \) integration. This gives us:

\[
\int_0^\infty \int_{-\infty}^{\infty} (x^2 + y^2 + 5) \phi_{xy} \, dydx = -\int_0^\infty \int_{-\infty}^{\infty} 2y\phi_x \, dydx = \int_{-\infty}^{\infty} 2y\phi(0, y) \, dy.
\]

Now all you have to do is convince yourself that there is at least one test function \( \phi \) for which the last integral here is non-zero. So \( v \) is not a weak solution.
5. (a) \( K(x) = -1/4\pi|x| \).

(b) \( u(x) = \int_{\mathbb{R}^3} K(x - y)f(y) \, dy \). Note that \( K(x - y) \to 0 \) as \( |x| \to \infty \), uniformly for \( y \) in the support of \( f \). Hence \( u(x) \to 0 \) as \( |x| \to \infty \).

(c) The Laplacian is self-adjoint, and so we consider
\[
\int u(x)\Delta \phi(x) \, dx = \int \int K(x - y)f(y) \, dy \, \Delta \phi(x) \, dx
= \int \int K(x - y)\Delta \phi(x) \, dx \, f(y) \, dy
= \int \int \Delta_x K(x - y)\phi(x) \, dx \, f(y) \, dy
= \int \int \delta(x - y)\phi(x) \, dx \, f(y) \, dy
= \int \phi(y) f(y) \, dy
= \int f(x) \phi(x) \, dx.
\]

(d) Now suppose \( f \) is radially symmetric, and is supported in the unit ball. For \( x \) outside the unit ball, spherical coordinates give
\[
u(x) = \int_{\mathbb{R}^3} K(x - y)f(y) \, dy
= \int_0^1 \int_{S^2} K(x - \rho x) \, dx \, \rho^2 F(\rho) \, d\rho
= \int_0^1 K(x) \int_{S^2} \rho^2 F(\rho) \, d\rho
\]
based on the mean value theorem applied to \( y \mapsto K(x - y) \), which is harmonic for \( |y| < 1 \) since \( |x| > 1 \). Hence
\[
u(x) = K(x) \int_{\mathbb{R}^3} f(y) \, dy = K(x).
\]

6. —